PRESERVING AND CONSTRUCTING MULTIPLE GAPS

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ABSTRACT. We analyze conditions to preserve with forcing multiple gaps, defined conditions to ensure their existence and study the minimal size of specific type of gaps.

1. Introduction

This work is divided mainly in two parts. However, before discussing them, we start establishing some notation: $\wp(X)$ denotes the power set of X. We write $X \subseteq^* Y$ and $X =^* Y$ to express that $X \setminus Y$ is finite and $X \triangle Y$ is finite respectively. Also we denote by $\wp(\omega)/fin$ the quotient of $\wp(\omega)$ with the finite subsets of natural numbers. It forms a boolean algebra with $\subseteq^*$ as order.

A pregap g is a pair of families $(\mathcal{A}, \mathcal{B})$ where both $\mathcal{A}$ and $\mathcal{B}$ are orthogonal families of infinite subsets of ω. This means that for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $A \cap B =^* \emptyset$. Furthermore, a pregap g is called a gap if there is no set of natural numbers C such that for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$, $A \subseteq^* C$ and $B \cap C =^* \emptyset$. When there exists a C with such properties (i.e., g is not a gap), we say that C fills or separates g. If $g = (\mathcal{A}, \mathcal{B})$ is a gap and both $\mathcal{A}$ and $\mathcal{B}$ can be well-ordered by $\subseteq^*$, g is called a (κ, λ)-gap where $(\mathcal{A} \subseteq^*)$ is isomorphic to κ and $(\mathcal{B} \subseteq^*)$ is isomorphic to λ. Thus, when we write $\mathcal{A}$ as $\{A_\alpha : \alpha < \kappa\}$ and $\mathcal{B}$ as $\{B_\alpha : \alpha < \lambda\}$, we assume that $\alpha < \beta < \kappa$ implies $A_\alpha \subseteq^* A_\beta$ and $\alpha < \beta < \lambda$ implies $B_\alpha \subseteq^* B_\beta$. Notice that if X and Y are cofinal subsets of κ and λ respectively, then the pregap $(\{A_\alpha : \alpha \in X\}, \{B_\alpha : \alpha \in Y\})$ is also a gap. Hausdorff showed that there exist an (ω_1, ω_1)-gap and Rothenberger showed that there is an $(\omega, \mathfrak{b})$-gaps, where $\mathfrak{b}$ is the bounding number (see [13]). However, there are no (ω, ω)-gaps.

Gaps appear frequently in infinite combinatorics. In general, the problem of extending homomorphisms between two Boolean algebras is deeply connected to the type of gaps present in each of them (see [8], Theorem 5). We are particularly interested in gaps in $\wp(\omega)/fin$. Typically, if one wishes to carry out any recursive construction in $\wp(\omega)/fin$, gaps represent potential obstructions to the construction. This is why every linear order of size ω_1 can be embedded into $\wp(\omega)/fin$, while it is consistent that this fails for 2^ω (see [11] and [14]).

The analysis of gaps plays a crucial role in Todorčević's proof that OGA implies $\mathfrak{b} = \omega_2$ (see [14]). There are also fascinating analogies and similarities between Aronszajn (Suslin) trees and gaps (specifically destructible gaps), see [1]. It is impossible to summarize all the applications of gaps in infinite combinatorics here. However, we emphasize that their importance cannot be overstated. The reader is encouraged to consult [16], [14], and [13] to learn more about this topic.

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This is also why the multidimensional gaps introduced by Avilés and Todorčević (which generalize the usual gaps) are of particular interest. The reader can see [3] and [2] for further details. No prior knowledge of [3] is required to follow this paper, as we will recall the necessary definitions and results as needed. However, we recommend consulting the aforementioned references for those interested in a deeper exploration of this subject.

Now we present the concept of multiple gap introduced by Avilés and Todorčević in [3], which will be the main focus of our papaer.

Definition 1.1. Let n a natural number.

(1) A family $\{\mathcal{A}_i : i < n\} \subseteq \wp([\omega]^\omega)$ is called an n-pregap if for all $i, j < n$ with $i \neq j$ $\mathcal{A}_i$ and $\mathcal{A}_j$ are orthogonal, meaning for every $I \in \mathcal{A}_i$ and $J \in \mathcal{A}_j$ we have $I \cap J =^* \emptyset$.
(2) A n-pregap $\{\mathcal{A}_i : i < n\}$ is an n-gap if for all $\{C_i : i < n\} \subseteq [\omega]^\omega$ such that for each $i < n$ and each $A \in \mathcal{A}_i$ $A \subseteq^* C_i$, then $\bigcap_{i<n} C_i \neq \emptyset$.
(3) If $\{\kappa_i : i < n\}$ is a set of cardinals, an n-gap (pregap) $\{\mathcal{A}_i : i < n\}$ is a $(\kappa_i : i < n)$-gap (pregap) when it satisfies that for every $i < n$ there is an enumeration $\{A^i_\alpha : \alpha < \kappa_i\}$ of $\mathcal{A}_i$ such that if $\alpha < \beta < \kappa_i$, then $A^i_\alpha \subset^* A^i_\beta$.

We fix $n \in \omega$ for the rest of our study. Also if $\{\mathcal{A}_i : i < n\}$ is an n-gap (or pregap) it will always be enumerated as in Definition 1.1. If g is an n-pregap and $\{C_i : i < n\} \subseteq [\omega]^\omega$ witnesses that g is not an n-gap, we say that $\{C_i : i < n\}$ fills or separates g.

It is easy to see that a $(\kappa_i : i < n)$-pregap $\{\mathcal{A}_i : i < n\}$ is a gap if and only if any pregap defined by any cofinal subset of each κ_i cannot be filled. That means, if for each $i < n$ X_i is a cofinal subset of κ_i and $\mathcal{B}_i := \{A^i_\alpha : \alpha \in X_i\}$, then $\{\mathcal{B}_i : i < n\}$ is a gap if and only if $\{\mathcal{A}_i : i < n\}$ is a gap. The following is easy:

Proposition 1.2. *There are no $(\omega : i < n)-$gaps.*

Another relevant result for this work (though counter intuitive because for $n = 2$ the statement is not true) is the following theorem, which is a consequence of Corollary 21 proved in [3].

