SACKS FORCING AND THE SHRINK WRAPPING PROPERTY

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Abstract. We consider a property stronger than the Sacks property, called the *shrink wrapping* property, which holds between the ground model and each Sacks forcing extension. Unlike the Sacks property, the shrink wrapping property does not hold between the ground model and a Silver forcing extension. We also show an application of the shrink wrapping property.

1. The Shrink Wrapping Property

Within this section, we will define the *shrink wrapping property*, which is a strengthening of the Sacks property, which holds between each Sacks forcing extension and the ground model, but not between any Silver forcing extension and the ground model.

Definition 1.1. Given a tree $T \subseteq {}^{<\omega}\omega$ (or $T \subseteq {}^{<\omega}2$) and a node $t \in T$, by $T|t$ we mean the tree T restricted to t:

$$T|t := \{s \in T : s \sqsubseteq t \text{ or } s \sqsupseteq t\}.$$

Definition 1.2. Given a function $f : \omega \to (\omega - \{0\})$, we say that a tree $T \subseteq {}^{<\omega}\omega$ *obeys* f iff for each $l \in \omega$, the set

$$\{n \in \omega : t^\frown n \in T \text{ for some } t \text{ on level } l \text{ of } T\}$$

has size $\leq f(l)$.

Definition 1.3. Let M be a transitive model of ZF. The *Sacks property* holds between V and M iff given any function $f : \omega \to (\omega - \{0\})$ in M satisfying $\lim_{l\to\omega} f(l) = \omega$ and given any $x \in {}^\omega\omega$ (in V), there is some tree $T \in M$ which obeys f such that $x \in [T]$.

The Sacks property as we have just defined it is equivalent to the version where we only consider a single such function f in M, instead of all such functions. Suppose that V is a Sacks forcing extension of a model M. Then the *Sacks property* holds between V and M. Now, if V is a Sacks forcing extension of M and $\langle x_n \in {}^\omega\omega : n < \omega\rangle$ is a sequence of reals (in V), then we cannot expect there to be a single function f in M and a sequence of trees $\langle T_n \subseteq {}^{<\omega}\omega : n \in \omega\rangle$ in M such that for each n, T_n obeys f and $x_n \in [T_n]$. Because of this, fix the following sequence of functions:

Definition 1.4. Fix $\langle f_i : i < \omega\rangle$, where each $f_i : \omega \to (\omega - \{0\})$ satisfies $\lim_{l\to\omega} f_i(l) = \omega$, and the sequence itself is such that $(i, l) \mapsto f_i(l)$ is an injection.

We can take this sequence to be computable, so that it is contained in every model of ZF. We have that if the Sacks property holds between V and M and $\langle x_n \in {}^\omega\omega : n \in \omega\rangle$ is any sequence of reals, then there is a sequence of trees $\langle T_n : n < \omega\rangle \in M$ such that $(\forall n < \omega)\ T_n$ obeys f_n and $x_n \in [T_n]$.

The following is a stronger property that we might want to hold between V and M: for every sequence $\mathcal{X} = \langle x_n \in {}^\omega\omega : n < \omega\rangle$ there exists a sequence of trees $\langle T_n \subseteq {}^{<\omega}\omega : n < \omega\rangle \in M$ such that

1) $(\forall n \in \omega)\ T_n$ obeys f_n and $x_n \in [T_n]$;
2) $(\forall n_1, n_2 \in \omega)$ one of the following holds:
   a) $x_{n_1} = x_{n_2}$;
   b) $[T_{n_1}] \cap [T_{n_2}] = \emptyset$.

Unfortunately, if the sequence $\mathcal{X}$ satisfies

$$\langle (n_1, n_2) : x_{n_1} = x_{n_2}\rangle \notin M,$$

then there can be no such sequence of trees in M. Thus, we need a weaker notion: a *shrink wrapper*.

Definition 1.5. Fix a canonical bijection $\eta : \omega \to [\omega]^2$ so that for each $\tilde{n} \in \omega$, we may talk about the $\tilde{n}$-th pair $\eta(\tilde{n}) \in [\omega]^2$.

The idea of a shrink wrapper is that for each $\{n_1, n_2\} = \eta(\tilde{n}) \in [\omega]^2$, the functions $F_{\tilde{n},n_1}$ and $F_{\tilde{n},n_2}$, together with the finite sets $I(n_1)$ and $I(n_2)$, will separate x_{n_1} and x_{n_2} as much as possible. For $n \in \eta(\tilde{n})$, the function $F_{\tilde{n},n} : {}^{\tilde{n}}2 \to \mathcal{P}({}^{<\omega}\omega)$ is shrink-wrapping $2^{\tilde{n}}$ possibilities for the value of x_n. We need to make sure that what contains one possibility for x_{n_1} is sufficiently disjoint from what contains another possibility for x_{n_2}, even if it is not possible that simultaneously both x_{n_1} and x_{n_2} are in the respective containers.

