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THERE ARE NO P-POINTS IN SILVER EXTENSIONS

BY

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The first author dedicates this work to his teacher, mentor and dear friend Bohuslav Balcar. The crucial result was proved on the day of his passing.

ABSTRACT

We prove that after adding a Silver real no ultrafilter from the ground model can be extended to a P-point, and this remains to be the case in any further extension which has the Sacks property. We conclude that there are no P-points in the Silver model. In particular, it is possible to construct a model without P-points by iterating Borel partial orders. This answers a question of Michael Hrušák. We also show that the same argument can be used for the side-by-side product of Silver forcing. This provides a model without P-points with the continuum arbitrary large, answering a question of Wolfgang Wohofsky.

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Introduction

Ultrafilters on countable sets have become of great importance in infinite combinatorics. A non-principal ultrafilter $\mathcal{U}$ is called a **P-point** if every countable subset of $\mathcal{U}$ has a pseudointersection in $\mathcal{U}$. Recall that a set $X \subseteq \omega$ is called a **pseudointersection** of a family $\mathcal{B} \subseteq [\omega]^\omega$ if $X \setminus B$ is finite for every $B \in \mathcal{B}$. Ultrafilters of this special type have been extensively studied in set theory and topology. Walter Rudin in 1956 (see [Rud56]) proved that the topological space

$$\omega^* = \beta\omega \setminus \omega$$

is not homogeneous assuming the continuum hypothesis CH. It is well known that the non-principal ultrafilters correspond in a natural way to points of ω^* and P-points are exactly points with neighborhoods closed under countable intersections. Rudin proved the non-homogeneity of ω^* using the following argument: CH implies that P-points exist, ultrafilters that are not P-points always exist, and a P-point and a non-P-point have different topological types. Frolík established in 1967 (see [Fro67]) that ω^* is not homogeneous without the need of the continuum hypothesis. Although Frolík's proof does not provide any specific types of ultrafilters, various distinct topological types of ultrafilters were later identified even without using any additional set-theoretic assumptions; see [Kun80, vM82, Ver13].

Nevertheless, P-points remain one of the central objects of research of set theorists and topologists. P-points are fundamental in forcing theory; most of the methods of preserving an ultrafilter in generic extensions require preserving some kind of a P-point; the reader may consult e.g., [BJ95, Zap09] for more details. They also appear in the study of the Tukey order [Dob15], partition calculus [BT78], model theory [Bla73], and other topics. The study of P-points is still a rich and active area of study; the reader may consult e.g., [Boo71, BHV13, RS17] for some more results and applications of P-points.

A well known result of Ketonen states that it is possible to construct a P-point if the dominating number is equal to the size of the continuum; $\mathfrak{d} = \mathfrak{c}$ [Ket76].[1] It is also possible to construct P-points if the parametrized diamond principle $\Diamond(\mathfrak{r})$

[1] The dominating number $\mathfrak{d}$ is the least cardinality of a set of functions in ω^ω such that every function is eventually dominated by a member of that set; $\mathfrak{c}$ is the cardinality of the continuum.

holds;[2] see [MHD04] for more information on parametrized diamond principles. On the other hand a remarkable theorem of Shelah states that the existence of P-points cannot be proved using just the axioms of ZFC alone. This result was proved in 1977 and first published in [Wim82]. The reader may find the proof in [She98]. The model of Shelah is obtained by iterating the Grigorieff forcing with parameters ranging over non-meager P-filters.

Independence results are often demonstrated in models obtained by employing forcing iterations of definable posets. One possible formalization of such canonical models is treated in [MHD04]. We say that a partial order $(P, \leq)$ is Borel if there is a Polish space X such that P is a Borel subset of X, and $\leq$ is a Borel subset of $X \times X$. A **canonical model** is a model obtained by performing a countable support forcing iteration of Borel proper partial orders of length ω_2. At the *Forcing and its applications retrospective workshop* held at the Fields Institute in 2015 Michael Hrušák posed the following problem.

Problem: Do P-points exist in every canonical model?

A canonical model will contain a P-point if the steps of the iteration add unbounded reals or if no splitting reals are added—in the resulting model either $\mathfrak{d} = \mathfrak{c}$ or $\Diamond(\mathfrak{r})$ does hold. Consequently, one only needs to consider Borel ω^ω-bounding forcing notions which do add splitting reals. The best known examples of this type of forcing are the random poset and the Silver poset. We answer the question of Hrušák in the negative; Theorem 6 states that there are no P-points in the Silver model.

In [Coh79] it was claimed that there is a P-point in the random model. Unfortunately, the presented proof is incorrect and the existence of P-points in the random model is presumably unknown. We will address this issue in the Appendix section.

Problem: Are there P-points in the random model?

The existence of a model without P-points with the continuum larger than ω_2 was an open question [Woh08]. Theorem 7 states that forcing with the side-by-side product of Silver forcing produces such a model.

[2] The reaping number $\mathfrak{r}$ is the smallest size of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that for every $X \in [\omega]^\omega$ there is $R \in \mathcal{R}$ such that either $R \subseteq X$ or $R \subseteq \omega \setminus X$. For more information on the reaping number and cardinal characteristics of the continuum in general, the reader may consult [Bla10]. The formulation of the associated diamond principle $\Diamond(\mathfrak{r})$ is somewhat involved and since it is not used in the present paper, it is omitted.