Theorem 1.3 (Avilés, Todorčević)**.** *MA implies there are no n-gaps of size ω_1 for $n \geq 3$.*

The paper is organized as follows: in the first section, we present some preservation results. We show that σ-n-links forcing notions do not destroy $(\kappa_i : i < n)$-gaps, in particular σ-centered forcing notions. Additionally, we prove Sacks forcing and Miller forcing are minimal in some sense of indestructibility (see Theorem 2.9 and Theorem 2.14). We also generalize an equivalence of $(\kappa_i : i < n)$-gaps using a colouring. However, the method for freezing a 2-gap does not remain valid.

In the second section, we explore methods for constructing multiple gaps. The first construction is based on [3], where we establish a cardinal condition to ensure their existence. In [9] the authors used a variant of $\diamondsuit$ to construct a 2-Hausdorff gap and we generalize these construction to $(\omega_1 : i < n)$-gaps. Finally, we use Capturing Axioms to build $(\omega_1 : i < n)$-gaps based on the construction done in [5] of a Hausdorff gap.

2. Preservation of multiple gaps.

Start with forcing preservation theorems.

2.1. **Forcing preservation.**

Definition 2.1. Let $\mathbb{P}$ be a forcing notion:

(1) Given g an n-gap we say $\mathbb{P}$ preserves g if for each $G \subseteq \mathbb{P}$ generic filter it holds that $V[G] \models g$ is an n-gap.
(2) Let $k \in \omega$. We say $\mathbb{P}$ is $\sigma - k-$linked if there are $\{\mathbb{P}_i : i < \omega\} \subseteq \wp(\mathbb{P})$ such that $\mathbb{P} = \bigcup_{i<\omega} \mathbb{P}_i$ and for all $i < \omega$ and $p_0, ..., p_{k-1} \in \mathbb{P}_i$ there is $q \in \mathbb{P}$ a common extension, that is $q \leq p_0, ..., p_{k-1}$.
(3) $\mathbb{P}$ is σ-centered if there is $\{\mathbb{P}_i : i \in \omega\} \subseteq \wp(\mathbb{P})$ such that $\mathbb{P} = \bigcup_{i<\omega} \mathbb{P}_i$ and each $\mathbb{P}_i$ is centered, that means, for each finite subset of conditions of $\mathbb{P}_i$ there is $q \in \mathbb{P}$ a common extension.

Lemma 2.2. *Let $\kappa_0 \geq ... \geq \kappa_{n-1} > \omega$ be regular cardinals, $c : \kappa_0 \times ... \times \kappa_{n-1} \to \omega$ a coloring. There exist $W \in [\kappa_0 \times ... \times \kappa_{n-1}]^{\kappa_0}$ a monochromatic set which is cofinal in each κ_i, that means for each $i < n$ and $\beta \in \kappa_i$ there is $(\alpha_0, ..., \alpha_{n-1}) \in W$ such that $\beta < \alpha_i$.*

Proof. Recursively, for each $0 < i < n$ and $\overline{\alpha} \in \prod_{j<i} \kappa_j$ we will define

(1) $c_{\overline{\alpha}} : \kappa_i \to \omega$,
(2) $W_i(\overline{\alpha}) \in [\kappa_i]^{\kappa_i}$ and
(3) $d_i : \prod_{j<i} \kappa_j \to \omega$

such that $W_i(\overline{\alpha})$ is a $c_{\overline{\alpha}}$-homogeneous set of color $d_i(\overline{\alpha})$.

We start by defining for each $\overline{\alpha} \in \prod_{j<n-1} \kappa_j$ the function $c_{\overline{\alpha}}$ as $c_{\overline{\alpha}}(\beta) = c(\overline{\alpha}, \beta)$. Assume for all $\overline{\alpha} \in \prod_{j<i+1} \kappa_j$ we have $W_{i+1}, c_{\overline{\alpha}}$ and d_{i+1} are already defined. Given $\overline{\alpha} \in \prod_{j<i} \kappa_j$ we define $c_{\overline{\alpha}}(\beta) = d_{i+1}(\overline{\alpha}, \beta)$. For $i = 0$ we define $c_{\emptyset} : \kappa_0 \to \omega$ given by $c_{\emptyset}(\beta) = d_1(\beta)$, $W_0 \in [\kappa_0]^{\kappa_0}$ a $c_{\emptyset}$-homogeneous set and $d_0 < \omega$ the image of W_0 under $c_{\emptyset}$. Let

$$W = \{(\alpha_0, ..., \alpha_{n-1}) \in \prod_{i<n} \kappa_i : \alpha_0 \in W_0,\ \alpha_1 \in W_1(\alpha_0),\ ..., \alpha_{n-1} \in W_{n-1}(\alpha_0, ..., \alpha_{n-2})\}.$$

By construction W is cofinal in each κ_i and $c(W) = d_0$. ☐

Theorem 2.3. *If $\mathbb{P}$ is a $\sigma - n$-linked forcing notion and g a $(\kappa_i : i < n)$-gap for some $\kappa_0, ..., \kappa_{n-1}$ uncountable regular cardinals, then $\mathbb{P}$ preserves g.*

Proof. Take $g = \{\mathcal{A}_i : i < n\}$ as in the hypothesis where $\mathcal{A}_i = \{A^i_\alpha : \alpha < \kappa_i\}$ and $\{\mathbb{P}_i : i < \omega\} \subseteq \wp(\mathbb{P})$ witnesses that $\mathbb{P}$ is $\sigma - k$-linked. By contradiction assume that g is not preserved by $\mathbb{P}$. Let $\dot{a}_i$ a $\mathbb{P}-$name for each $i < n$ such that $\mathbb{P} \Vdash$ "$(\dot{a}_i : i < n)$ fills g". For each $(\alpha_0, ..., \alpha_{n-1}) \in \kappa_0 \times ... \times \kappa_{n-1}$ we can find an $m_{(\alpha_0,...,\alpha_{n-1})} < \omega$ and a $p_{(\alpha_0,...,\alpha_{n-1})} \in \mathbb{P}$ such that

$$p_{(\alpha_0,...,\alpha_{n-1})} \Vdash \text{“}\forall i < n\, (A^i_{\alpha_i} \setminus m_{(\alpha_0,...,\alpha_{n-1})} \subseteq \dot{a}_i)\text{”}.$$

Fix $\ell_{(\alpha_0,...,\alpha_{n-1})}$ that $p_{(\alpha_0,...,\alpha_{n-1})} \in \mathbb{P}_{\ell_{(\alpha_0,...,\alpha_{n-1})}}$. Without lost of generality we can assume $\kappa_0 \geq \kappa_1 \geq ... \geq \kappa_{n-1} > \omega$. Also, there is a natural coloring $c : \kappa_0 \times ... \times \kappa_{n-1} \to \omega \times \omega$ given by $c(\alpha_0, ..., \alpha_{n-1}) = (\ell_{(\alpha_0,...,\alpha_{n-1})}, m_{(\alpha_0,...,\alpha_{n-1})})$. Thus, there is $W \in [\kappa_0 \times ... \times \kappa_{n-1}]^{\kappa_0}$ as in Lemma 2.2 of color (m, ℓ) for some $\ell, m < \omega$.