Fix $\tilde{n} \in \omega$ and consider the $\tilde{n}$-th pair $\{n_1, n_2\}$. If $x_{n_1} = x_{n_2}$, they certainly cannot be separated and this is a special case. Also, there are finitely many "isolated" points which might prevent the separation of x_{n_1} from x_{n_2}. In fact, we can get a finite set $I(k)$ of isolated points associated to each x_k as opposed to each pair $\{x_{n_1}, x_{n_2}\}$.

When we construct a shrink wrapper for a sequence of reals in a Sacks forcing extension, we can easily get the trees that occur in the shrink wrapper to obey functions in the ground model. To facilitate this, we do the following:

Definition 1.6. Fix an injection $\Phi : {}^{<\omega}2 \times \omega \to \omega$.

In the definition of a shrink wrapper, we will have each $F_{\tilde{n},n}(s)$ be a tree which obeys $f_{\Phi(s,n)}$. Thus, the definition of a shrink wrapper depends on the injection Φ and the injection $(i,l) \mapsto f_i(l)$. However, the reader can check that the choice of these two injections is not important, as long as they are both in the ground model.

Definition 1.7. A *shrink wrapper* $\mathcal{W}$ for $\mathcal{X} = \langle x_n \in {}^\omega\omega : n \in \omega \rangle$ is a pair $\langle \mathcal{F}, I \rangle$ such that $I : \omega \to [{}^\omega\omega]^{<\omega}$ and $\mathcal{F}$ is a collection of functions $F_{\tilde{n},n}$ for $\tilde{n} \in \omega$ and $n \in \eta(\tilde{n})$ which satisfy the following conditions.

1) Given $\tilde{n}$ and $n \in \eta(\tilde{n})$, $F_{\tilde{n},n} : {}^{\tilde{n}}2 \to \mathcal{P}({}^{<\omega}\omega)$ and for each $s \in {}^{\tilde{n}}2$, $F_{\tilde{n},n}(s) \subseteq {}^{<\omega}\omega$ is a leafless tree that obeys $f_{\Phi(s,n)}$.
2) Given $\tilde{n}$ and $n \in \eta(\tilde{n})$, $(\exists s \in {}^{\tilde{n}}2)\, x_n \in [F_{\tilde{n},n}(s)]$.
3) Given $\{n_1, n_2\} = \eta(\tilde{n})$, $(\forall s_1, s_2 \in {}^{\tilde{n}}2)$ one of the following relationships holds between the sets $C_1 := [F_{\tilde{n},n_1}(s_1)]$ and $C_2 := [F_{\tilde{n},n_2}(s_2)]$:
   3a) $C_1 = C_2$ and if either $x_{n_1} \in C_1$ or $x_{n_2} \in C_2$, then $x_{n_1} = x_{n_2}$;
   3b) $(\exists x \in I(n_1) \cap I(n_2))\, C_1 = C_2 = \{x\}$;
   3c) $C_1 \cap C_2 = \emptyset$, and therefore $(\exists l \in \omega)(\forall (y_1, y_2) \in C_1 \times C_2)\, y_1$ and y_2 differ before level l.

The therefore part of 3c) is because if for each l there was a node on level l of the tree $T := F_{\tilde{n},n_1}(s_1) \cap F_{\tilde{n},n_2}(s_2)$, then because T has finite branching, by Konig's lemma it would have an infinite branch. When we construct a shrink wrapper, we can usually ensure that it satisfies the following additional property:

4) Given $\tilde{n}$ and $n \in \eta(\tilde{n})$, $(\forall s_1, s_2 \in {}^{\tilde{n}}2)$ one of the following relationships holds between the sets $C_1 := [F_{\tilde{n},n}(s_1)]$ and $C_2 := [F_{\tilde{n},n}(s_2)]$:
   4a) $(\exists x \in I(n))\, C_1 = C_2 = \{x\}$;
   4b) $C_1 \cap C_2 = \emptyset$, and therefore $(\exists l \in \omega)(\forall (y_1, y_2) \in C_1 \times C_2)\, y_1$ and y_2 differ before level l.

Note this is a requirement on the single function $F_{\tilde{n},n}$ where $n \in \eta(\tilde{n})$, and not a requirement on the pair of functions $(F_{\tilde{n},n_1}, F_{\tilde{n},n_2})$ where $\{n_1, n_2\} = \eta(\tilde{n})$.