Our notation and terminology is mostly standard, including some folklore abuse of notation. When p is a partial function from ω to 2, we denote this by $p;\omega \to 2$ and we write $p^{-1}(1)$ instead of $p^{-1}[\{1\}]$. We say that $\mathcal{I} = \{I_n \mid n \in \omega\}$ is an **interval partition** if there is an increasing sequence of natural numbers $\langle m_n \rangle_{n\in\omega}$ such that $m_0 = 0$ and $I_n = [m_n, m_{n+1})$.

We say that a forcing notion $\mathbf{P}$ has the **Sacks property** if for every $p \in \mathbf{P}$ and for every f such that $p \Vdash \dot{f} \in \omega^\omega$ there is $q \le p$ and $\{X_n \mid n \in \omega\}$ such that $X_n \in [\omega]^{n+1}$ for every $n \in \omega$, and $q \Vdash \dot{f}(n) \in X_n$ for each $n \in \omega$. It is a common practise to require in the definition of the Sacks property that $X_n \in [\omega]^{2^n}$, instead of $X_n \in [\omega]^{n+1}$ as we demanded. Nevertheless, both resulting notions are equivalent; see, e.g., [GQ04, section 3].

If $p;\omega \to 2$ is a partial function we denote by $[p]$ the set of all total function extending p, i.e., $[p] = \{f \in 2^\omega \mid p \subseteq f\}$.

Destroying P-points with Silver reals

For a partial function $p;\omega \to 2$ we denote $\operatorname{dom} p$ the domain of p and $\operatorname{cod} p = \omega \setminus \operatorname{dom} p$. We denote the **Silver forcing** (after Jack Howard Silver, see [Mat79]) by $\mathbf{PS}$. Some authors also call this poset the Prikry–Silver forcing. It consists of all partial functions $p;\omega \to 2$ such that $\operatorname{cod} p$ is infinite, and relation $p \le q$ is defined as $q \subseteq p$. We will always assume that $p^{-1}(1)$ is infinite for each $p \in \mathbf{PS}$; such conditions form a dense subset of the poset. If G is a generic filter for the Silver forcing, the **Silver generic real** is defined as $r = \bigcap\{[p] \mid p \in G\}$. It is well known and easy to see that G and r can be defined from each other. A typical application of the Silver forcing is to demonstrate that the inequality $\operatorname{cof}\mathcal{N} < \mathfrak{r}$ is consistent.[3] The reader may consult [Hal17] for an introduction and more information regarding the Silver forcing. It is well known that the Silver forcing is proper and has the Sacks property.

For a partial (or total) function $p;\omega \to 2$ define an interval partition of ω by letting $I_n(p) = \{k \in \omega \mid |k \cap p^{-1}(1)| = n\}$ for $n \in \omega$ and $\mathcal{I}(p) = \{I_n(p) \mid n \in \omega\}$. Note that if q extends p, then $\mathcal{I}(q)$ refines $\mathcal{I}(p)$, i.e., every interval of $\mathcal{I}(p)$ is the union of intervals of $\mathcal{I}(q)$. Moreover, if r is the generic real, then $\mathcal{I}(r)$ refines $\mathcal{I}(p)$ for every p in the generic filter. The proofs of the following simple observations are left for the reader.

[3] $\mathcal{N}$ is the ideal of Lebesgue null subsets of the real line.

Lemma 1: *Let $p \in \mathbf{PS}$ and $k \in \omega$ be such that $I_k(p) \subseteq \operatorname{dom} p$.*

(1) *If $q \leq p$, then $I_k(p) \in \mathcal{I}(q)$.*
(2) *$p \Vdash I_k(p) \in \mathcal{I}(\dot{r})$ (where $\dot{r}$ is the name for the generic real).*

Lemma 2: *Assume that $p, q \in \mathbf{PS}$ and $k, n \in \omega$ are such that*

(1) *$q \leq p$,*
(2) *$I_k(p) \in \mathcal{I}(q)$, and*
(3) *$|q^{-1}(1) \cap \min I_k(p)| = |p^{-1}(1) \cap \min I_k(p)| + n$.*

Then $I_k(p) = I_{k+n}(q)$.

As a consequence of these observations we conclude the following.

Corollary 3: *If $p \in \mathbf{PS}$ and $k \in \omega$ are such that $I_k(p) \subseteq \operatorname{dom} p$, then p forces that: There is $n \in \omega$, $n \leq |\operatorname{cod} p \cap \min I_k(p)|$ such that $I_k(p) = I_{k+n}(\dot{r})$ (where $\dot{r}$ is the name for the generic real).*

By $-_n$ and $=_n$ we denote the subtraction operation and congruence relation modulo n. The notation $k \in_n X$ is interpreted as 'there is $x \in X$ such that $k =_n x$.' For $X, Y \subset n$ we write $X -_n Y = \{x -_n y \mid x \in X, y \in Y\}$.