For $i < n$ define $D_i = \bigcup_{(\alpha_0,...,\alpha_{n-1}) \in W} A^i_{\alpha_i} \setminus m$. We claim that $\{D_i : i < n\}$ fills g in V. Clearly, for $i < n$ and $\beta < \kappa_i$, $A^i_\beta \subseteq^* D_i$. In order to prove that

$\bigcap_{i<n} D_i$ is empty we proceed by contradiction. Let $x \in \bigcap_{i<n} D_i$, there exist $(\alpha_0^0, ..., \alpha_{n-1}^0), ..., (\alpha_0^{n-1}, ..., \alpha_{n-1}^{n-1}) \in W$ such that for all $j < n$

$$p_{(\alpha_0^j, ..., \alpha_{n-1}^j)} \Vdash \text{“}\forall i < n\, (x \in \dot{a}_i)\text{”}.$$

Then we can find $q \leq p_{(\alpha_0^0, ..., \alpha_{n-1}^0)}, ..., p_{(\alpha_0^{n-1}, ..., \alpha_{n-1}^{n-1})}$. Thus, $q \Vdash$ “$x \in \bigcap_{i<n} \dot{a}_i$”, a contradiction. □

Corollary 2.4. Let g be a $(\kappa_i : i < n)$-gap for κ_i uncountable regular cardinals for each $i < n$.

(1) Any σ-centered forcing preserves g.
(2) Cohen forcing preserves g.

Now we need to introduce notation for trees.

Definition 2.5. (1) $(T, <_T)$ is a tree if T has minimum and for all $s \in T$ the set $pred_T(s) := \{t \in T : t <_T s\}$ is well ordered,
(2) $[T]$ denotes the sets of cofinal branches of T,
(3) for all $n < \omega$ $T_n = \{s \in T : |pred_T(s)| = n\}$,
(4) $T_{\leq n} = \bigcup_{i \leq n} T_i$
(5) for all $s \in T$ $succ_T(s)$ is the set of immediate successors of s in T.
(6) For a family $\mathcal{B}$ of subsets of ω, we say that a subset T is a $\mathcal{B}$-tree if for each $s \in T$ the set $succ_T(s)$ is included in a element of $\mathcal{B}$

The next two theorems deal with Sacks forcing and Miller forcing. Given a tree p, $x \in p$ is a splitting node of p if it has at least two immediate successors in p. Sacks forcing, denoted by $\mathbb{S}$, is the poset of subtrees of 2^ω such that below each node there is a splitting node, this trees are called perfect trees. Miller forcing, denoted by $\mathbb{PT}$, is the poset of subtrees of ω^ω such that below each node there is a splitting node, and the size of the set of immediate successors of a node is either 1 or ω. This trees are called super perfect trees.

Proposition 2.6. *Both $\mathbb{S}$ and $\mathbb{PT}$ have continuous reading of names. This means: If $\mathbb{P} \in \{\mathbb{S}, \mathbb{PT}\}$, $q \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $q \Vdash$ “$\dot{x} \in \omega^\omega$”, then there is $p \leq q$ and $F : [p] \to 2^\omega$ a continuous ground model function such that*

$$p \Vdash \text{“}F(\dot{r}_{gen}) = \dot{x}\text{”}$$

where $\dot{r}_{gen}$ is the name of the generic real.

To learn more about the continuous reading of names property, see [20].
In [19] Yorioka proved the result following which inspires our next theorem.

Proposition 2.7 (Yorioka)**.** *Sacks forcing does not fill 2-gaps.*

Theorem 2.8. *Let $g = \{\mathcal{A}_i : i < n\}$ be an n-gap. The next sentences are equivalent:*

(1) $\mathbb{S} \Vdash$ “g is not a gap”
(2) Any forcing $\mathbb{P}$ that adds new reals forces that g is not a gap.
(3) If $W \supseteq V$ is a ZFC model such that $(2^\omega \cap W) \setminus V \neq \emptyset$, then $W \vDash g$ is not a gap.

Proof. Clearly (2) implies (1) and (3) implies (2). So, we will prove (1) implies (3). Assume $\mathbb{S} \Vdash$ "g is not a gap". By the continuous reading of names there are $p \in \mathbb{S}$ and a $F_i : [p] \to 2^\omega$ with $i < n$ continuous functions in the ground model such that

$$p \Vdash \text{“}\{F_i(\dot{r}_{gen}) : i < n\} \text{ fills } g\text{”}$$

where $\dot{r}_{gen}$ is a name for the generic real. Assume without lost of generality that for each $x \in [p]$ we have $\bigcap_{i<n} F_i(x) = \emptyset$ extending p if is necessary. For each $i < n$ and $A \in A_i$ we define

$$Y_A = \{x \in [p] : A \not\subseteq^* F_i(x)\}. \tag{2.1}$$

We claim that all Y_A are countable. First, note that Y_A can be expressed as

$$\begin{aligned} Y_A =& \{x \in [p] : \forall k \in \omega \, \exists m > k \, (m \in A \wedge m \notin F_i(x))\} \\ =& \bigcap_{k\in\omega} \bigcup_{m>k} \bigcup_{m\in A} \{x \in [p] : m \notin F_i(x)\} \end{aligned}$$

Since F_i is a continuous function and $\{y : m \notin y\}$ is a close set for all m, $\{x \in [p] : m \notin F_i(x)\} = F_i^{-1}(\{y : m \notin y\})$ is close. Thus, Y_A is Borel. If Y_A is not countable, by the perfect set property there is a perfect set included in Y_A (see Theorem 13.6 of [10]). Also, it is well known that all perfect set in 2^ω are the branches of a Sacks tree. Then we can find $q \in \mathbb{S}$ such that $[q] \subseteq Y_A$ (and $q \leq p$). Thus

$$q \Vdash \text{“}A \not\subseteq^* F_i(\dot{r}_{gen})\text{”}.$$

We get a contradiction.

