ON $(1,\omega_1)$-WEAKLY UNIVERSAL FUNCTIONS

OSVALDO GUZMÁN

Abstract. A function $U : [\omega_1]^2 \longrightarrow \omega$ is called $(1,\omega_1)$-*weakly universal* if for every function $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective function $h : \omega_1 \longrightarrow \omega_1$ and a function $e : \omega \longrightarrow \omega$ such that $F(\alpha,\beta) = e(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta \in \omega_1$. We will prove that it is consistent that there are no $(1,\omega_1)$-weakly universal functions, this answers a question of Shelah and Steprāns. In fact, we will prove that there are no $(1,\omega_1)$-weakly universal functions in the Cohen model and after adding ω_2 Sacks reals side-by-side. However, we show that there are $(1,\omega_1)$-weakly universal functions in the Sacks model. In particular, the existence of such graphs is consistent with ♣ and the negation of the Continuum Hypothesis.

Introduction and Preliminaries. A graph $U : [\omega_1]^2 \longrightarrow 2$ is called *universal* if for every graph $F : [\omega_1]^2 \longrightarrow 2$ there is an injective function $h : \omega_1 \longrightarrow \omega_1$ such that $F(\alpha,\beta) = U(h(\alpha), h(\beta))$ for each $\alpha, \beta \in \omega_1$. It is easy to see that universal graphs exist assuming the Continuum Hypothesis, and in [14] and [15] Shelah showed that the existence of universal functions is consistent with the failure of CH. In [10] Mekler showed that the existence of universal functions $U : [\omega_1]^2 \longrightarrow \omega$ is also consistent with the failure of the Continuum Hypothesis. Universal graphs and functions were recently studied by Shelah and Steprāns in [13], where they showed that the existence of universal graphs is consistent with several values of $\mathfrak{b}$ and $\mathfrak{d}$. They also considered several variations of universal functions, in particular, the following notion was studied:

Definition 1. A function $U : [\omega_1]^2 \longrightarrow \omega$ is $(1,\omega_1)$-weakly universal if for every $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective function $h : \omega_1 \longrightarrow \omega_1$ and a function $e : \omega \longrightarrow \omega$ such that $F(\alpha,\beta) = e(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta \in \omega_1$.

Evidently, every universal function is $(1,\omega_1)$-weakly universal. In [13] it was proved that a function $U : [\omega_1]^2 \longrightarrow \omega$ is $(1,\omega_1)$-weakly universal if and only if for every $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective function $h : \omega_1 \longrightarrow \omega_1$

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such that if $F(\alpha,\beta) \neq F(\alpha_1,\beta_1)$ then $U(h(\alpha), h(\beta)) \neq U(h(\alpha_1), h(\beta_1))$ for every $\alpha, \beta, \alpha_1, \beta_1 \in \omega_1$.

In an unpublished note of Tanmay Inamdar, it was proved that $(1,\omega_1)$-weakly universal functions exist assuming Martin's axiom for Knaster forcings (see [13]). In [13] Shelah and Steprāns asked the following:

Problem 2 ([13])**.** *Is there (in ZFC) a $(1,\omega_1)$-weakly universal function?*

In this note, we answer the previous question in the negative. For more on universal graphs and functions, the reader may consult [9] and [13].

Recall that $\clubsuit$ is the following statement:

- **$\clubsuit$):** There is a family $\{S_\alpha \mid \alpha \in LIM(\omega_1)\}$ such that each S_α is an unbounded subset of α and for every $X \in [\omega_1]^{\omega_1}$ the set $\{\alpha \mid S_\alpha \subseteq X\}$ is stationary.

The principle $\clubsuit$ is a weakening of the $\diamondsuit$ principle. It is well known that $\clubsuit$ is consistent with the failure of the Continuum Hypothesis (see [16], [5], [8] or [1]). The *stick principle* (introduced in [2]) is a weakening of $\clubsuit$:

- **$\bullet\!\!|$):** There is a family $\{S_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega_1]^\omega$ such that for every $X \in [\omega_1]^{\omega_1}$ there is an $\alpha \in \omega_1$ such that $S_\alpha \subseteq X$.

It is easy to see that the stick principle is a consequence of both $\clubsuit$ and CH. For more on $\clubsuit$ and $\bullet\!\!|$ the reader may consult [1], [5], [7] and [4].

We say that a tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* if for every $s \in p$ there is $t \in p$ extending s such that $t^\frown 0, t^\frown 1 \subseteq p$. The set of all Sacks trees is denoted by $\mathbb{S}$ and we order it by extension. Given an ordinal α, by $\mathbb{S}^\alpha$ we will denote the countable support product of α copies of Sacks forcing and by $\mathbb{S}_\alpha$ we denote α-iteration of $\mathbb{S}$ with countable support. By the *Sacks model* we mean the model obtained after forcing with $\mathbb{S}_{\omega_2}$ and by the *side-by-side Sacks model* we mean the model obtained after forcing with $\mathbb{S}^{\omega_2}$ to a model of GCH. Although the partial orders $\mathbb{S}^{\omega_2}$ and $\mathbb{S}_{\omega_2}$ are not forcing equivalent, they share very similar features. It is then interesting to point out differences between this two forcing notions. Some of the main differences between them are the following:

(1) In the Sacks model every subset of reals of size ω_2 can be mapped continuously onto the reals, while in the side-by-side Sacks model this is not the case (see [12]).
(2) The cardinal invariant $\mathfrak{hm}$[1] is evaluated differently on the Sacks model and in the side-by-side Sacks model (see [6]).
(3) The CPA axioms hold in the Sacks model but not in the side-by-side Sacks model (see [3]).