Definition 1.8. Given a model M of ZFC, we say that the *shrink wrapping property* holds between M and V iff every sequence $\mathcal{X} = \langle x_n \in {}^\omega\omega : n \in \omega \rangle$ has a shrink wrapper $\mathcal{W}$ in M. A forcing $\mathbb{P}$ has the shrink wrapping property iff the shrink wrapping property holds between the ground model and each forcing extension.

In Theorem 3.7 we will show that Sacks forcing has the shrink wrapping property. If a forcing has the shrink wrapping property, then it automatically has the Sacks property. That is, consider any real x in the forcing extension. Now consider any sequence $\mathcal{X} = \langle x_n : n \in \omega$ such that $x_0 = x$. Let $\tilde{n}$ be such that $0 \in \eta(\tilde{n})$. Let $\mathcal{W}$ be a shrink wrapper for $\mathcal{X}$ in the ground model. We have that x_0 is a path through the tree

$$\bigcup \{F_{\tilde{n},0}(s) : s \in {}^{\tilde{n}}2\},$$

and this tree obeys the function

$$l \mapsto \sum_{s \in {}^{\tilde{n}}2} f_{\Phi(s,0)}(l),$$

which is in M (and does not depend on $\mathcal{X}$).

2. Application to Pointwise Eventual Domination

Before we show that there is always a shrink wrapper in the ground model after doing Sacks forcing, let us discuss an application of shrink wrappers themselves. Given two functions $f, g : {}^\omega\omega \to {}^\omega\omega$, let us write $f \leq^* g$ and say that g *pointwise eventually dominates* f iff

$$(\forall x \in {}^\omega\omega)(\forall^\infty n)\, f(x)(n) \leq g(x)(n).$$

One may ask what is the cofinality of the set of Borel functions from ${}^\omega\omega$ to ${}^\omega\omega$ ordered by $\leq^*$. The answer is 2^ω, which follows from the result in [1] that given any $A \subseteq \omega$, there is a Baire class one (and therefore Borel) function $f_A : {}^\omega\omega \to {}^\omega\omega$ such that given any Borel $g : {}^\omega\omega \to {}^\omega\omega$ satisfying $f_A \leq^* g$, we have that A is Δ^1_1 in any code for g. One may ask what functions f_A have such a property.

Being precise, say that a function $f : {}^\omega\omega \to {}^\omega\omega$ *sufficiently encodes* $A \subseteq \omega$ iff whenever $g : {}^\omega\omega \to {}^\omega\omega$ is Borel and satisfies $f \leq^* g$, then $A \in L[c]$ where c is any code for g. What must a function do to sufficiently encode A? Given a sequence $\mathcal{X} = \langle x_n \in {}^\omega\omega : n < \omega \rangle$, let us write $f_{\mathcal{X}} : {}^\omega\omega \to {}^\omega\omega$ for the function

$$f_{\mathcal{X}}(x)(n) := \begin{cases} \min\{l : x(l) \neq x_n(l)\} & \text{if } x \neq x_n, \\ 0 & \text{otherwise.} \end{cases}$$

Given $A \subseteq \omega$, is there always some $\mathcal{X}$ such that $f_{\mathcal{X}}$ sufficiently encodes A? It might seem like the answer is yes, because if a Borel function $g : {}^\omega\omega \to \omega$ everywhere dominates one of the sections $x \mapsto f_{\mathcal{X}}(x)(n)$, then x_n is Δ^1_1 in any code for g [1].

However, using a shrink wrapper, we can show that consistently there is not always a function of the form $f_{\mathcal{X}}$ that sufficiently encodes A. Specifically, suppose V is a Sacks forcing extension of an inner model M, $A \notin M$, and $\mathcal{X}$ is a sequence of reals. In the next section, we will show that there is a shrink wrapper $\mathcal{W} \in M$ for $\mathcal{X}$. In this section we will show how to build from $\mathcal{W}$ a Borel function $g : {}^\omega\omega \to {}^\omega\omega$, with a code $c \in M$, satisfying $f_{\mathcal{X}} \leq^* g$. Since $c \in M$, also $L[c] \subseteq M$, which implies $A \notin L[c]$. Hence, $f_{\mathcal{X}}$ does not sufficiently encode A.

To facilitate the discussion, let us make the following definitions:

Definition 2.1. Give $x \in {}^\omega\omega$, $[[x]] \subseteq {}^{<\omega}\omega$ is the set of all finite initial segments of x.