Now, let W be a model of ZFC which extends V and has a new real. In W we take $x \in [p]$ such that $x \notin V$. Since Y_A is countable, $Y_A = Y_A^V$ for each A. Therefore, for each $i < n$ and $A \in \mathcal{A}_i$ $x \notin Y_A$. That proves $\{F_i(x) : i < n\}$ fills g. $\square$

Corollary 2.9. An n-gap is $\mathbb{S}$-indestructible if and only if it is indestructible for some forcing that adds reals.

Join Corollary 2.4 and Corollary 2.9 we have next result.

Corollary 2.10. Sacks forcing preserves any $(\kappa_i : i < n)$-gap when each κ_i is uncountable.

We can also state a theorem similar to the one above for Miller forcing. However, we need to recall some previous results first.

Denote by K_σ the σ-ideal generated by σ-compact sets over ω^ω.

Next lemma can be found in [10] as Corollary 21.23.

Lemma 2.11. *[Kechris, Saint Raymond] Let $A \subseteq \omega^\omega$ be an analytic set. Either $A \in K_\sigma$ or there is q a super prefect tree such that $[q] \subseteq A$.*

Now we will fix some notation to work with $\mathbb{PT}$. If $m < \omega$, $split_m(p)$ is the set of splitting nodes of p that have exactly m splitting nodes below and we call stem to the unique element of $split_0(p)$, it will be denoted by $stem(p)$. Recursively we can define a map π_p between $\omega^{<\omega}$ and the splitting nodes of any Miller tree p as follows:

(1) $\pi_p(\emptyset) = stem(p)$

(2) Assume π_p is defined on $\omega^{\leq m}$ and $\sigma \in \omega^m$. We consider the increasing enumeration of $succ_p(\pi_p(\sigma))$ and define $\pi_p(\sigma^\frown i)$ as the first splitting node bellow the i-th element of $succ_p(\pi_p(\sigma))$.

Note that π can be extended in an unique way to a homeomorphism $\overline{\pi}_p : \omega^\omega \to [p]$.

Lemma 2.12. *Let W be a ZFC model which extends V and has an unbounded real over V, call it x. If we fix $p \in \mathbb{PT} \cap V$, then there is $y \in [p] \cap W$ which is unbounded over V.*

Proof. Let $y = \overline{\pi}_p(x)$. To see that y is unbounded over V, we take K a σ-compact set coded in V. Without lost of generality we can assume there is $g \in V$ such that $K = \{f : f \leq^* g\}$. If we assume $y \in K$, then $x \in \{f : f \leq \overline{\pi}_p^{-1}(g)\}$. Since $\overline{\pi}_p$ is defined in V, we have that x is bounded by a function in V. Thus, we conclude that y is unbounded over V. □

Theorem 2.13. *Let $g = \{\mathcal{A}_i : i < n\}$ an n-gap. The next sentences are equivalent:*

(1) $\mathbb{PT} \Vdash$ "*g is not a gap*"
(2) Any forcing $\mathbb{P}$ that adds unbounded reals force g is not a gap.
(3) If $W \supseteq V$ is a ZFC model such that W has a unbounded real over V, then $W \vDash g$ is not a gap.

Proof. Clearly (2) implies (1) and (3) implies (2). So, we will prove (1) implies (3). Employ the continuous reading of names to fix $p \in \mathbb{PT}$ and for each $i < n$ $F_i : [p] \to 2^\omega$ continuous functions such that

$$p \Vdash \text{“}\{F_i(\dot{r}_{gen}) : i < n\} \text{ fills } g\text{”}$$

where $\dot{r}_{gen}$ is a name for the generic real. Assume without lost of generality that for each $x \in [p]$ we have $\bigcap_{i<n} F_i(x) = \emptyset$ extending p if is necessary.

Given $i < n$ and $A \in A_i$ define Y_A as in 2.1, we know that is a Borel set. Note that Y_A is σ-compact: if not, by Lemma 2.11 there is $q \in \mathbb{PT}$ such that $[q] \subseteq Y_A$ and

$$q \Vdash \text{“}A \not\subseteq^* F_i(\dot{r}_{gen})\text{”}.$$

But p forces the opposite.

Take W as in (3). By Lemma 2.12 there is x an unbounded real over V in $[p]$. For each $i < n$ and $A \in \mathcal{A}_i$, we have that $x \notin Y_A$ because every compact set in ω^ω is a finite-branching tree. Therefore, σ-compact sets are bounded by any function that dominates the countable family from the ground model, which bounds each finite-branching tree covering σ-compact set. Consequently , for each $i < n$ and $A \in \mathcal{A}_i$ $x \notin Y_A$. That proves $(F_i(x))_{i<n}$ fills g. □

Corollary 2.14. An n-gap is $\mathbb{PT}$-indestructible if and only if it is indestructible for some forcing that adds unbounded reals.

Corollary 2.15. Miller forcing preserves any $(\kappa_i : i < n)$-gap for κ_i a uncountable cardinal for each $i < n$.

With these results we finish the forcing preservation section.

2.2. **Combinatoric preservation.** The next definition was inspired in a partition defined in [15] (Definition 9.4). Also can be seen [16] and [19] .

Definition 2.16. Let $g = \{\mathcal{A}_i : i < n\}$ be an $(\omega_1 : i < n)$-pregap such that for all $i, j < n$ and $\alpha < \omega_1$ $A^i_\alpha \cap A^j_\alpha = \emptyset$. We define $c_g : [\omega_1]^n \to 2$ such that $c_g(\alpha_0, ..., \alpha_{n-1}) = 0$ if and only if $\forall i < n\, \exists j_i < n(\{j_i : i < n\} = n \wedge \bigcap_{i<n} A^{j_i}_{\alpha_i} \neq \emptyset)$. This type of pregaps will be called normal pregaps.

Theorem 2.17. *Let $g = \{\mathcal{A}_i : i < n\}$ be a normal $(\omega_1 : i < n)$- pregap, the next sentences are equivalent:*

(1) g is a not gap.
(2) There is an uncountable set X that is c_g-homogeneous of color 1.

Proof. We start by proving that (2) implies (1). Let be $X \subseteq \omega_1$ be homogeneous of size ω_1 and color 1. For each $i < n$ define the set $a_i = \bigcup\{A^i_\alpha : \alpha \in X\}$. Thus, $\{a_i : i < n\}$ fills g. Clearly for $A \in \mathcal{A}_i$ we have $A \subseteq^* a_i$. If $m \in \bigcap_{i<n} a_i$, for each $i < n$, there is $\alpha_i \in X$ such that $m \in A^i_{\alpha_i}$, then $\bigcap_{i<n} A^i_{\alpha_i} \neq \emptyset$. But X is homogeneous of color 1.