In this note, we will point out another difference: There are $(1, \omega_1)$-weakly universal functions in the Sacks model, while there are no such graphs in the side-by-side Sacks model. In [9] it was proved that $\overset{\bullet}{|} + \mathfrak{c} > \omega_1$ implies that there is no universal function $U : [\omega_1]^2 \longrightarrow 2$. However, our results show that the existence of $(1, \omega_1)$-weakly universal functions is even consistent with $\clubsuit + \mathfrak{c} > \omega_1$.

The countable support product of Sacks forcing. The *Sacks side-by-side model* is the model obtained by forcing with $\mathbb{S}^{\omega_2}$ over a model of the Generalized Continuum Hypothesis. We will prove that there are no $(1, \omega_1)$-weakly universal graphs in the Sacks side-by-side model.

We will need the following lemma:

Lemma 3. *There is a function $\pi : 2^\omega \longrightarrow \omega^\omega$ such that for every $r \in 2^\omega$, if f is an infinite partial function such that $f \subseteq \pi(r)$, then r is definable from f.*

Proof. Let $h : \omega \longrightarrow 2^{<\omega}$ be a definable bijection. We define $\pi : 2^\omega \longrightarrow \omega^\omega$ as follows: if $r \in 2^\omega$ and $n \in \omega$ then $\pi(r)(n) = m$ if m is the least natural number such that $h(m)$ is an initial segment of r and $h(m)$ has length at least n. It is easy to see that π has the desired property. □

Note that if M is a transitive model of ZFC and $r \notin M$ then $\pi(r)$ does not contain infinite partial functions from M. We will use the following unpublished result of Baumgartner (the reader may consult [8] for a proof):

Proposition 4 (Baumgartner). *The principle $\clubsuit$ holds in the Sacks side-by-side model.*

[1]The cardinal invariant $\mathfrak{hm}$ is the smallest size of a family of c_{min}-monochromatic sets required to cover the Cantor space (where $c_{min}(x, y)$ is the parity of the largest initial segment common to both x and y). It is known that $\mathfrak{c}^-, cof(\mathcal{N}) \leq \mathfrak{hm}$ (see [6]). It is an open question of Geschke if the inequality $\mathfrak{hm} < \mathfrak{r}$ is consistent. In a yet unpublished work, the author proved that the inequality $\mathfrak{hm} < \mathfrak{u}$ is consistent.

In fact, we will only use that every uncountable subset of ω_1 in the Sacks side-by-side model contains a countable ground model set. Given a function $F : [\omega_1]^2 \longrightarrow \omega$ and $U : [\omega_1]^2 \longrightarrow \omega$, we say that (h, e) is an $(1, \omega_1)$-*weakly universal embedding from* F *to* U if $h : \omega_1 \longrightarrow \omega_1$ is an injective function, $e : \omega \longrightarrow \omega$ and $F(\alpha, \beta) = e(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta \in \omega_1$. We can now prove the following result, answering the problem of Shelah and Steprāns:

Proposition 5. *There are no* $(1, \omega_1)$*-weakly universal graphs in the Sacks side-by-side model.*

Proof. Let $p_0 \in \mathbb{S}^{\omega_2}$ and $\dot{U}$ such that $p_0 \Vdash$ "$\dot{U} : [\omega_1]^2 \longrightarrow \omega$". Since the product of Sacks forcing has the ω_2-chain condition, we may find $\omega_1 \leq \beta < \omega_2$ such that $p_0 \in \mathbb{S}^{\beta}$ and $\dot{U}$ is a $\mathbb{S}^{\beta}$-name. Given $\alpha < \omega_1$, let $\dot{d}_\alpha$ be name for $\pi(\dot{r}_{\beta+\alpha})$ where $\dot{r}_{\beta+\alpha}$ is the name for the $(\beta+\alpha)$-generic real. For every infinite $\alpha < \omega_1$, we fix an enumeration $\alpha = \{\alpha_n \mid n \in \omega\}$.

If $G \subseteq \mathbb{S}^{\omega_2}$ is a generic filter, in $V[G]$ we define a function $F : [\omega_1]^2 \longrightarrow \omega$ as follows: given $\omega \leq \alpha < \omega_1$ we define $F(\alpha_n, \alpha) = d_\alpha(n)$. Let $\dot{F}$ be a name for F and let $\dot{h}$ be a $\mathbb{S}^{\omega_2}$-name for an injective function from ω_1 to ω_1 and $\dot{e}$ be a $\mathbb{S}^{\omega_2}$-name for a function from ω to ω. We will see that we can find an extension q of p_0 that forces that $(\dot{h}, \dot{e})$ is not a $(1, \omega_1)$-embedding of $\dot{F}$ in $\dot{U}$.

We can first find $p_1 \leq p_0$ and a ground model injective function $g : S \longrightarrow \omega_1$ such that $p_1 \Vdash$ "$g \subseteq \dot{h}$" where $S \in [\omega_1]^\omega$ (this is possible since the stick principle holds in the Sacks side-by-side model, witnessed by the ground model countable sets). Let M be a countable elementary submodel such that $p_1, \dot{U}, \beta, \dot{F}, g, \dot{h}, \dot{e} \in M$. Let $q \leq p_1$ be a $(M, \mathbb{S}^{\omega_2})$-generic condition. We claim that q forces that $(\dot{h}, \dot{e})$ is not an $(1, \omega_1)$-embedding of $\dot{F}$ in $\dot{U}$. Assume this is not the case, so there is $q_1 \leq q$ that forces that $(\dot{h}, \dot{e})$ is an $(1, \omega_1)$-embedding of $\dot{F}$ in $\dot{U}$.