Lemma 4: *For each $n \in \omega$ there exists $k(n) \in \omega$ such that for each set $C \in [k(n)]^n$ there exists $s \in k(n)$ such that $C \cap (C -_{k(n)} \{s\}) = \emptyset$.*

Proof. If s does not satisfy the conclusion of the lemma, then $s \in C -_{k(n)} C$. As $|C -_{k(n)} C| \leq n^2$, any choice of $k(n) > n^2$ works as desired. ∎

The following proposition contains the main technical argument central for the results of this paper.

Proposition 5: *Let $\mathcal{U}$ be a non-principal ultrafilter and $\dot{Q}$ be a $\mathbf{PS}$-name for a forcing such that $\mathbf{PS} * \dot{Q}$ has the Sacks property. If $G \subset \mathbf{PS} * \dot{Q}$ is a generic filter over V, then $\mathcal{U}$ cannot be extended to a P-point in $V[G]$.*

Before giving the formal proof of the Proposition, let us sketch the core idea of the argument. The basic approach is the same as in the no-P-points proof of Shelah from [She98]. We will show that in order to extend $\mathcal{U}$ to an ultrafilter in the generic extension, one would need to add to $\mathcal{U}$ a particular countable set $\mathcal{D}$ of newly introduced subsets of ω, and at the same time there is no way to add to $\mathcal{U}$ also the pseudointersection of $\mathcal{D}$; for each pseudointersection Z of $\mathcal{D}$ there is $U \in \mathcal{U}$ such that $Z \cap U = \emptyset$.

The sets in $\mathcal{D}$ will be chosen as the typical independent reals added by the Silver forcing. Let r be the generic real; define d_i^n as the union of intervals $I_j(r)$ such that $j =_n i$. Although it is easy to see that each such d_i^n is an independent real, this fact will not be explicitly needed in our argument and is therefore left for the interested reader to observe. For a fixed n the sets d_i^n form a partition of ω into n pieces, and consequently each ultrafilter extending $\mathcal{U}$ needs to contain one element of this partition; denote this set $d^n_{y(n)}$. We will show that the set $\mathcal{D} = \{d^n_{y(n)} \mid n \in \omega\}$ works as desired.

The argument for non-existence of pseudointersections will go along the lines of the following simple claim.

Claim: *Suppose $\mathcal{D} = \{d^n \mid n \in \omega\}$ is a subset of an ultrafilter $\mathcal{U}$ with the following property. For every function $f\colon \omega \to \omega$ there is an interval partition $\{a_n \mid n \in \omega\}$ such that*

- *$f(n) < \min a_{n+1}$ for each $n \in \omega$, and*
- *$\bigcup\{a_n \cap d^n \mid n \in \omega\} \notin \mathcal{U}$.*

Then $\mathcal{D}$ does not have a pseudointersection in $\mathcal{U}$, and consequently $\mathcal{U}$ is not a P-point.

Although the details of the sketched idea will be for technical reasons somewhat adjusted, e.g., we will use only a subset of the above defined set $\mathcal{D}$, the formal proof of Proposition 5 will roughly follow the described argument.

Proof. First we use the function k from Lemma 4 to inductively construct two increasing sequences of integers. Put $v(0) = 0$ and $m(0) = k(2)$. Assume $v(n-1)$, $m(n-1)$ are defined, put

$$v(n) = \sum\{m(i) \mid i \in n\} \quad \text{and} \quad m(n) = k((n+1)(v(n)+2)).$$

Let r be the **PS** generic real in $V[G]$ added by the first stage of the iteration. For $n \in \omega$ and $i \in m(n)$ let

$$D_i^n = \bigcup\{I_j(r) \mid j \in \omega, j =_{m(n)} i\}.$$

For a fixed n the set $\{D_i^n \mid i < m(n)\}$ is a finite partition of ω. We will show that in $V[G]$, for every function $y\colon \omega \to \omega$ which satisfies $y(n) < m(n)$ for every $n \in \omega$, and every pseudointersection Z of $\{D^n_{y(n)} \mid n \in \omega\}$ there is a set $U \in \mathcal{U}$ such that $U \cap Z = \emptyset$. This implies that $\mathcal{U}$ cannot be extended to a P-point in $V[G]$.

Let $(p, \dot{q})$ be any condition in $\mathbf{PS} * \dot{Q}$, and let $\dot{Z}$, $\dot{y}$ be the corresponding names for Z and y. Utilizing the Sacks property we can assume that there are $f\colon \omega \to \omega$ and $\{X_n \in [m(n)]^{n+1} \mid n \in \omega\}$ in V such that

$$(p, \dot{q}) \Vdash (\dot{Z} \setminus f(n)) \subseteq D^n_{\dot{y}(n)} \quad \text{and} \quad \dot{y}(n) \in X_n.$$

Choose an interval partition $\mathcal{A} = \{A_n \mid n \in \{-1\} \cup \omega\}$ of ω ordered in the natural way such that

(1) $f(n) < \min A_{2n}$ for each $n \in \omega$,
(2) $m(n) < |A_{2n+j} \cap \operatorname{cod} p|$ for each $n \in \omega$, $j \in 2$, and
(3) $\mathcal{I}(p)$ refines $\mathcal{A}$.