Definition 2.2. Given a tree $T \subseteq {}^{<\omega}\omega$, $\mathrm{Exit}(T) : {}^\omega\omega \to \omega$ is the function

$$\mathrm{Exit}(T)(x) := \begin{cases} \min\{l : x \restriction l \notin T\} & \text{if } x \notin [T], \\ 0 & \text{if } x \in [T]. \end{cases}$$

For the remainder of this section we will show that if M is a transitive model of ZF and a sequence $\mathcal{X}$ of reals has a shrink wrapper in M, then there is a Borel function g with a code in M such that $f_{\mathcal{X}} \leq^* g$. We will illustrate the main ideas by considering a situation where M contains something stronger than a shrink wrapper for $\mathcal{X}$.

Proposition 2.3. *Let M be a transitive model of ZF. Let*

$$\mathcal{X} = \langle x_n \in {}^\omega\omega : n \in \omega \rangle.$$

Suppose

$$\mathcal{T} = \langle T_n : n \in \omega \rangle \in M$$

is a sequence of subtrees of ${}^{<\omega}\omega$ satisfying the following:

1) $(\forall n \in \omega)\, x_n \in [T_n]$.
2) $(\forall n_1, n_2 \in \omega)$ *one of the following holds:*
   a) $x_{n_1} = x_{n_2}$;
   b) $[T_{n_1}] \cap [T_{n_2}] = \emptyset$.

Then there is a Borel function $g : {}^\omega\omega \to {}^\omega\omega$ that has a Borel code in M satisfying

$$(\forall x \in {}^\omega\omega)\, f_{\mathcal{X}}(x) \leq^* g(x).$$

Proof. Let $g : {}^\omega\omega \to {}^\omega\omega$ be defined by

$$g(x)(n) := \max\{\mathrm{Exit}(T_n)(x), n\}.$$

Certainly g is Borel, with a code in M (because $\mathcal{T} \in M$). The "$\text{Exit}(T_n)(x)$" part of the definition is doing most of the work. Specifically, for any $n \in \omega$ and $x \notin [T_n]$,

$$f_{\mathcal{X}}(x)(n) = \text{Exit}([[x_n]])(x) \leq \text{Exit}(T_n)(x).$$

This is because since x_n is a path through the tree T_n, $x \notin [T_n]$ implies the level where x exits T_n is not before the level where x differs from x_n. Thus, we have

$$(\forall n \in \omega)\, x \notin [T_n] \Rightarrow f_{\mathcal{X}}(x)(n) \leq g(x)(n).$$

Suppose, towards a contradiction, that there is some $x \in {}^{\omega}\omega$ satisfying $f_{\mathcal{X}}(x) \not\leq^* g(x)$. Fix such an x. Let A be the infinite set

$$A := \{n \in \omega : f_{\mathcal{X}}(x)(n) > g(x)(n)\}.$$

It must be that $x \in [T_n]$ for each $n \in A$. By hypothesis, this implies $x_{n_1} = x_{n_2}$ for all $n_1, n_2 \in A$. Thus, $f_{\mathcal{X}}(x)(n)$ is the same constant for all $n \in A$. This is a contradiction, because $g(x)(n) \geq n$ for all n. □

Here is the stronger result where we only assume that M has a shrink wrapper for $\mathcal{X}$:

Theorem 2.4. *Let M be a transitive model of ZF. Let*

$$\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n \in \omega \rangle.$$

Suppose $\mathcal{W} = \langle \mathcal{F}, I \rangle \in M$ is a shrink wrapper for $\mathcal{X}$. Then there is a Borel function $g : {}^{\omega}\omega \to {}^{\omega}\omega$ that has a Borel code in M satisfying

$$(\forall x \in {}^{\omega}\omega)\, f_{\mathcal{X}}(x) \leq^* g(x).$$

Proof. For each $n \in \omega$, let $T_n \subseteq {}^{<\omega}\omega$ be the tree

$$T_n := \bigcap \{ \bigcup \text{Im}(F_{\tilde{n},n}) : \tilde{n} \in \omega \wedge n \in \eta(\tilde{n}) \}.$$

That is, for each $t \in {}^{<\omega}\omega$, $t \in T_n$ iff

$$(\forall \tilde{n} \in \omega)[n \in \eta(\tilde{n}) \Rightarrow t \in \bigcup_{s \in {}^{\tilde{n}}2} F_{\tilde{n},n}(s)].$$

By part 2) of the definition of a shrink wrapper,

$$(\forall n \in \omega)\, x_n \in [T_n].$$

Let $e(n_2)$ be the least level l such that if $n_1 < n_2$, $\tilde{n}$ satisfies $\eta(\tilde{n}) = \{n_1, n_2\}$, and $s_1, s_2 \in {}^{\tilde{n}}2$ satisfy $[F_{\tilde{n},n_1}(s_1)] \cap [F_{\tilde{n},n_2}(s_2)] = \emptyset$, then all elements of $[F_{\tilde{n},n_1}(s_1)]$ differ from all elements of $[F_{\tilde{n},n_2}(s_2)]$ before level l.