Let $G \subseteq \mathbb{S}^{\omega_2}$ be a generic filter such that $q_1 \in G$. Let $X = \beta \cup (M \cap \omega_2)$ and define G_X to be the restriction of G to $\mathbb{S}^X$. Since q_1 is a $(M, \mathbb{S}^{\omega_2})$-generic condition, it follows that $\dot{U}[G], \dot{e}[G] \in V[G_X]$. Fix $\delta \in \omega_1$ such that $S \subseteq \delta$ and $\beta + \delta \notin X$, let $A = \{n \in \omega \mid \delta_n \in S\}$. For every $\alpha \in \omega_1$, we define $f_\alpha : A \longrightarrow \omega$ the function given by $f_\alpha(n) = \dot{e}[G]\left(\dot{U}[G](g(\delta_n), \alpha)\right)$ and note that $f_\alpha \in V[G_X]$ for every $\alpha \in \omega_1$. Let $\alpha \in \omega_1$ such that $\dot{h}[G](\delta) = \alpha$.

Since $(\dot{h}[G], \dot{e}[G])$ is forced to be an $(1,\omega_1)$-embedding, if $n \in A$ then we have the following:

$$\begin{aligned} d_\delta(n) &= \dot{F}[G](\delta_n, \delta) \\ &= \dot{e}[G]\Big(\dot{U}[G]\Big(\dot{h}(\delta_n), \dot{h}(\delta)\Big)\Big) \\ &= \dot{e}[G]\Big(\dot{U}[G](g(\delta_n), \alpha)\Big) \\ &= f_\alpha(n) \end{aligned}$$

Hence $f_\alpha \subseteq d_\delta$, but this is a contradiction since $r_{\beta+\delta} \notin V[G_X]$. □

The Cohen model. The Cohen model is the model obtained after adding ω_2-Cohen reals with finite support to a model of the Generalized Continuum Hypothesis. We will show that there are no $(1,\omega_1)$-weakly universal graphs in the Cohen model.

Lemma 6. *If $U : [\omega_1]^2 \longrightarrow \omega$ then U is not $(1,\omega_1)$-weakly universal after adding ω_2 Cohen reals.*

Proof. Let $U : [\omega_1]^2 \longrightarrow \omega$. We define the function $H : [\omega^\omega]^2 \longrightarrow \omega$ given by $H(x,y) = |x \wedge y|$ (where $x \wedge y$ denotes the largest initial segment which both x and y have in common). Let $\dot{c}_\alpha$ be the name for the α-Cohen real. Let $\dot{F}$ be a name of a function from $[\omega_1]^2$ to ω such that $\mathbb{C}_{\omega_2} \Vdash$ "$\dot{F}(\alpha,\beta) = H(\dot{c}_\alpha, \dot{c}_\beta)$".

Let $p \in \mathbb{C}_{\omega_2}$, $\dot{h}$ a name for an injective function from ω_1 to ω_1 and $\dot{e}$ a name for a function from ω to ω. We must find $q \leq p$ and $\alpha, \beta \in \omega_1$ such that $q \Vdash$ "$\dot{F}(\alpha,\beta) \neq \dot{e}U(\dot{h}(\alpha), \dot{h}(\beta))$". For every $\alpha < \omega_1$ we find $p_\alpha \leq p$ and δ_α such that $p_\alpha \Vdash$ "$\dot{h}(\alpha) = \delta_\alpha$" and $\alpha \in dom(p_\alpha)$. By the usual pruning arguments, we may find $X \in [\omega_1]^{\omega_1}$, $R \in [\omega_2]^{<\omega}$, $\overline{p} \in \mathbb{C}_{\omega_2}$ and $s \in \omega^{<\omega}$ such that the following holds:

1. $\{dom(p_\alpha) \mid \alpha \in X\}$ forms a Δ-system with root R.
2. $p_\alpha \restriction R = \overline{p}$ for every $\alpha \in X$.
3. $\alpha \notin R$ for every $\alpha \in X$.
4. $p(\alpha) = s$ for every $\alpha \in X$.

It is clear that $\{p_\alpha \mid \alpha \in \omega_1\}$ is a centered set (any finite set of conditions are compatible). Let M be a countable elementary submodel such that $\overline{p}, \{p_\alpha \mid \alpha \in \omega_1\}, \dot{e}, R \in M$. Since $M \cap \omega_2$ is countable, we may find $\alpha, \beta \in X \setminus M$ such that $\alpha \neq \beta$ and $dom(p_\alpha) \cap M = dom(p_\beta) \cap M = R$. Let

$m = U(\delta_\alpha, \delta_\beta)$. We may now find $\overline{q}$ and i such that the following conditions hold:

(1) $\overline{q} \in M$ and $\overline{q} \leq \overline{p}$.
(2) $\overline{q} \Vdash$ "$\dot{e}(m) = i$".