We will assume that $U_0 = \bigcup\{A_{2n+1} \mid n \in \omega\} \in \mathcal{U}$, otherwise take the interval partition $\mathcal{A}' = \langle A_{-1} \cup A_0, A_1, A_2, \ldots\rangle$ instead.[4] The plan is to use the trace of extensions of p on the interval A_{2n} to control the possible behavior of the set $D^n_{y(n)} \cap A_{2n+1}$ for all $n \in \omega$ simultaneously.

Let $p_1 \in \mathbf{PS}$ be any extension of p such that $A_{2n-1} \subseteq \operatorname{dom} p_1$ and $|\operatorname{cod} p_1 \cap A_{2n}| = m(n)$ for each $n \in \omega$. Note that for any $j \in \omega$ if $I_j(p_1) \subseteq A_{2n-1}$, then $p_1 \Vdash I_j(p_1) \in \mathcal{I}(\dot{r})$. Also note that $|\operatorname{cod} p_1 \cap \min A_{2n}| = v(n)$ for each $n \in \omega$. Let

$$C_n = X_n -_{m(n)} \{i \mid i \in v(n) + 2\}$$

and notice that $|C_n| \leq (n+1)(v(n)+2)$. For $n \in \omega$ put

$$H_n = A_{2n+1} \cap \bigcup\{I_j(p_1) \mid j \in \omega, j \in_{m(n)} C_n\}.$$

We will now distinguish two cases.

Case 1; $\bigcup\{H_n \mid n \in \omega\} \notin \mathcal{U}$, hence $U = \bigcup\{A_{2n+1} \setminus H_n \mid n \in \omega\} \in \mathcal{U}$. Pick any $p_2 < p_1$, $p_2 \in \mathbf{PS}$ such that $p_2^{-1}(1) = p_1^{-1}(1)$ and $|\operatorname{cod} p_2 \cap A_{2n}| = 1$ for each $n \in \omega$. Notice that $\mathcal{I}(p_1) = \mathcal{I}(p_2)$, $|\operatorname{cod} p_2 \cap \min(A_{2n+1})| = n+1$, and if $j \in \omega$ is such that $I_j(p_2) \subseteq A_{2n+1}$, then $I_j(p_2) \subseteq \operatorname{dom} p_2$. For each $n \in \omega$ Corollary 3 states that p_2 forces: There is $i \leq n+1$ such that for each $j \in \omega$ if $I_j(p_1) \subseteq A_{2n+1}$, then

$$I_j(p_1) = I_{j+i}(\dot{r}).$$

[4] In the following proof, we will use the second assumption on the interval partition $\mathcal{A}$ only for $j = 0$. Notice, however, that assuming it only for $j = 0$ at the moment of choosing $\mathcal{A}$ would not have been sufficient as if it were the case that $U_0 \notin \mathcal{U}$, we would be working with the partition $\mathcal{A}'$ instead, and $\mathcal{A}'$ would not be fulfilling the necessary requirement. The observant reader may also notice that the last assumption on $\mathcal{A}$ will in fact not be necessary in the proof.

As $(p_2, \dot{q})$ forces $\dot{y}(n) \in X_n$, it follows that if $I_j(p_1) = I_{j+i}(r) \subseteq D^n_{y(n)} \cap A_{2n+1}$, then $j \in_{m(n)} (X_n -_{m(n)} \{i \mid i \in n+2\}) \subseteq C_n$. We can conclude that

$$(p_2, \dot{q}) \Vdash D^n_{\dot{y}(n)} \cap A_{2n+1} \subset H_n.$$

This together with

$$(p, \dot{q}) \Vdash (\dot{Z} \setminus \min A_{2n}) \subseteq D^n_{\dot{y}(n)}$$

implies that $(p_2, \dot{q}) \Vdash \dot{Z} \cap U = \emptyset$.

Case 2; $U = \bigcup\{H_n \mid n \in \omega\} \in \mathcal{U}$. Applying Lemma 4, for each $n \in \omega$ there exists $s_n \in m(n)$ such that $C_n \cap (C_n -_{m(n)} \{s_n\}) = \emptyset$. Put

$$t(n) = \sum\{s_i \mid i \in n\} \leq v(n) - n.$$

Pick a condition $p_2 < p_1$, $p_2 \in \mathbf{PS}$ such that $|\operatorname{cod} p_2 \cap A_{2n}| = 1$ and

$$|p_2^{-1}(1) \cap A_{2n}| = |p_1^{-1}(1) \cap A_{2n}| + s_n$$

for each $n \in \omega$. Such p_2 exists as $|\operatorname{cod} p_1 \cap A_{2n}| = m(n)$. Note that in this case $|\operatorname{cod} p_2 \cap \min(A_{2n+1})| = n+1$, and if $j \in \omega$ is such that $I_j(p_2) \subseteq A_{2n+1}$, then $I_j(p_2) \subseteq \operatorname{dom} p_2$ and $I_j(p_2) = I_{j-t(n+1)}(p_1)$. For each $n \in \omega$ Corollary 3 implies that p_2 forces: There is $i \leq n+1$ such that for each $j \in \omega$ if $I_j(p_1) \subseteq A_{2n+1}$, then $I_j(p_1) = I_{j+t(n+1)+i}(\dot{r})$. As $(p_2, \dot{q})$ forces $\dot{y}(n) \in X_n$, it follows that if