To prove that (1) implies (2). Take $a_0, ..., a_{n-1}$ witnesses that g is not a gap. We can find $m \in \omega$ and $W \in [\omega_1]^{\omega_1}$ such that for all $\alpha \in W$ and $i < n$ $A^i_\alpha \setminus m \subseteq a_i$. Since $A^i_\alpha \cap A^j_\alpha = \emptyset$ when $i \neq j$, we can assume there are $M_0, ..., M_{n-1} \in \wp(m)$ disjoint such that for all $i < n$ and $\alpha \in W$ $A^i_\alpha \cap m = M_i$. Now we prove W is homogeneous of color 1 by contradiction. Take as hypothesis that $\alpha_0, ..., \alpha_{n-1} \in W$ have color 0 with c_g. Then find $k \in \bigcap_{i<n} A^i_{\alpha_i} \neq \emptyset$. If $k < m$, $k \in \bigcap_{i<n} M_i = \emptyset$. If $k \geq m$, then $k \in \bigcap_{i<n}(A^i_{\alpha_i} \setminus m) \subseteq \bigcap_{i<n} a_i$. An absurd.

□

Corollary 2.18. Let $g = \{\mathcal{A}_i : i < n\}$ be a normal $(\omega_1 : i < n)$- pregap. The next sentences are equivalents:

(1) g is a gap.
(2) There is no uncountable set X that is c_g-homogeneous of color 1.

Definition 2.19. Given g an $(\omega_1 : i < n)$-gap we define the forcing ordered by $\supseteq$

$$\mathbb{F}(g) = \{p \in [\omega_1]^{<\omega} : p \text{ is } c_g - \text{ homogeneous of color } 0\}.$$

For $n = 2$ the previous corollary holds. Even more, both conditions of the corollary are equivalent to the sentence:

$$\mathbb{F}(g) \text{ is ccc.}$$

However, by Theorem 1.3, we know that:

Corollary 2.20. For any g normal $n-$gap, for $n > 2$, $\mathbb{F}(g)$ is not ccc.

3. Existence of multiple gaps

In this section we start by building some variants of the gap constructed in [3] [section 3, Theorem 6]. Given two finite sequences of natural numbers s and t, we denote the concatenation of s followed by t as $s^\frown t$. If t is a sequences of just one number, for example i, we write $s^\frown i$ instead of $s^\frown(i)$. Also, if $s \in n^{<\omega}$, $\langle s\rangle$ is the set of all $x \in n^\omega$ which extends s. In the product topology of n^ω, $\langle s\rangle$ is a basic clopen set.

We denote by $\mathcal{M}$ the σ-idea of meager set of 2^ω. Thus, $non(\mathcal{M})$ is the least size of a set $A \subseteq 2^\omega$ such that $A \notin \mathcal{M}$. Is well known that $\omega < non(\mathcal{M}) \leq \mathfrak{c}$, see [4].

Proposition 3.1. *If $non(\mathcal{M}) = \kappa$, there is an n-gap $\{\mathcal{A}_i : i < n\}$ such that each $\mathcal{A}_i$ has size κ.*

Proof. Fix $B \subseteq n^\omega$ a non meager dense set of size κ (this is possible because n^ω has a countable dense set). Let $T = \{x \restriction k : k < \omega \wedge x \in B\}$. We define for $i < n$ and $x \in n^\omega$

$$a_x^i = \{s \in T : s^\frown i \in x\}$$

Let $\mathcal{A}_i = \{a_x^i : x \in B\}$ and $g = \{\mathcal{A}_i : i < n\}$. We will prove that g is an n-gap. In the sake of contradiction assume that there are $c_0, ..., c_{n-1}$ sets that fill g. For each $m < \omega$ let

$$A(m) = \{x \in n^\omega : \forall i < n\, (a_x^i \setminus n^{\leq m} \subseteq c_i)\}$$

Notice that $A(m)$ is close: Take $y \notin A(m)$, then there is $i < n$ and $t \subseteq y$ with $|t| > m$ such that $t \in a_x^i \setminus c_i$. Thus, $y \in \langle t^\frown i\rangle$ and $\langle t^\frown i\rangle \cap A(m) = \emptyset$.

Since $B \subseteq \bigcup_{m<\omega} A(m)$ and B is not meager, there exists m_0 such that the interior of $A(m_0)$ is not empty. Choose $t \in n^{<\omega}$ with $\langle t\rangle \subseteq A(m_0)$. With out lost of generality we can ask $|t| > m_0$. As B is dense for each $i < n$ we can find $x_i \in \langle t^\frown i\rangle \cap B$. Thus, $t \in a_{x_i}^i \subseteq c_i$. Therefore $t \in \bigcap_{i<n} c_i$ is the contradiction we wanted to get. □

With this result it is natural to define the next cardinal invariant. First, given $B \subseteq n^\omega$ and $i < n$, let $A_i^B := \{a_x^i : x \in B\}$. Define

$$\mu_{n-gap} = min\{|B| : \{A_i^B : i < n\} \text{ is an n-gap}\}$$

Thus, we have already proved that $\mu_{n-gap} \leq non(\mathcal{M})$.

In [17] Todorčević introduces the concept of a countably separated gap. A 2-gap $(\mathcal{A}_0, \mathcal{A}_1)$ is said to be countably separated if there is a family $\{c_k : k \in \omega\} \subseteq [\omega]^\omega$ such that for all $A_0 \in \mathcal{A}_0$ and $A_1 \subseteq \mathcal{A}_1$ there is $k < \omega$ such that $A_0 \subseteq^* c_k$ and $A_1 \cap c_k =^* \emptyset$. In [3] Avilés and Todorčević mention different ways to generalize countably separated gaps to multiple gaps. The next definition is one of them.

Definition 3.2. Given $g = \{\mathcal{A}_i : i < n\}$ an n-pregap we say that g is countably separated if there exists $\{c_i^k : i < n\,, k \in \omega\} \subseteq \wp(\omega)$ such that:

(1) for all $k < \omega$ $\bigcap_{i<n} c_i^k = \emptyset$ and
(2) for all $x_0 \in \mathcal{A}_0, ..., x_{n-1} \in \mathcal{A}_{n-1}$ there is $k < \omega$ such that for all $i < n$ $x_i \subseteq c_i^k$.