This is possible since $\overline{p}, \dot{e} \in M$. Let $t \in \omega^{<\omega}$ be such that $|t| > i$ and $s \subseteq t$. We now define a condition q as follows:

$$q(\xi) = \left\langle \begin{array}{cl} \overline{q}(\xi) & \text{if } \xi \in dom(\overline{q}) \\ p_\alpha(\xi) & \text{if } \xi \in dom(p_\alpha) \setminus dom(\overline{q}) \text{ and } \xi \neq \alpha \\ p_\beta(\xi) & \text{if } \xi \in dom(p_\beta) \setminus dom(\overline{q}) \text{ and } \xi \neq \beta \\ t & \text{if } \xi = \alpha \text{ or } \xi = \beta \end{array} \right.$$

Note that this is possible since $dom(\overline{q}) \subseteq M$. Clearly $q \Vdash$ "$\dot{F}(\alpha, \beta) > i$" and $q \Vdash$ "$\dot{e}(U(\dot{h}(\alpha), \dot{h}(\beta))) = i$" so $q \Vdash$ "$\dot{F}(\alpha, \beta) \neq \dot{e}(U(\dot{h}(\alpha), \dot{h}(\beta)))$". □

Since Cohen forcing has the countable chain condition, we conclude the following:

Proposition 7. *There are no $(1, \omega_1)$-weakly universal graphs in the Cohen model.*

The Sacks model. The proof that there are no $(1, \omega_1)$-weakly universal graph in the Side by Side Sacks model uses that the stick principle holds in such model. It is then natural to wonder if the stick principle is enough to get the non-existence of such graphs (under the failure of the Continuum Hypothesis). Moreover, the stick principle already forbids the existence of some universal graphs, as the following result of Shelah and Steprāns shows:

Proposition 8 ([13]). *stick $+ \mathfrak{c} > \omega_1$ implies that there is no universal function $U : [\omega_1]^2 \longrightarrow 2$.*

By the *Sacks model* we mean a model obtained by forcing with $\mathbb{S}_{\omega_2}$ over a model of the Generalized Continuum Hypothesis. In this section, we will prove that there is a $(1, \omega_1)$-weakly universal graph in the Sacks model. The following is a result of Mildenberger:

Proposition 9 ([11]). *♣ holds in the Sacks model.*

In particular, we will be able to conclude that the existence of a $(1,\omega_1)$-weakly universal graph is consistent with ♣. As usual, if $T \subseteq 2^{<\omega}$ is a tree, we denote by $[T]$ the set of all branches (i.e. maximal linearly order sets) through T. Given $f \in 2^\omega$ and $T \subseteq 2^{<\omega}$ a finite tree, we say that $f \in^* [T]$ if there is $n \in \omega$ such that $f \restriction n \in [T]$. If $f \in^* [T]$, we define by $f \restriction T$ to be the unique $t \in 2^{<\omega}$ such that there is n for which $t = f \restriction n \in [T]$. For this section, we fix W as the set of all (T, f) such that $T \subseteq 2^{<\omega}$ is a finite tree and $f : [T] \longrightarrow \omega$. It is easy to see that W is a countable set.

We will need some definition and lemmas regarding iterated Sacks forcing. The following is based on [12] and [8]. If $p \in \mathbb{S}$ and $s \in 2^{<\omega}$ we define $p_s = \{t \in p \mid t \subseteq s \vee s \subseteq t\}$. Note that p_s is a Sacks tree if and only if $s \in p$. By *supp* (p) we will denote the support of p.

Definition 10. Let $p \in \mathbb{S}_\alpha$, $F \in [supp\,(p)]^{<\omega}$ and $\sigma : F \longrightarrow 2^n$. We define p_σ as follows:

(1) $supp\,(p_\sigma) = supp\,(p)$.
(2) Letting $\beta < \alpha$ the following holds:
  (a) $p_\sigma(\beta) = p(\beta)$ if $\beta \notin F$.
  (b) $p_\sigma(\beta) = p(\beta)_{\sigma(\beta)}$ if $\beta \in F$.

Similar to previous situation, p_σ is not necessarily a condition of $\mathbb{S}_\alpha$. We will say that $\sigma : F \longrightarrow 2^n$ is *consistent with* p if $p_\sigma \in \mathbb{S}_\alpha$. A condition p is (F,n)-*determined* if for every $\sigma : F \longrightarrow 2^n$ either σ is consistent with p or there is $\beta \in F$ such that $\sigma \restriction (F \cap \beta)$ is consistent with p and $(p \restriction \beta)_{\sigma \restriction (F\cap\beta)} \Vdash$ "$\sigma(\beta) \notin p(\beta)$".

We say that $p \in \mathbb{S}_\alpha$ is *determined* if for every $F \in [supp\,(p)]^{<\omega}$ and for every $n \in \omega$ there are G and m such that the following holds:

(1) $G \in [supp\,(p)]^{<\omega}$.
(2) $F \subseteq G$.
(3) $n < m$.
(4) p is (G,m)-determined.

The following result is well known:

Lemma 11 ([12])**.** *For every $p \in \mathbb{S}_\alpha$ there is a determined $q \leq p$.*

Let p be a determined condition. We say that $\langle (F_i, n_i, \Sigma_i) \mid i \in \omega \rangle$ is *a representation of* p if the following holds:

(1) $F_i \in [supp\,(p)]^{<\omega}$, $n_i \in \omega$.
(2) $F_i \subseteq F_{i+1}$ and $n_i < n_{i+1}$.
(3) $supp\,(p) = \bigcup\limits_{i \in \omega} F_i$.
(4) p is (F_i, n_i)-determined for every $i \in \omega$.
(5) Σ_i is the set of all $\sigma : F_i \longrightarrow 2^{n_i}$ such that σ is consistent with p.