Let $g : {}^{\omega}\omega \to {}^{\omega}\omega$ be defined by

$$g(x)(n) := \max\{\text{Exit}(T_n)(x), e(n), n\}.$$

Certainly g is Borel, with a code in M (because $\mathcal{W} \in M$). Just like in the previous proposition, since $x_n \in [T_n]$, for all $x \in {}^\omega\omega$ and $n \in \omega$ we have

$$x \notin [T_n] \Rightarrow f_{\mathcal{X}}(x)(n) \leq g(x)(n).$$

Suppose, towards a contradiction, that there is some $x \in {}^\omega\omega$ satisfying $f_{\mathcal{X}}(x) \not\leq^* g(x)$. Fix such an x. Let A be the infinite set

$$A := \{n \in \omega : f_{\mathcal{X}}(x)(n) > g(x)(n)\}.$$

It must be that $x \in [T_n]$ for each $n \in A$. Since A is infinite, we may fix $n_1, n_2 \in A$ satisfying the following:

i) $n_1 < n_2$;
ii) $f_{\mathcal{X}}(x)(n_1) \leq n_2$.

Let $\tilde{n}$ satisfy $\eta(\tilde{n}) = \{n_1, n_2\}$. Since $x \in [T_{n_1}]$, fix some $s_1 \in {}^{\tilde{n}}2$ satisfying

$$x \in [F_{\tilde{n},n_1}(s_1)] =: C_1.$$

Also, since $x_{n_2} \in [T_{n_2}]$, fix some $s_2 \in {}^{\tilde{n}}2$ satisfying

$$x_{n_2} \in [F_{\tilde{n},n_2}(s_2)] =: C_2.$$

Because $f_{\mathcal{X}}(x)(n_2) > g(x)(n_2)$, we have $f_{\mathcal{X}}(x)(n_2) > e(n_2)$, so

$$\text{Exit}([[x_{n_2}]])(x) > e(n_2).$$

This, combining with the definition of $e(n_2)$ and the fact that $x \in C_1$ and $x_{n_2} \in C_2$ tells us that $C_1 \cap C_2 \neq \emptyset$ (because otherwise $x \in C_1$ and $x_{n_2} \in C_2$ would differ before level $e(n_2)$, which by definition of $e(n_2)$ would mean that $\text{Exit}([[x_{n_2}]])(x) \leq e(n_2)$). Thus, by part 3) of the definition of a separation device, one of the following holds:

a) $x_{n_1} = x_{n_2}$;
b) $C_1 = C_2 = \{x\}$.

Now, b) cannot be the case because $C_2 = \{x\}$ implies $x_{n_2} = x$, which implies $f_{\mathcal{X}}(x)(n_2) = 0$, which contradicts the fact that $f_{\mathcal{X}}(x)(n_2) > g(x)(n_2)$. On the other hand, a) cannot be the case because $x_{n_1} = x_{n_2}$ implies $f_{\mathcal{X}}(x)(n_1) = f_{\mathcal{X}}(x)(n_2)$, which by ii) implies

$$f_{\mathcal{X}}(x)(n_2) = f_{\mathcal{X}}(x)(n_1) \leq n_2 \leq g(x)(n_2) < f_{\mathcal{X}}(x)(n_2),$$

which is impossible. □

3. Sacks Forcing

In this section, we will show that the shrink wrapping property holds between the ground model and any Sacks forcing extension.

Definition 3.1. A tree $p \subseteq {}^{<\omega}2$ is *perfect* iff it is nonempty and for each $t \in p$, there are incompatible $t_1, t_2 \in p$ extending t. Sacks forcing $\mathbb{S}$ is the poset of all perfect trees $p \subseteq {}^{<\omega}2$, where $p_1 \leq p_2$ iff $p_1 \subseteq p_2$.

Given $p_1, p_2 \in \mathbb{S}$, $p_1 \perp p_2$ means that p_1 and p_2 are incompatible.

Definition 3.2. Let $p \subseteq {}^{<\omega}2$ be a perfect tree. A node $t \in p$ is called a *branching node* iff $t^\frown 0, t^\frown 1 \in p$. Stem$(p)$ is the unique branching node t of p such that all elements of p are comparable to t. A node $t \in p$ is said to be an *n-th branching node* iff it is a branching node and there are exactly n branching nodes that are proper initial segments of it. In particular, Stem(p) is the unique 0-th branching node of p. Given Sacks conditions p, q, we write $q \leq_n p$ iff $q \leq p$ and all of the k-th branching nodes, for $k \leq n$, of p are in q and are branching nodes.