Also, we call g strong gap if g is not countably separated.

Then, we can define next cardinal.

$$\mu_{stong-n-gap} = min\{|B| : (A_i := \{a_x^i : x \in B\})_{i<n} \text{ is a strong n-gap}\}$$

Proposition 3.3. *The gap defined in Proposition 3.1 is actually a strong gap.*

Proof. By contradiction. Let g be the gap defined in Proposition 3.1 and $\{c_i^k : i < n\,, k \in \omega\}$ be the witness that g is countably separated. Thus, for each $m, k < \omega$ let

$$A(m, k) = \{x : \forall i < n\, (a_x^i \setminus n^{\leq m} \subseteq c_i^k\}$$

Now we prove that for all m, k $A(m, k)$ is close. Let $y \notin A(m, k)$, then there is $i < n$ and $t \subseteq y$ with $|t| > m$ such that $t \in a_y^i \setminus c_i^k$. Thus, $y \in \langle t^\frown i\rangle$ and $\langle t^\frown i\rangle \cap A(m, k) = \emptyset$ because $t \in a_x^i \setminus c_i^k$ for all $x \in \langle t^\frown i\rangle$.

Using the fact of $B \subseteq \bigcup_{m,k<\omega} A(m,k)$, B is not meager and dense there are $m_0, k_0 < \omega$, $t \in n^{<\omega}$ with $|t| > m_0$ and $x_0 \in \langle t^\frown 0\rangle \cap B, ..., x_{n-1} \in \langle t^\frown n-1\rangle \cap B$. Thus, $t \in \bigcap_{i<n} c_i^{k_0}$, a contradiction. □

By the last proposition we have $\mu_{strog-n-gap} \leq non(\mathcal{M})$.

Given a tree T, we say that T is k-branching if for each $t \in T$, $|succ_T(t)| = k$. For $n > 2$, let $\mathcal{I}_n$ the σ-ideal on ω^n generated by $\{[T] : T$ is a (n-1)-branching tree$\}$. Notice that if $T \in \mathcal{I}_n$, $[T]$ is nowhere dense because for each $t \in succ_T(t)$ has size n. Thus, $\mathcal{I}_n$ is a proper ideal.

Theorem 3.4. *Let $B \subseteq \omega^n$. $B \notin \mathcal{I}_n$ if and only if $g = (\mathcal{A}_i := \{a_x^i : x \in B\})_{i<n}$ is a strong n-gap.*

Proof. First take $B \in \mathcal{I}_n$. Let $\{T_k : k < \omega\}$ be (n-1)-branching trees such that $B \subseteq \bigcup_{k<\omega} [T_k]$. For each $i < n$ recursively we build a family $\{c_i^k : k < \omega\}$ such that

$$c_i^k = \bigcup \{a_x^i \setminus (\bigcup_{m<k} c_i^m) : x \in B \cap ([T_k] \setminus \bigcup_{m<k} [T_m])\}.$$

Claim 3.5. For all $m < k$, $i < n$ and $x \in B \cap ([T_k] \setminus [T_m])$, the set $a_x^i \cap c_i^m$ is finite.

Proof of the claim. By contradiction. Take $\{t_j : j < \omega\} \subseteq a_x^i \cap c_i^m$. So, $t_j {}^\frown i \subseteq x$ and there are $\{y_j : j < \omega\} \subseteq B \cap [T_k]$ such that $t_k \subseteq y_k$. Then (y_k) converges to x but $[T_k]$ is close, then $x \in [T_k]$, an absurd. ■

Thus, if we take $c_i = \bigcup_{k<\omega} c_i^k$ for each $i < n$, they show that g is not a gap (then is not strong gap).

Now fix B that can not be covered by countable many (n-1)-branching trees and assume, to get an absurd, that g is not a strong n-gap. Let $\{c_i^m : i < n, m \in \omega\}$ be a family witness. We define for each $m < \omega$,

$$A(m) = \{x \in B : \forall i < n\, (a_x^i \subseteq c_i^m)\}.$$

As in Proposition 3.1, B is covered by $\bigcup_{m<\omega} A(m)$. Then there is $m < \omega$ such that $A(m)$ is not in $\mathcal{I}_n$. There exists $T \subseteq n^{<\omega}$ such that $[T] \subseteq A(m)$ and there are infinite many $t \in T$ such that $|succ_T(t)| = n$. Fix t as in previous sentences such that $|t| > m$. Then, for all $i < n$ there is $x_i \in [T]$ such that $t^\frown i \subseteq x_i$. Thus, the contradiction we were looking for is given by $t \in \bigcap_{i<n} c_i^m$. □

Corollary 3.6. For all $n > 2$ $\mu_{n-gap} = \mu_{strong-n-gap} = non(\mathcal{I}_n) \leq non(\mathcal{M})$.

Now we will focus on building gaps with non ZFC axioms.

Definition 3.7. Suppose that A is a Borel subset of some Polish space. A map $F : 2^{<\omega_1} \to A$ is Borel if for every δ the restriction of F to 2^δ is a Borel map.

The following statement is known as $\diamondsuit(n, =)$.

(3.1)

$$\text{For every Borel map } F : 2^{<\omega_1} \to n \text{ there is } d : \omega_1 \to n \text{ such that for all } f : \omega_1 \to n$$
$$\text{it satisfies that } \{\alpha < \omega_1 : F(f \restriction \alpha) = d(\alpha)\} \text{ is a stationary set of } \omega_1$$

It is a parametrized diamond principle introduced by Džamonjain, Hrušák and Moore in [12]. In [9] the authors used $\diamondsuit(2, =)$ to build a non metrizable Fréchet group. There are many models where $\diamondsuit(n, =)$ holds, in particular after forcing with a Suslin tree. In particular $\diamondsuit(n, =)$ is consistent with the continuum being arbitrary large.

Theorem 3.8. *$\lozenge(n,=)$ implies that there exists an $(\omega_1 : i<n)$-gap.*

Proof. We associate the tree $2^{<\omega_1}$ with

$$\{(a_0,...,a_{n-1},\mathcal{A}^0_\alpha,...,\mathcal{A}^{n-1}_\alpha) : \alpha<\omega_1\,, \forall i<n(a_i\subseteq\omega\wedge \\ \wedge\, \mathcal{A}^i_\alpha=\{A^i_\beta\in\wp(\omega):\beta<\alpha\})\}.$$

Call T this encoding of $2^{<\omega_1}$. Fix $t=(a_0,...,a_{n-1},\mathcal{A}^0_\alpha,...,\mathcal{A}^{n-1}_\alpha)\in T$ for some $\alpha<\omega_1$. If $(\mathcal{A}^0_\alpha,...,\mathcal{A}^{n-1}_\alpha)$ constituted a pregap, it is countable, then, by making a slight variation to the proof of Proposition 1.2, we can obtain a partition of ω in n+1 many infinite $B_0(t),...,B_n(t)$ such that for each $i<n$ for all $A\in\mathcal{A}^i_\alpha$ holds $A\subseteq^* B_i(t)$.