We will also need the following definition:

Definition 12. Let $p \in \mathbb{S}_\alpha$ be a determined condition and $\dot{r}$ an $\mathbb{S}_\alpha$-name for an element of 2^ω. We say that p is *$\dot{r}$-canonical* if there are two sequences $\langle (F_i, n_i, \Sigma_i) \mid i \in \omega \rangle$ and $\langle \mathcal{C}_i \mid i \in \omega \rangle$ with the following properties:

(1) $\{(F_i, n_i, \Sigma_i) \mid i \in \omega\}$ is a representation of p.
(2) $\mathcal{C}_i = \{C_\sigma \mid \sigma \in \Sigma_i\}$ is a collection of disjoint clopen subsets of 2^ω.
(3) For every $\sigma \in \Sigma_i$ there is $s_\sigma \in 2^{n_i}$ such that $C_\sigma \subseteq \langle s_\sigma \rangle$.[2]
(4) If $i \in \omega$ and $\sigma \in \Sigma_i$, then $p_\sigma \Vdash$ "$\dot{r} \in C_\sigma$" (in particular, p_σ determines $\dot{r} \upharpoonright n_i$).

In the above situation, we say that $\langle (F_i, n_i, \Sigma_i, \mathcal{C}_i) \mid i \in \omega \rangle$ is an *$\dot{r}$-canonical representation for* p. The following is lemma 6 of [12]:

Lemma 13 ([12])**.** *Let $\alpha \leq \omega_2$, $p \in \mathbb{S}_\alpha$ and $\dot{r}$ an $\mathbb{S}_\alpha$-name for an element of 2^ω such that $p \Vdash$ "$\dot{r} \notin \bigcup\limits_{\beta<\alpha} V[G_\beta]$". There is $q \leq p$ such that q is $\dot{r}$-canonical.*

With the same proof of the previous lemma, it is possible to prove the following:

Lemma 14. *Let $\alpha \leq \omega_2$, $p \in \mathbb{S}_\alpha$, $\dot{r}$ an $\mathbb{S}_\alpha$-name for an element of 2^ω such that $p \Vdash$ "$\dot{r} \notin \bigcup\limits_{\beta<\alpha} V[G_\beta]$" and $\dot{g}$ an $\mathbb{S}_\alpha$-name for an element of ω^ω. There is $q \leq p$ such that q is $\dot{r}$-canonical with $\dot{r}$-canonical representation $\langle (F_i, n_i, \Sigma_i, \mathcal{C}_i) \mid i \in \omega \rangle$ and there is $\langle h_i \mid i \in \omega \rangle$ such that the following conditions:*

(1) *$h_i : \Sigma_i \longrightarrow \omega$ for every $i \in \omega$.*
(2) *If $i \in \omega$ and $\sigma \in \Sigma_i$ then $q_\sigma \Vdash$ "$\dot{g}(i) = h_i(\sigma)$".*

[2]If $t \in 2^{<\omega}$ we define $\langle t \rangle = \{x \in 2^\omega \mid t \subseteq x\}$.

The lemma 6 of [12] is proved using a fusion argument. To prove the previous lemma we use the same fusion argument, with the extra step of deciding the respective value of $\dot{g}$ at each step. We leave the details for the reader. As before, in the above situation we say that $\langle (F_i, n_i, \Sigma_i, \mathcal{C}_i, h_i) \mid i \in \omega \rangle$ is an $(\dot{r}, \dot{g})$-*canonical representation for* q.

We can now prove the following:

Proposition 15. *Let $\eta < \omega_2, \dot{g}$ and $p \in \mathbb{S}_{\eta+1}$ such that $p \Vdash$ "$\dot{g} : \omega \longrightarrow \omega$". There is a determined $q \in \mathbb{S}_{\eta+1}$ and $\{(n, T_n, f_n) \mid n \in \omega\}$ with the property that $\{(T_n, f_n) \mid n \in \omega\} \subseteq W$ such that the following holds:*

(1) $q \leq p$.
(2) $q \Vdash$ "$\dot{r}_\eta \in^* [T_n]$" *for each* $n \in \omega$.
(3) $q \Vdash$ "$\dot{g}(n) = f_n(\dot{r}_\eta \restriction T_n)$" *for every* $n \in \omega$.

Proof. By the previous lemma, we can find $p_1 \leq p$ that has an $(\dot{r}_\eta, \dot{g})$-canonical representation $\{(F_i, n_i, \Sigma_i, \mathcal{C}_i, h_i) \mid i \in \omega\}$. We now have the following interesting property: If $G \subseteq \mathbb{S}_{\eta+1}$ is a generic filter with $p_1 \in G$ and $\langle r_\alpha \rangle_{\alpha \leq \eta}$ is the generic sequence, then the following holds in $V[G]$:

***):** For every $i \in \omega$ and $\sigma \in \Sigma_i$, if $r_\eta \in C_\sigma$ then $\sigma(\beta) \subseteq r_\beta$ for every $\beta \in F_i$.

This property holds because $\mathcal{C}_i = \{C_\sigma \mid \sigma \in \Sigma_i\}$ is a collection of disjoint sets. In this way, r_η is able to "code" each of the previews generic reals. Let Y be the set of all maximal $z \in 2^{<\omega}$ with the property that $\langle z \rangle \subseteq \bigcup \mathcal{C}_i$. Note that since $\mathcal{C}_i$ is a finite set of clopen sets, Y is a finite set. Let T_i be the smallest finite tree such that $Y \subseteq T_i$. Note that T_i has the following properties:

(1) $\bigcup\limits_{s \in [T_i]} \langle s \rangle = \bigcup \mathcal{C}_i$.
(2) For every $s \in [T_i]$ there is exactly one $\sigma \in \Sigma_i$ for which $\langle s \rangle \subseteq C_\sigma$ (where $\mathcal{C}_i = \{C_\sigma \mid \sigma \in \Sigma_i\}$).