Lemma 3.3 (Fusion Lemma)**.** *Let $\langle p_n : n \in \omega \rangle$ be a sequence of Sacks conditions such that*

$$p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots.$$

Then $p_\omega := \bigcap_{n\in\omega} p_n$ is a Sacks condition below each p_n.

Proof. This is standard and can be found in introductory presentations of Sacks forcing. See, for example, [2]. □

The sequence $\langle p_n : n \in \omega \rangle$ in the lemma above is known as a *fusion sequence*. The following will help in the construction of fusion sequences.

Lemma 3.4 (Fusion Helper Lemma)**.** *Let $R : {}^{<\omega}2 \to \mathbb{S}$ be a function with the following properties:*

1) $(\forall s_1, s_2 \in {}^{<\omega}2)$ $s_2 \sqsupseteq s_1$ *implies* $R(s_2) \leq R(s_1)$*;*
2) $(\forall s \in {}^{<\omega}2)$ *Stem*$(R(s^\frown 0)) \perp$ *Stem*$(R(s^\frown 1))$*.*

For each $n \in \omega$, let p_n be the Sacks condition

$$p_n := \bigcup \{R(s) : s \in {}^n2\}.$$

Then

$$R(\emptyset) = p_0 \geq p_1 \geq_0 p_2 \geq_1 p_3 \geq_2 \ldots$$

is a fusion sequence.

Proof. Consider any $n \geq 1$. Certainly $p_n \supseteq p_{n+1}$, because for each $s \in {}^n2$, $R(s) \supseteq R(s^\frown 0) \cup R(s^\frown 1)$. To show that $p_n \geq_{n-1} p_{n+1}$, consider a k-th branching node t of p_n for some $k \leq n-1$. One can check that there is some $s \in {}^k2$ such that t is the largest common initial segment of Stem$(R(s^\frown 0))$ and Stem$(R(s^\frown 1))$. Since

$$\text{Stem}(R(s^\frown 0)) \cup \text{Stem}(R(s^\frown 1)) \subseteq R(s^\frown 0) \cup R(s^\frown 1) \subseteq p_{n+1},$$

we have that t is a branching node of p_{n+1}. Thus, we have shown that for each $k \leq n-1$, each k-th branching node of p_n is a branching node of p_{n+1}. Hence, $p_n \geq_{n-1} p_{n+1}$. □

We present a forcing lemma that is a basic building block for separating x_{n_1} from x_{n_2}. Combining this with a fusion argument gives us the result.

Lemma 3.5. *Let $\mathbb{P}$ be any forcing. Let $p_0, p_1 \in \mathbb{P}$ be conditions. Let $\dot{\tau}_0, \dot{\tau}_1$ be names for elements of ${}^{\omega}\omega$. Suppose that there is no $x \in {}^{\omega}\omega$ satisfying the following two statements:*

1) $p_0 \Vdash \dot{\tau}_0 = \check{x}$;
2) $p_1 \Vdash \dot{\tau}_1 = \check{x}$.

Then there exist $p_0' \leq p_0$; $p_1' \leq p_1$; and $t_0, t_1 \in {}^{<\omega}\omega$ satisfying the following:

3) $t_0 \perp t_1$,
4) $p_0' \Vdash \dot{\tau}_0 \sqsupseteq \check{t}_0$,
5) $p_1' \Vdash \dot{\tau}_1 \sqsupseteq \check{t}_1$.

Proof. There are two cases to consider. The first is that there exists some $x \in {}^{\omega}\omega$ such that 1) is true. When this happens, 2) is false. Hence, there exist $t_1 \in {}^{<\omega}\omega$ and $p_1' \leq p_1$ such that 5) is true and $x \not\sqsupseteq t_1$. Letting $p_0' := p_0$ and t_0 be some initial segment of x incompatible with t_1, we see that 3) and 4) are true.