$$I_j(p_1) = I_{j+t(n+1)+i}(r) \subseteq D^n_{y(n)} \cap A_{2n+1},$$

then

$$\begin{aligned} j \in_{m(n)} (X_n -_{m(n)} \{t(n+1)\}) -_{m(n)} \{i \mid i \in n+2\} \\ =& ((X_n -_{m(n)} \{t(n)\}) -_{m(n)} \{i \mid i \in n+2\}) -_{m(n)} \{s_n\} \\ \subseteq& (X_n -_{m(n)} \{i \mid i \in v(n)+2\}) -_{m(n)} \{s_n\} = C_n -_{m(n)} \{s_n\}. \end{aligned}$$

For $n \in \omega$ put

$$\bar{H}_n = A_{2n+1} \cap \bigcup\{I_j(p_1) \mid j \in \omega, j \in_{m(n)} (C_n -_{m(n)} \{s_n\})\};$$

$H_n \cap \bar{H}_n = \emptyset$, because if $j \in_{m(n)} C_n$, then $j \notin_{m(n)} C_n -_{m(n)} \{s_n\}$.

Now

$$(p_2, \dot{q}) \Vdash D^n_{\dot{y}(n)} \cap A_{2n+1} \subset \bar{H}_n.$$

Again, together with

$$(p, \dot{q}) \Vdash (\dot{Z} \setminus \min A_{2n}) \subseteq D^n_{\dot{y}(n)}$$

we get $(p_2, \dot{q}) \Vdash \dot{Z} \cap U = \emptyset$. ∎

The Silver model is the result of a countable support iteration of Silver forcing of length ω_2.

THEOREM 6: *There are no P-points in the Silver model.*

Proof. Denote by $\mathbf{PS}_\alpha$ the countable support iteration of Silver forcing of length α for $\alpha \le \omega_2$. Assume V is a model of CH and let $G \subset \mathbf{PS}_{\omega_2}$ be a generic filter. Let $\mathcal{U} \in V[G]$ be a non-principal ultrafilter. For $\alpha < \omega_2$ let

$$\mathcal{U}_\alpha = \mathcal{U} \cap V[G_\alpha],$$

where G_α is the restriction of G to $\mathbf{PS}_\alpha$. By the standard reflection argument, there is $\alpha < \omega_2$ such that $\mathcal{U}_\alpha \in V[G_\alpha]$ and it is an ultrafilter in that model. Since the next step of the iteration adds a Silver real and the tail of the iteration has the Sacks property, Proposition 5 states that $\mathcal{U}_\alpha$ cannot be extended to a P-point in $V[G]$, in particular, $\mathcal{U}$ is not a P-point. ∎

We show that forcing with the side-by-side product of Silver forcing also produces a model without P-points.

THEOREM 7: *Assume* GCH, *let* $\kappa > \omega_1$ *be a cardinal with uncountable cofinality. If* $\bigotimes_\kappa \mathbf{PS}$ *is the countable support product of* κ *many Silver posets and* $G \subset \bigotimes_\kappa \mathbf{PS}$ *is a generic filter, then*

$$V[G] \models \text{ there are no P-points and } \mathfrak{c} = \kappa.$$

Proof. It is well known that under GCH the poset $\bigotimes_\kappa \mathbf{PS}$ is an ω_2-c.c. proper forcing notion, has the Sacks property (see, e.g., [Kos92]), and $V[G] \models \mathfrak{c} = \kappa$. Assume $\mathcal{U}$ is an ultrafilter in $V[G]$. Since $\bigotimes_\kappa \mathbf{PS}$ is ω_2-c.c., there is $J \subset \kappa$ of size ω_1 such for every $A \in \mathcal{P}(\omega) \cap V$ and $q \in \bigotimes_\kappa \mathbf{PS}$ the statement $A \in \mathcal{U}$ is decided by a condition with support contained in J and compatible with q. Choose $\alpha \in \kappa \setminus J$ and let r be the $\mathbf{PS}$ generic real added by the α-th coordinate of the product.

The theorem is now proved in the same way as Proposition 5; as the proof follows most parts of the proof of Proposition 5 in verbatim, we will focus in detail only on the points where adjustments are necessary.

Start with defining the functions v and m, consider sets D_i^n defined from the generic real r, and pick any $\bigotimes_\kappa \mathbf{PS}$ names $\dot{Z}, \dot{y}$. Let (p, q) be any condition in

$$\mathbf{PS} \times \bigotimes_{\kappa \setminus \{\alpha\}} \mathbf{PS} = \bigotimes_\kappa \mathbf{PS}$$

which forces that $\mathcal{U}$ is a non-principal ultrafilter; we interpret $p \in \mathbf{PS}$ as the α-th coordinate and q as the other coordinates of a condition in the full product poset. We invoke the Sacks property of $\bigotimes_\kappa \mathbf{PS}$ to assume the existence of an appropriate function f and a sequence $\{X_n \mid n \in \omega\}$. Choose the interval partition $\mathcal{A}$ satisfying properties (1)–(3) with respect to p and consider

$$U_0 = \bigcup\{A_{2n+1} \mid n \in \omega\}.$$

As $U_0 \in V$, there is a condition $(p, q_1) < (p, q)$ deciding whether U_0 is an element of $\mathcal{U}$, because of the choice of coordinate α. We will assume that $(p, q_1) \Vdash U_0 \in \mathcal{U}$, otherwise take the interval partition $\mathcal{A}'$ instead. Follow with choosing the condition p_1 extending p, define the sets C_n and H_n for $n \in \omega$.