For every i we have the following properties:

(1) $p_1 \Vdash$ "$\dot{r}_\eta \in [T_i]$".
(2) Let $G \subseteq \mathbb{S}_{\eta+1}$ be a generic filter with $p_1 \in G$ and $\sigma \in \Sigma_i$. If $r_\eta \in C_\sigma$ then $(p_1)_\sigma \in G$.

We now have the following claim:

Claim 16. *If $i \in \omega$, $s \in [T_i]$ and q_0, q_1 are two conditions extending p_1 such that $q_i \Vdash$ "$s \subseteq \dot{r}_\eta$" for $j \in \{0,1\}$ then there is $k \in \omega$ such that $q_0 \Vdash$ "$\dot{g}(i) = k$" and $q_1 \Vdash$ "$\dot{g}(i) = k$".*

We will prove the claim. Let $\sigma \in \Sigma_i$ such that $\langle s \rangle \subseteq C_\sigma$ and let $j < 2$. Note that since $q_j \leq p_1$ and $q_j \Vdash$ "$\dot{r}_\eta \in C_\sigma$", it follows that $q_j \Vdash$ "$(p_1)_\sigma \in \dot{G}$" (where $\dot{G}$ is a name for the generic filter), hence $q_j \Vdash$ "$\dot{g}(i) = h_i(\sigma)$", the claim follows.

For every $n \in \omega$, we define a function $f_n : [T_n] \longrightarrow \omega$ as follows: for every $s \in [T_n]$, let $f_n(s)$ such that for every $q \leq p_1$ if $q \Vdash$ "$s \subseteq \dot{r}_\eta$" then $q \Vdash$ "$\dot{g}(n) = f_n(s)$". Note that f_n is well defined by the previous claim. It is easy to see that $\{(n, T_n, f_n) \mid n \in \omega\}$ has the desired properties. $\square$

We will say that a graph $U : [\omega_1]^2 \longrightarrow W$ is $(1, \omega_1)$-*weakly universal* if for every $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective $h : \omega_1 \longrightarrow \omega_1$ and a function $e : W \longrightarrow \omega$ such that $F(\alpha, \beta) = e(U(h(\alpha), h(\beta)))$. As expected, we have the following result:

Lemma 17. *If there is a $U : [\omega_1]^2 \longrightarrow W$ which is $(1, \omega_1)$-weakly universal, then there is $U_1 : [\omega_1]^2 \longrightarrow \omega$ that is $(1, \omega_1)$-weakly universal.*

Proof. Let $U : [\omega_1]^2 \longrightarrow W$ be a $(1, \omega_1)$-weakly universal graph. Fix $g : W \longrightarrow \omega$ a bijective function. We define $U_1 : [\omega_1]^2 \longrightarrow \omega$ where $U_1(\alpha, \beta) = g(U(\alpha, \beta))$. It is easy to see that U_1 is $(1, \omega_1)$-weakly universal. $\square$

For the rest of this section, we will assume the Continuum Hypothesis. Fix a large enough regular cardinal $\theta > (2^{\omega_2})^+$. We will now fix $\overline{M} = \{(M_\alpha, \in, \Vdash_{\mathbb{S}_{\eta_\alpha+1}}, p_\alpha, \eta_\alpha, \xi_\alpha, \dot{g}_\alpha) \mid \alpha \in \omega_1\}$ with the following properties:

(1) M_α is a countable elementary submodel of $H(\theta)$ with the property that $p_\alpha, \eta_\alpha, \xi_\alpha, \dot{g}_\alpha \in M_\alpha$.
(2) $\eta_\alpha < \omega_2$ and $p_\alpha \in \mathbb{S}_{\eta_\alpha+1}$.
(3) $\xi_\alpha < \omega_1$ and $p_\alpha \Vdash$ "$\dot{g}_\alpha : \xi_\alpha \longrightarrow \omega$".
(4) For every $(N, \in, \Vdash_{\mathbb{S}_{\eta+1}}, p, \eta, \xi, \dot{g})$ if the following properties hold:
    (a) N is a countable elementary submodel of $H(\theta)$ with the property that $p, \eta, \xi, \dot{g} \in N$.
    (b) $\eta < \omega_2$ and $p \in \mathbb{S}_{\eta+1}$.
    (c) $\xi < \omega_1$ and $p \Vdash$ "$\dot{g} : \xi \longrightarrow \omega$".

Then, there is $\alpha < \omega_1$ such that $(M_\alpha, \in, \Vdash_{\mathbb{S}_{\eta_\alpha+1}}, p_\alpha, \eta_\alpha, \xi_\alpha, \dot{g}_\alpha)$ and $(N, \in, \Vdash_{\mathbb{S}_{\eta+1}}, p, \eta, \xi, \dot{g})$ are isomorphic.