The second case is that there is no $x \in {}^{\omega}\omega$ satisfying 1). When this happens, there exist conditions $p_0^a, p_0^b \leq p_0$ and incompatible nodes $s_a, s_b \in {}^{<\omega}\omega$ satisfying both $p_0^a \Vdash \dot{\tau}_0 \sqsupseteq \check{s}_a$ and $p_0^b \Vdash \dot{\tau}_0 \sqsupseteq \check{s}_b$. Now, it cannot be that both $p_1 \Vdash \dot{\tau}_1 \sqsupseteq \check{s}_a$ and $p_1 \Vdash \dot{\tau}_1 \sqsupseteq \check{s}_b$. Assume, without loss of generality, that $p_1 \nVdash \dot{\tau}_1 \sqsupseteq \check{s}_a$. This implies that there exist $p_1' \leq p_1$ and $t_1 \in {}^{<\omega}\omega$ such that $s_a \perp t_1$ and $p_1' \Vdash \dot{\tau}_1 \sqsupseteq \check{t}_1$. Letting $p_0' := p_0^a$ and $t_0 := s_a$, we are done. □

At this point, the reader may want to think about how to use this lemma to prove that if V is a Sacks forcing extension of a transitive model M of ZF and $\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n \in \omega \rangle$ satisfies

$$(\forall n \in \omega)\, x_n \notin M$$

and

$$\{(n_1, n_2) : x_{n_1} = x_{n_2}\} \in M,$$

then there is a sequence $\mathcal{T}$ of subtrees of ${}^{<\omega}\omega$ satisfying the hypotheses of Proposition 2.3.

The next lemma explains the appearance of I in the definition of a shrink wrapper. We are intending the name $\dot{\tau}$ to be such that $\dot{\tau}(n)$ refers to the x_n in the sequence $\mathcal{X} = \langle x_n : n \in \omega \rangle$.

Lemma 3.6. *Consider Sacks forcing* $\mathbb{S}$*. Let* $p \in \mathbb{S}$ *be a condition and* $\dot{\tau}$ *a name satisfying* $p \Vdash \dot{\tau} : \omega \to {}^\omega\omega$*. Then there exists a condition* $p' \leq p$ *and there exists a function* $I : \omega \to [{}^\omega\omega]^{<\omega}$ *satisfying*

$$p' \Vdash (\forall n \in \omega)\, \dot{\tau}(n) \in \check{V} \to \dot{\tau}(n) \in \check{I}(n).$$

Proof. We may easily construct a function $R : \omega \to \mathbb{S}$ that satisfies the conditions of Lemma 3.4 such that $R(\emptyset) \leq p$ and for each $s \in {}^n2$, either $R(s) \Vdash \dot{\tau}(n) \notin \check{V}$ or $(\exists x \in {}^\omega\omega)\, R(s) \Vdash \dot{\tau}(n) = \check{x}$. Define I as follows:

$$I(n) := \{x \in {}^\omega\omega : (\exists s \in {}^n2)\, R(s) \Vdash \dot{\tau}(n) = \check{x}\}.$$

Let $p' := \bigcap_n \bigcup\{R(s) : s \in {}^n2\}$. The condition p' and the function I are as desired. □

We are now ready for the main forcing argument of this section.

Theorem 3.7. *Consider Sacks forcing* $\mathbb{S}$*. Let* $p \in \mathbb{S}$ *be a condition and* $\dot{\tau}$ *be a name satisfying* $p \Vdash \dot{\tau} : \omega \to {}^\omega\omega$*. Then there exists a condition* $q \leq p$ *and there exists* $\mathcal{W} = \langle \mathcal{F}, I \rangle$ *satisfying*

$$q \Vdash \check{\mathcal{W}} \text{ is a shrink wrapper for } \langle \dot{\tau}(n) : n \in \omega \rangle.$$

Proof. First, let $p' \leq p$ and $I : \omega \to [{}^\omega\omega]^{<\omega}$ be given by the lemma above. That is, for each $n \in \omega$,

$$p' \Vdash \dot{\tau}(\check{n}) \in \check{V} \to \dot{\tau}(\check{n}) \in \check{I}(\check{n}).$$

We will define a function $R : {}^{<\omega}2 \to \mathbb{S}$ with $R(\emptyset) \leq p'$ satisfying conditions 1) and 2) of Lemma 3.4. At the same time, we will construct a family of functions

$$\mathcal{F} = \langle F_{\tilde{n},n} : \tilde{n} \in \omega, n \in \eta(\tilde{n}) \rangle.$$

Let $\mathcal{W} = \langle \mathcal{F}, I \rangle$. Our q will be

$$q := \bigcap_{\tilde{n}} \bigcup_{s \in {}^{\tilde{n}}2} R(s).$$

The function $F_{\tilde{n},n}$ will return leafless subtrees of ${}^{<\omega}\omega$. Moreover, each tree $F_{\tilde{n},n}(s)$ will obey the function $f_{\Phi(s,n)}$. We will have it so for all $n \in \omega$ and all $\tilde{n}$ satisfying $n \in \eta(\tilde{n})$,

$$(\forall s \in {}^{\tilde{n}}2)\, R(s) \Vdash \dot{\tau}(\check{n}) \in [\check{F}_{\tilde{n},n}(\check{s})].$$