Now consider the set

$$H = \bigcup\{H_n \mid n \in \omega\}.$$

As $H \in V$, there is $(p_1, q_2) < (p_1, q_1)$ deciding whether $H \in \mathcal{U}$.

Case 1; $(p_1, q_2) \Vdash H \notin \mathcal{U}$. Now proceed again in verbatim as in case 1 of the proof of Proposition 5; define U, choose $p_2 < p_1$, and conclude that $(p_2, q_2) \Vdash \dot{Z} \cap U = \emptyset$.

Case 2; $(p_1, q_2) \Vdash H \in \mathcal{U}$. Proceed again as in case 2 of the proof of Proposition 5; define U, find s_n for each $n \in \omega$, and choose $p_2 < p_1$. And finally conclude $(p_2, q_2) \Vdash \dot{Z} \cap U = \emptyset$. ∎

Concluding remarks

Theorem 6 can be stated in an axiomatic manner. Recall that $\mathcal{N}$ denotes the ideal of Lebesgue null sets and let v_0 be the ideal associated with the Silver forcing;

$$v_0 = \{A \subset 2^\omega \mid \forall p \in \mathbf{PS}\ \exists q \in \mathbf{PS}, q < p, [q] \cap A = \emptyset\}.$$

This ideal was introduced in [CRSW93] and studied in [Bre95, DPH00]. The proof of Proposition 5 can be reformulated to yield the following theorem, while the detailed proof is provided in [Guz17].

Theorem 8: *The inequality* $\operatorname{cof}\mathcal{N} < \operatorname{cov} v_0$ *implies that there are no P-points.*

An alternative version of results of this paper was suggested by Jonathan Verner. The side-by-side product $\bigotimes_\omega \mathbf{PS}$ adds a Silver generic real r_α for each coordinate $\alpha \in \omega$. Consider the pair of complementary splitting reals

$$X^i_\alpha = \bigcup \{I_{2n+i}(r_\alpha) \mid n \in \omega\}, \quad i \in 2;$$

an argument similar to (and less technical than) the proof of Proposition 5 demonstrates the following.

Claim: *Let $\mathcal{U}$ be a non-principal ultrafilter. The product $\bigotimes_\omega \mathbf{PS}$ forces that no pseudo-intersection of $\{X^{i(\alpha)}_\alpha \mid \alpha \in \omega\}$ is $\mathcal{U}$-positive, and this remains to be the case in each further Sacks property extension.*

Furthermore, it is possible to reason along the lines of the proof of Theorem 7 to obtain a stronger version of the theorem. These results are to be included in forthcoming publications.

Announcement 9: *Assume* GCH*; let $\kappa > \omega_1$ be a cardinal with uncountable cofinality. If $\bigotimes_\kappa \mathbf{PS}$ is the countable support product of κ many Silver posets and $G \subset \bigotimes_\kappa \mathbf{PS}$ is a generic filter, then*

$$\begin{aligned} V[G] \models\ & \textit{For every non-principal ultrafilter } \mathcal{U} \textit{ there exists} \\ & \{X_\alpha \mid \alpha \in \mathfrak{c}\} \subset \mathcal{U} \textit{ such that for each } y \in [\mathfrak{c}]^\omega \cap V \\ & \textit{no pseudointersection of } \{X_\alpha \mid \alpha \in y\} \textit{ is an element of } \mathcal{U}. \end{aligned}$$

The motivation for stating this theorem comes from the problem of Isbell [Isb65] which asks for the existence of two Tukey non-equivalent ultrafilters on ω. The problem can be equivalently formulated as a statement resembling the conclusion of Announcement 9; see [DT11].

Problem (Isbell): Is it consistent that for each non-principal ultrafilter $\mathcal{U}$ on ω there exists $\mathcal{X} \in [\mathcal{U}]^\mathfrak{c}$ such that for each $\mathcal{Y} \in [\mathcal{X}]^\omega$ is $\bigcap \mathcal{Y} \notin \mathcal{U}$?

Appendix

At the request of the referee, we address here the situation concerning the random model. We point out the issue in the argument in [Coh79] used to reason for the existence of P-points in the random model. The reader may consult [FBH17, FB] for more information.

It is an unpublished result of K. Kunen that if ω_1 many Cohen reals are added to the ground model followed by adding ω_2 many random reals, the resulting random model will contain a P-point. Recently A. Dow proved that

P-points exist in the random model provided CH and $\square_{\omega_1}$ does hold in the ground model [Dow18].

The construction in [Coh79] uses the notion of a pathway. For a recent development and general treatment of pathways see [FB].