This is possible since $\mathbb{S}_{\eta+1}$ is proper and we are assuming the Continuum Hypothesis. For every $\alpha < \omega_1$, let $\delta_\alpha = M_\alpha \cap \omega_1$. We now choose $\{\beta_\alpha \mid \alpha \in \omega_1\} \subseteq \omega_1$ such that $\delta_\alpha < \beta_\alpha$ and if $\alpha_1 \neq \alpha_2$ then $\beta_{\alpha_1} \neq \beta_{\alpha_2}$. For every $\alpha < \omega_1$, we also fix an enumeration $\xi_\alpha = \{\xi_\alpha(n) \mid n \in \omega\}$. By the previous lemmas, for every $\alpha < \omega_1$, we can find q_α, $\{(n, T_n^\alpha, f_n^\alpha) \mid n \in \omega\}$ such that the following holds:

(1) $q_\alpha \in \mathbb{S}_{\eta_\alpha+1} \cap M_\alpha$ and $q_\alpha \leq p_\alpha$.
(2) $\{(T_n^\alpha, f_n^\alpha) \mid n \in \omega\} \subseteq W$.
(3) $q_\alpha \Vdash$ "$\dot{r}_{\eta_\alpha} \in^* [T_n^\alpha]$" for each $n \in \omega$.
(4) $q_\alpha \Vdash$ "$\dot{g}_\alpha(\xi_\alpha(n)) = f_n^\alpha(\dot{r}_{\eta_\alpha} \upharpoonright T_n^\alpha)$" for every $n \in \omega$.

We now define the graph $U : [\omega_1]^2 \longrightarrow W$ as follows: given $\alpha < \omega_1$ and $n \in \omega$ we define $U(\xi_\alpha(n), \beta_\alpha) = (T_n^\alpha, f_n^\alpha)$ (the value of U is not important in any other case, so if a pair (ν_1, ν_2) is not of the form $(\xi_\alpha(n), \beta_\alpha)$, we can let $U(\nu_1, \nu_2)$ be any element of W otherwise). We will show that U is forced to be $(1,\omega_1)$-weakly universal. Given $\eta < \omega_2$, in the forcing extension, we define the function $e_\eta : W \longrightarrow \omega$ given by $e_\eta(T, f) = f(\dot{r}_\eta \upharpoonright T)$ if $\dot{r}_\eta \in^* [T]$ and $e_\eta(T, f) = 0$ in other case. We need the following lemma:

Lemma 18. *Let $G \subseteq \mathbb{S}_{\omega_2}$ be a generic filter. Let $\eta < \omega_2, \xi < \omega_1$ and $g : \xi \longrightarrow \omega$ such that $g \in V[G_{\eta+1}]$. There is $\alpha \in \omega_1$ such that the following holds:*

(1) $\xi_\alpha = \xi$.
(2) $g(\xi_\alpha(n)) = e_\eta(U(\xi_\alpha(n), \beta_\alpha))$ *for every* $n \in \omega$.

Proof. It is enough to show that the conditions that force the above properties are dense, in this way, there will be such condition in the generic filter. Let $p \in \mathbb{S}_{\eta+1}$, we will see we can extend p to get the desired conclusion. Let N be a countable elementary submodel such that $p, \eta, \xi, \dot{g} \in N$. We first find $\alpha < \omega_1$ such that $(M_\alpha, \in, \Vdash_{\mathbb{S}_{\eta_\alpha+1}}, p_\alpha, \eta_\alpha, \xi_\alpha, \dot{g}_\alpha)$ and $(N, \in, \Vdash_{\mathbb{S}_{\eta+1}}, p, \eta, \xi, \dot{g})$ are isomorphic. Let $\pi : M_\alpha \longrightarrow N$ be the isomorphism and let $q = \pi(q_\alpha)$. Note that the isomorphism fixes every ordinal smaller than δ_α (in particular each $\xi_\alpha(n)$) as well as each element in W. By the isomorphism, the following conditions hold:

(1) $q \in \mathbb{S}_{\eta+1} \cap N$ and $q \leq p$.
(2) $q \Vdash$ "$\dot{r}_\eta \in^* [T_n^\alpha]$" for each $n \in \omega$.

(3) $q \Vdash$ "$\dot{g}(\xi_\alpha(n)) = f_n^\alpha(\dot{r}_\eta \restriction T_n^\alpha)$" for every $n \in \omega$.

By the last clause, it follows that $q \Vdash$ "$\dot{g}(\xi_\alpha(n)) = e_\eta(U(\xi_\alpha(n), \beta_\alpha))$". □

We can then prove the following:

Proposition 19. *There is a $(1, \omega_1)$-weakly universal graph in the Sacks model.*

Proof. We will show that U is forced to be a $(1, \omega_1)$-weakly universal graph (note that this is enough by lemma 17). Let $p \in \mathbb{S}_{\omega_2}$ and $\dot{F}$ such that $p \Vdash$ "$\dot{F} : [\omega_1]^2 \longrightarrow \omega$". Since Sacks forcing has the ω_2-chain condition, we may assume that there is $\eta < \omega_2$ such that $p \in \mathbb{S}_\eta$ and $\dot{F}$ is an $\mathbb{S}_\eta$-name.

Given $\gamma \leq \omega_1$, we will say that an injective function $h : \gamma \longrightarrow \omega_1$ is a *partial e_η-embedding* if $F(\alpha, \beta) = e_\eta(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta < \gamma$. Let G be a generic filter such that $p \in G$. We claim that in $V[G_{\eta+1}]$ the following holds:

***):** If $h : \gamma \longrightarrow \omega_1$ is a partial e_η-embedding with $\gamma < \omega_1$, then there is a partial e_η-embedding $\overline{h} : \gamma + 1 \longrightarrow \omega_1$ extending h.