We define a function F for each t as follows:

$$F(t)=i \text{ if } i \text{ is the minimum such that } B_n(t)\not\subseteq^* a_i$$

if there is an i as above. In case there is no such i or $(\mathcal{A}^0_\alpha,...,\mathcal{A}^{n-1}_\alpha)$ is not a gap $F(t)=0$.

We can verify that F is Borel because if $0<i<n$, then $F^{-1}(i)=\{t\in T : \forall j<i\, B_n(t)\subseteq^* a_j\wedge B_n(t)\not\subseteq^* a_i\}$. As we showed in Theorem 2.8 the condition $B_n(t)\not\subseteq^* a_i$ is Borel and an analogous argument shows that $B_n(t)\subseteq^* a_j$ also is a Borel condition. For $i=0$ is enough notice that $F^{-1}(0)$ is the complement of $\bigcup_{0<i<n}F^{-1}(i)$ which is a finite union of Borel sets. We fix d as in 3.1. Now, we fix an $(\omega : i<n)$-pregap $(\mathcal{A}^0_\omega,...,\mathcal{A}^{n-1}_\omega)$ where for all $i<n$ $\mathcal{A}^i_\omega=\{A^i_m : m<\omega\}$. We need to define A^i_α for all $\alpha<\omega_1\setminus\omega$ and $i<n$:

By recursion on α. Fix $\alpha<\omega_1\setminus\omega$ and assume $(\mathcal{A}^0_\alpha,...,\mathcal{A}^{n-1}_\alpha)$ is already defined where $\mathcal{A}^i_\alpha=\{A^i_\beta : \beta<\alpha\}$ for all $i<n$. Fix a $t_\alpha\in T$ such that $(\mathcal{A}^0_\alpha,...,\mathcal{A}^{n-1}_\alpha)$ is the second half of t. If $d(\alpha)=i$, for $j\neq i$, let $A^j_\alpha\subseteq^* B_j(t_\alpha)$ co-infinite in $B_j(t_\alpha)$ such that $A^i_\beta\subseteq^* A^i_\alpha$ and take $A^i_\alpha=B_i(t_\alpha)\cup B_n(t_\alpha)$.

Let $\mathcal{A}_i=\{A^i_\alpha : \alpha<\omega_1\}$. We will prove that $g=\{\mathcal{A}_i : i<n\}$ is an $(\omega_1 : i<n)$-gap.

Take $a_0,...a_{n-1}\subseteq\omega$. We will prove they do not fill g. Define

$$f=\{(a_0,...,a_{n-1},\mathcal{A}^0_\alpha,...,\mathcal{A}^{n-1}_\alpha) : \alpha<\omega_1\}.$$

Choose $\alpha<\omega_1$ such that $F(f\restriction\alpha)=d(\alpha)$. Assume that $d(\alpha)=i$. If there is no $i<n$ such that $B_n(f\restriction\alpha)\not\subseteq^* a_i$. Then $B_n(f\restriction\alpha)\subseteq^*\bigcap_{j<n}a_j$. In this case $a_0,...a_{n-1}$ do not fill our pregap. So assume there is an $i<n$ such that $B_n(f\restriction\alpha)\not\subseteq^* a_i$. Since $F(f\restriction\alpha)=g(\alpha)$, $B_i(f\restriction\alpha)\cup B_n(f\restriction\alpha)=A^i_\alpha\not\subseteq^* a_i$. We conclude $a_0,...a_{n-1}$ do not fill g. $\square$

Our last construction of an n-gap is going hand in hand with Construction Schemes which we define below.

Definition 3.9. We call a sequence $\tau=(m_k,n_{k+1},r_{k+1})_{k\in\omega}\subseteq\omega^3$ a *type* if:

(1) $m_0=1$,
(2) $\forall k\in\omega\setminus 1\big(\,n_k\geq 2\,\big)$,
(3) $\forall k\in\omega\big(\,m_k>r_{k+1}\,\big)$,
(4) $\forall k\in\omega\setminus 1\big(\,m_{k+1}=r_{k+1}+(m_k-r_{k+1})n_{k+1}\,\big)$.
(5) $\forall\ell\in\omega$ the set $\{k : r_k=\ell\}$ is infinite.

Constructions schemes were introduced in [18] by Todorčevíc. The intuition of these construction schemes is to amalgamate many finite isomorphic structures to obtain a structure of size ω_1. Similar to the forcing method, construction schemes allow you to take care of more than ω_1 "tasks" in "few steps" unlike recursion constructions. In [18] and [5] there are many examples of objects obtained with constructions schemes.

Given $\mathcal{F} \subseteq [\omega_1]^{\leq\omega}$, $(\mathcal{F}, \subseteq)$ is a well-founded partial order. That means we can define a rank function $rank : \mathcal{F} \to \omega$ given by $rank(x) = sup\{rank(y) + 1 : y \subset x\}$, that induce a partition of $\mathcal{F}$ as $\mathcal{F}_k = rank^{-1}\{k\}$.

Definition 3.10. Let $\tau = \langle m_k, n_{k+1}, r_{k+1}\rangle_{k\in\omega}$ be a type. We say that $\mathcal{F} \subseteq [\omega_1]^{<\omega}$ is a construction scheme of type τ if:

(1) $\mathcal{F}$ is cofinal in $[\omega_1]^{<\omega}$,
(2) $\forall k \in \omega\ \forall F \in \mathcal{F}_k\big(\ |F| = m_k\ \big)$,
(3) $\forall k \in \omega\ \forall F, E \in \mathcal{F}_k\big(\ E \cap F \sqsubseteq E, F\ \big)$
(that means if $\alpha \in E \cap F$ and $\beta \in (E \cup F) \setminus (E \cap F)$, then $\alpha < \beta$),
(4) $\forall k \in \omega\ \forall F \in \mathcal{F}_{k+1}\ \exists! F_0, \ldots, F_{n_{k+1}-1} \in \mathcal{F}_k$ such that $F = \bigcup\limits_{i<n_{k+1}} F_i$.
Moreover, $\langle F_i\rangle_{i<n_{k+1}}$ forms a Δ-system with root $R(F)$ such that $|R(F)| = r_{k+1}$ and
$$R(F) < F_0\backslash R(F) < \cdots < F_{n_{k+1}-1}\backslash R(F).$$

The decomposition viewed in the last point is known as the canonical decomposition of F. For the rest of the paper, when we write first F in a construction scheme and $F_0, ..., F_{n_{k+1}-1}$ we always are talking about the canonical decomposition of F. Also $R(F)$ is the root of the Δ-system.