Definition 10: A sequence $\{A_\alpha \mid \alpha \in \kappa\}$ is a **pathway** if the following conditions hold.

(1) $\omega^\omega = \bigcup\{A_\alpha \mid \alpha \in \kappa\}$,
(2) $A_\alpha \subseteq A_\beta$ for $\alpha < \beta$,
(3) A_α does not dominate $A_{\alpha+1}$,[5]
(4) if $f, g \in A_\alpha$, then $(f \operatorname{join} g) \in A_\alpha$ (where $(f_0 \operatorname{join} f_1) \in \omega^\omega$ is defined by $(f_0 \operatorname{join} f_1)(2n+i) = f_i(n)$),
(5) if g is Turing reducible to f and $f \in A_\alpha$, then $g \in A_\alpha$.

The following is [Coh79, Theorem 1.1].

Theorem 11: *The existence of a pathway implies the existence of a P-point.*

This result is a useful tool for proving the existence of P-points in certain models. In order to prove that there is a P-point in the random model (i.e., the model obtained by adding ω_2 random reals to a model of CH), the author of [Coh79] aims to construct a pathway in the generic extension. We do not know whether there are pathways in this model. The construction from [Coh79] does not work, as we will demonstrate.

We denote $\mathbf{B}$ the random forcing and $\mathbf{B}(\omega_2)$ the poset for adding ω_2 many random reals. It is well known that if M is a countable elementary submodel of $H(\theta)$ (for some sufficiently large cardinal θ), $r\colon \omega_2 \to 2$ is a $\mathbf{B}(\omega_2)$-generic function over V, and $\pi\colon \omega \to \omega_2$ is an injective function in V (but not necessarily M), then $M[r \circ \pi]$ is a $\mathbf{B}$-generic extension of M (see [Coh79] for more details).

We outline the construction in [Coh79]. Using CH in the ground model V find $\{M_\alpha \mid \alpha \in \omega_1\}$, an increasing chain of countable elementary submodels of $H(\theta)$ such that

$$\omega^\omega \subset \bigcup\{M_\alpha \mid \alpha \in \omega_1\}.$$

[5] I.e., there is a function in $A_{\alpha+1}$ not eventually dominated by any element of A_α.

Let $r\colon \omega_2 \to 2$ be a $\mathbf{B}(\omega_2)$-generic function over V. Work in $V[r]$; let Π be the set of all injective functions from ω to ω_2 in V. For every $\alpha < \omega_1$ define

$$A_\alpha = \bigcup\{\omega^\omega \cap M_\alpha[r \circ \pi] \mid \pi \in \Pi\}.$$

The argument in [Coh79] relies on $\{A_\alpha \mid \alpha \in \omega_1\}$ being a pathway. We show that this is not the case.

Fix $\mathcal{P} = \{P_n \mid n \in \omega\} \subseteq [\omega]^\omega$ a partition of ω and let $\mathrm{Q} = \{q_n \mid n \in \omega\}$ be an enumeration of the rational numbers. Furthermore, we take both $\mathcal{P}$ and the enumeration of Q to be definable. For $f, g\colon \omega \to 2$ we define $f \star g\colon \mathrm{Q} \to 2$ by declaring

$$f \star g(q_n) = 1$$

if and only if $f \restriction P_n = g \restriction P_n$. The following proposition implies that no A_α is closed under the join operation.

Proposition 12: *Let $r\colon \omega_2 \to 2$ be a $\mathbf{B}(\omega_2)$-generic function over V, and let M be a countable elementary submodel of $H(\theta)$. There are $\pi_0, \pi_1 \in \Pi$ such that there is no $\sigma \in \Pi$ for which $M[r \circ \pi_0] \cup M[r \circ \pi_1] \subseteq M[r \circ \sigma]$.*

Proof. Let $\delta = M \cap \omega_1$. Since δ is countable ordinal, there is $S \subseteq \mathrm{Q}$ order isomorphic to δ. Now choose the functions $\pi_0, \pi_1 \in \Pi$ such that the following holds:

- If $q_n \in S$, then $\pi_0 \restriction P_n = \pi_1 \restriction P_n$.
- If $q_n \notin S$, then $\pi_0[P_n] \cap \pi_1[P_n] = \emptyset$.

Recall that both $M[r \circ \pi_0]$ and $M[r \circ \pi_1]$ are $\mathbf{B}$-generic extensions of M. Assume that $\{r \circ \pi_0, r \circ \pi_1\} \subset M[r \circ \sigma]$ for some $\sigma \in \Pi$. Then also

$$(r \circ \pi_0) \star (r \circ \pi_1) \in M[r \circ \sigma],$$

and a simple genericity argument implies

$$((r \circ \pi_0) \star (r \circ \pi_1))^{-1}(1) = S \in M[r \circ \sigma].$$

Now $\delta \in M[r \circ \sigma]$ is a contradiction with $M[r \circ \sigma]$ being a generic extension of M. ∎

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References

[BHV13] A. Blass, M. Hrušák and J. Verner, *On strong P-points*, Proceedings of the American Mathematical Society **141** (2013), 2875–2883.

[BJ95] T. Bartoszyński and H. Judah, *Set Theory*, A K Peters, Wellesley, MA, 1995.