We argue in $V[G_{\eta+1}]$. Let $\xi = \bigcup h[\gamma] + 1$ and note we can find $g \in V[G_{\eta+1}]$ such that $g : \xi \longrightarrow \omega$ and $g(h(\delta)) = F(\delta, \gamma)$ for all $\delta < \gamma$. By the previous lemma, there is $\alpha \in \omega_1$ such that $\xi_\alpha = \xi$ and $g(\xi_\alpha(n)) = e_\eta(U(\xi_\alpha(n), \beta_\alpha))$. We now define $\overline{h} = h \cup \{(\gamma, \beta_\alpha)\}$. Note that $\beta_\alpha \notin h[\gamma]$ since $h[\gamma] \subseteq \xi = \xi_\alpha < \delta_\alpha < \beta_\alpha$. We only need to prove that $\overline{h}$ is a partial e_η-embedding. Let $\delta < \gamma$, we can find $n \in \omega$ such that $h(\delta) = \xi_\alpha(n)$. It then follows that:

$$\begin{aligned} e_\eta(U(\overline{h}(\delta), \overline{h}(\gamma))) &= e_\eta(U(\xi_\alpha(n), \beta_\alpha)) \\ &= g(\xi_\alpha(n)) \\ &= g(h(\delta)) \\ &= F(\delta, \gamma) \end{aligned}$$

This finishes the claim. It is clear that any maximal e_η-embedding will embed F into U. □

Open questions. In general, a function $U : [\omega_1]^2 \longrightarrow \omega$ is $(1, \kappa)$-*weakly universal* if for every $F : [\omega_1]^2 \longrightarrow \omega$ there is an injective function h :

$\omega_1 \longrightarrow \omega_1$ and a function $e : \omega \longrightarrow \omega$ such that $|e^{-1}(n)| < \kappa$ for every $n \in \omega$ and $F(\alpha, \beta) = e(U(h(\alpha), h(\beta)))$ for every $\alpha, \beta \in \omega_1$. It would be interesting to know the answer of the following question:

Problem 20. *Are there $(1,\omega)$-weakly universal functions (or even $(1,2)$-weakly universal functions) in the Sacks model?*

In fact, we conjecture that $\overset{\bullet}{|} + \mathfrak{c} > \omega_1$ implies that there are no $(1,\omega)$-weakly universal functions.

We would like to mention that there are no $(1,\omega_1)$-weakly universal functions after performing a pseudo-iteration of Cohen forcing, as described in [5]. It would be interesting to know what kind of universal graphs exist on the "canonical models" of set theory.

Problem 21. *Are there $(1,\omega_1)$-weakly universal functions in the random, Hechler, Laver, Miller and Mathias models?*

The purpose of the CPA axioms introduced in [3] is to provide an axiomatization of the Sacks model. In light of this work, it is then natural to ask the following:

Problem 22. *Does the existence of $(1,\omega_1)$-weakly universal functions follow from one of the CPA axioms?*

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References

[1] Jörg Brendle. Cardinal invariants of the continuum and combinatorics on uncountable cardinals. *Ann. Pure Appl. Logic*, 144(1-3):43–72, 2006.

[2] S. Broverman, J. Ginsburg, K. Kunen, and F. D. Tall. Topologies determined by σ-ideals on ω_1. *Canad. J. Math.*, 30(6):1306–1312, 1978.

[3] Krzysztof Ciesielski and Janusz Pawlikowski. *The covering property axiom, CPA*, volume 164 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2004. A combinatorial core of the iterated perfect set model.

[4] Mirna Džamonja and Saharon Shelah. Similar but not the same: various versions of ♣ do not coincide. *J. Symbolic Logic*, 64(1):180–198, 1999.

[5] Sakaé Fuchino, Saharon Shelah, and Lajos Soukup. Sticks and clubs. *Ann. Pure Appl. Logic*, 90(1-3):57–77, 1997.

[6] S. Geschke, M. Kojman, W. Kubiś, and R. Schipperus. Convex decompositions in the plane and continuous pair colorings of the irrationals. *Isr. J. Math.*, 131:285–317, 2002.

[7] Michael Hrušák. Another $\diamondsuit$-like principle. *Fund. Math.*, 167(3):277–289, 2001.

[8] Michal Hrušák. Life in the Sacks model. *Acta Univ. Carolin. Math. Phys.*, 42(2):43–58, 2001. 29th Winter School on Abstract Analysis (Lhota nad Rohanovem/Zahrádky u České Lípy, 2001).

[9] Paul B. Larson, Arnold W. Miller, Juris Steprāns, and William A. R. Weiss. Universal functions. *Fund. Math.*, 227(3):197–246, 2014.

[10] Alan H. Mekler. Universal structures in power $\aleph_1$. *J. Symbolic Logic*, 55(2):466–477, 1990.

[11] Heike Mildenberger. The club principle and the distributivity number. *The Journal of Symbolic Logic*, 76(1):3446, 2011.

[12] Arnold W. Miller. Mapping a set of reals onto the reals. *J. Symbolic Logic*, 48(3):575–584, 1983.

[13] Shelah Saharon and Juris Steprāns. Universal graphs and functions. *preprint*.

[14] Saharon Shelah. On universal graphs without instances of CH. *Ann. Pure Appl. Logic*, 26(1):75–87, 1984.

[15] Saharon Shelah. Universal graphs without instances of CH: revisited. *Israel J. Math.*, 70(1):69–81, 1990.

[16] Saharon Shelah. *Proper and Improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

University of Toronto, Toronto, ON M5S 3H7, Canada
E-mail address: oguzman@math.toronto.edu