Definition 3.11. Let $\mathcal{F}$ be a construction scheme of type τ. We define $\rho_{\mathcal{F}} : \omega_1^2 \longrightarrow \omega$ as:
$$\rho_{\mathcal{F}}(\alpha, \beta) = \min\{k \in \omega\ :\ \exists F \in \mathcal{F}_k(\ \{\alpha, \beta\} \subseteq F\ )\}.$$
If $\mathcal{F}$ is clear from context, we will write $\rho_{\mathcal{F}}$ simply as ρ.

Definition 3.12. Let $\alpha \in \omega_1$. $\Sigma_\alpha : \omega \longrightarrow \omega \cup \{-1\}$ is the function defined as follows: $\Sigma_\alpha(0) = 0$ and if $k \in \omega\backslash 1$ and $F \in \mathcal{F}_k$ is such that $\alpha \in F$ then:
$$\Sigma_\alpha(k) = \begin{cases} -1 & \text{\textit{if} } \alpha \in R(F) \\ i & \text{\textit{if} } \alpha \in F_i\backslash R(F) \end{cases}$$

And

We need next result.

Proposition 3.13. *Let $\alpha < \beta \in \omega_1$ and $k \in \omega\backslash 1$. Then:*

(1) If $k = \rho(\alpha, \beta)$, then $0 \leq \Sigma_\alpha(k) < \Sigma_\beta(k)$.
(2) If $k > \rho(\alpha, \beta)$, then either $\Sigma_\alpha(k) = -1$ or $\Sigma_\alpha(k) = \Sigma_\beta(k)$.

The proof of this fact can be found in [5].

Now, we will talk about capturing axioms. Given $F \in \mathcal{F}$ we let
$\rho^F = max\{\rho(\alpha, \beta) : \alpha, \beta \in F\}$.

Definition 3.14. Let $\mathcal{F}$ be a construction scheme and $C = \{c_0, ..., c_{k-1}\} \subseteq [\omega_1]^{<\omega}$. For each $F \in \mathcal{F}$, we say that F captures C if:

(1) $n_{\rho^F} \geq k$,
(2) for all $i < k$ $c_i \subseteq F_i$ and $c_i \setminus R(F)$,

(3) for all $i < k$ $\phi_i(c_0) = c_i$ where ϕ_i is the increasing bijection from F_0 to F_i.

Definition 3.15. Let $\mathcal{F}$ be a construction scheme, and $k \in \omega \setminus 1$. We say that $\mathcal{F}$ is k-capturing if for each uncountable $S \subseteq [\omega_1]^{<\omega}$ and $m \in \omega$ there are $C \in [S]^k$ and $F \in \mathcal{F}_m$ which captures C.

This notions were introduced by Todorčevíc in [18], and in [5] it was proved that $\Diamond$ implies the existence of a k-capturing construction scheme for arbitrary $k < \omega$.

Finally, for each $k < \omega$ CA_k represents next sentence.

There is an k-capturing construction scheme of every possible good type satisfying that $k \leq n_j$ for each $j \in \omega \setminus 1$.

Theorem 3.16. *CA_n implies there exists an $(\omega_1 : i < n)$-gap for $n > 1$.*

Proof. Let $\mathcal{F}$ be an n-capturing construction scheme of type $(m_k, n, r_k)_{k<\omega}$. For each $k \in \omega$ let $X_k = \{nk, nk+1, ..., nk+n-1\}$. Notice that for all $k < \omega$ X_k has size n and for $k \neq m$ $X_k \cap X_m = \emptyset$. For all $\alpha < \omega_1$, $i < n$ and $k < \omega$ we define $A^i_\alpha(k)$ as follows:

If $\Sigma_\alpha(k) = -1$, then for all $i < n$, $A^i_\alpha(k) = \emptyset$. If $\Sigma_\alpha(k) = j$, then $A^i_\alpha(k) = \{nk + b : b \equiv i + j (mod\, n) \wedge b < n\}$.

Now, let $A^i_\alpha = \bigcup_{k \in \omega} A^i_\alpha(k)$ and $\mathcal{A}_i = \{A^i_\alpha : \alpha < \omega\}$. We need to prove that $g = \{\mathcal{A}_i : i < n\}$ is an $(\omega_1 : i < n)$-gap.

We start noting that for $\alpha < \omega_1$ and $i < n$, A^i_α is infinite because by definition of construction schemes $\Sigma_\alpha(k) \neq -1$ for infinite many k. Also, for $i \neq j$ and $\alpha < \omega_1$ $A^i_\alpha \cap A^j_\alpha = \emptyset$ because $\{X_k : k \in \omega\}$ is a partition ω. Furthermore, for all $i < n$ and $\alpha < \beta$ notice that

$$A^i_\alpha \setminus A^i_\beta \subseteq \{nk + b : k \leq \rho(\alpha, \beta) \wedge \Sigma_\alpha(k) \geq 0 \wedge b < n \wedge b \equiv i + \Sigma_\alpha(k)(mod\, n)\}.$$

Notice that last set is finite. Therefore, g consists of orthogonal families, that means, g is an $(\omega_1 : i < n)$-pregap. Now we will prove g is a gap. Fix $S \in [\omega_1]^{\omega_1}$. Since $\mathcal{F}$ is an n-capturing construction scheme, we can find $D \in [S]^n$, $k < n$ and $F \in \mathcal{F}_k$ such that F captures D. Thus, if $D = \{\alpha_i : i < n\}$, for all $i < n$ $\alpha_i \in F_i \setminus R(F)$. Then $A^i_{\alpha_i}(k) = \{nk\}$. Consider c the coloring defined in Definition 2.16, then $c(D) = 0$ by definition. That proves S is not homogeneous of color 1 under c. By Corollary 2.18, we finish the proof. $\square$

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