[Bla73] A. Blass, *The Rudin–Keisler ordering of P-points*, Transactions of the American Mathematical Society **179** (1973), 145–166.

[Bla10] A. Blass, *Combinatorial cardinal characteristics of the continuum*, in *Handbook of Set Theory. Vols. 1, 2, 3*, Springer, Dordrecht, 2010, pp. 395–489.

[Boo71] D. Booth, *Ultrafilters on a countable set*, Annals of Mathematical Logic **2** (1970/1971), 1–24.

[Bre95] J. Brendle, *Strolling through paradise*, Fundamenta Mathematicae **148** (1995), 1–25.

[BT78] J. E. Baumgartner and A. D. Taylor, *Partition theorems and ultrafilters*, Transactions of the American Mathematical Society **241** (1978), 283–309.

[Coh79] P. E. Cohen, *P-points in random universes*, Proceedings of the American Mathematical Society **74** (1979), 318–321.

[CRSW93] J. Cichoń, A. Rosłanowski, J. Steprāns and B. Węglorz, *Combinatorial properties of the ideal* $\mathfrak{P}_2$, Journal fo Symbolic Logic **58** (1993), 42–54.

[Dob15] N. Dobrinen, *Survey on the Tukey theory of ultrafilters*, Zbornik Radova (Beograd) **17(25)** (2015), 53–80.

[Dow18] A. Dow, *P-filters and Cohen, Random, and Laver forcing*, preprint,

[DPH00] C. A. Di Prisco and J. M. Henle, *Doughnuts, floating ordinals, square brackets, and ultraflitters*, Journal of Symbolic Logic **65** (2000), 461–473.

[DT11] N. Dobrinen and S. Todorcevic, *Tukey types of ultrafilters*, Illinois Journal fo Mathematics **55** (2011), 907–951.

[FB] D. Fernández-Bretón, *Generized pathways*, unpublished note, `https://arxiv.org/abs/1810.06093`.

[FBH17] D. Fernández-Bretón and M. Hrušák, *Corrigendum to "Gruff ultrafilters" [Topol. Appl. 210 (2016) 355–365]*, Topology and its Applications **231** (2017), 430–431.

[Fro67] Z. Frolík, *Sums of ultrafilters*, Bulletin of the American Mathematical Society **73** (1967), 87–91.

[GQ04] S. Geschke and S. Quickert, *On Sacks forcing and the Sacks property*, in *Classical and New Paradigms of Computation and their Complexity Hierarchies*, Trends in Logic—Studia Logica Library, Vol. 23, Kluwer, Dordrecht, 2004, pp. 95–139.

[Guz17] O. Guzmán, *P-points, mad families and cardinal invariants*, Ph.D. thesis, Universidad Nacional Autónoma de México, 2017, `https://arxiv.org/abs/1810.09680`.

[Hal17] L. J. Halbeisen, *Combinatorial Set Theory*, Springer Monographs in Mathematics, Springer, Cham, 2017.

[Isb65] J. R. Isbell, *The category of cofinal types. II*, Transactions of the American Mathematical Society **116** (1965), 394–416.

[Ket76] J. Ketonen, *On the existence of P-points in the Stone–Čech compactification of integers*, Fundamenta Mathematicae **92** (1976), 91–94.

[Kos92] P. Koszmider, *A formalism for some class of forcing notions*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **38** (1992), 413–421.

[Kun80] K. Kunen, *Weak P-points in* $\mathbf{N}^*$, in *Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978)*, Colloquia Mathematica Societatis János Bolyai, Vol. 23, North-Holland, Amsterdam–New York, 1980, pp. 741–749.

[Mat79] A. R. D. Mathias, *Surrealist landscape with figures (a survey of recent results in set theory)*, Periodica Mathematica Hungarica **10** (1979), 109–175.

[MHD04] J. T. Moore, M. Hrušák and M. Džamonja, *Parametrized* $\diamondsuit$ *principles*, Transactions of the American Mathematical Society **356** (2004), 2281–2306.

[RS17] D. Raghavan and S. Shelah, *On embedding certain partial orders into the P-points under Rudin–Keisler and Tukey reducibility*, Transactions of the American Mathematical Society **369** (2017), 4433–4455.

[Rud56] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Mathematical Journal **23** (1956), 409–419.

[She98] S. Shelah, *Proper and Improper Forcing*, Perspectives in Mathematical Logic, Springer, Berlin, 1998.

[Ver13] J. L. Verner, *Lonely points revisited*, Commentationes Mathematicae Universitatis Carolinae **54** (2013), 105–110.

[vM82] J. van Mill, *Sixteen topological types in* $\beta\omega - \omega$, Topology and its Applications **13** (1982), 43–57.

[Wim82] E. L. Wimmers, *The Shelah P-point independence theorem*, Israel Journal of Mathematics **43** (1982), 28–48.

[Woh08] W. Wohofsky, *On the existence of p-points and other ultrafilters in the Stone–Čech-compactification of* $\mathbf{N}$, Master's thesis, Vienna University of Technology, 2008.

[Zap09] J. Zapletal, *Preserving P-points in definable forcing*, Fundamenta Mathematicae **204** (2009), 145–154.