Thus, q will force that $\mathcal{W}$ satisfies conditions 1) and 2) of the definition of a shrink wrapper. To show that q forces condition 3) of that definition, it suffices to show that for all $\{n_1, n_2\} = \eta(\tilde{n})$ and all $s_1, s_2 \in {}^{\tilde{n}}2$, one of the following holds, where $T_1 := F_{\tilde{n},n_1}(s_1)$ and $T_2 := F_{\tilde{n},n_2}(s_2)$:

3a′) $T_1 = T_2$ and $(\forall s \in {}^{\tilde{n}}2)$,

$$R(s) \Vdash (\dot{\tau}(\check{n}_1) \in [\check{T}_1] \vee \dot{\tau}(\check{n}_2) \in [\check{T}_2]) \to \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2);$$

3b′) $(\exists x \in I(n_1) \cap I(n_2))\, [T_1] = [T_2] = \{x\}$;
3c′) $[T_1] \cap [T_2] = \emptyset$, and moreover $\mathrm{Stem}(T_1) \perp \mathrm{Stem}(T_2)$.

We will define the functions $F_{\tilde{n},n}$ and the conditions $R(s)$ for $s \in {}^{\tilde{n}}2$ by induction on $\tilde{n}$. Beginning at $\tilde{n} = 0$, let $\{n_1, n_2\} = \eta(0)$. We will define F_{0,n_1}, F_{0,n_2}, and $R(\emptyset) \leq p'$. If $p' \Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then let $R(\emptyset) := p'$ and define $F_{0,n_1}(\emptyset) = F_{0,n_2}(\emptyset) = T$, where $T \subseteq {}^{<\omega}\omega$ is a tree that obeys both $f_{\Phi(\emptyset,n_1)}$ and $f_{\Phi(\emptyset,n_2)}$, such that $p' \Vdash \dot{\tau}(\check{n}_1) \in [\check{T}]$. Such a T is guaranteed to exist because $\mathbb{S}$ has the Sacks property. This causes 3a′) to be satisfied. If $p' \not\Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then let $t_1, t_2 \in {}^{<\omega}\omega$ be incomparable nodes and let $R(\emptyset) \leq p'$ satisfy $R(\emptyset) \Vdash \dot{\tau}(\check{n}_1) \sqsupseteq \check{t}_1$ and $R(\emptyset) \Vdash \dot{\tau}(\check{n}_2) \sqsupseteq \check{t}_2$. Then we may define $F_{0,n_1}(\emptyset) = T_1$ and $F_{0,n_2}(\emptyset) = T_2$ where T_1 and T_2 are leafless trees that obey $f_{\Phi(\emptyset,n_1)}$ and $f_{\Phi(\emptyset,n_2)}$ respectively such that $\mathrm{Stem}(T_1) \sqsupseteq t_1$, $\mathrm{Stem}(T_2) \sqsupseteq t_2$, $R(\emptyset) \Vdash \dot{\tau}(\check{n}_1) \in [\check{T}_1]$, and $R(\emptyset) \Vdash \dot{\tau}(\check{n}_2) \in [\check{T}_2]$. This causes 3c′) to be satisfied.

We will now handle the successor step of the induction. Let $\{n_1, n_2\} = \eta(\tilde{n})$ for some $\tilde{n} > 0$. We will define $R(s)$ for each $s \in {}^{\tilde{n}}2$, and we will define both $F_{\tilde{n},n_1}$ and $F_{\tilde{n},n_2}$ assuming $R(s')$ has been defined for each $s' \in {}^{<\tilde{n}}2$. To keep the construction readable, we will start with initial values for the $R(s)$'s and the $F_{\tilde{n},n}$'s, and we will modify them as the construction progresses until we arrive at their final values. That is, we will say "replace $R(s)$ with a stronger condition..." and "shrink the tree $F_{\tilde{n},n}(s)$...". When we make these replacements, it is understood that still $R(s) \Vdash \dot{\tau}(\check{n}) \in [\check{F}_{\tilde{n},n}(\check{s})]$. The construction consists of 5 steps.

Step 1: First, for each $s \in {}^{(\tilde{n}-1)}2$, let $R(s^\frown 0)$ and $R(s^\frown 1)$ be arbitrary extensions of $R(s)$ such that $\mathrm{Stem}(R(s^\frown 0)) \perp \mathrm{Stem}(R(s^\frown 1))$. Also, for each $n \in \{n_1, n_2\}$ and $s \in {}^{\tilde{n}}2$, let $F_{\tilde{n},n}(s)$ be a leafless subtree of ${}^{<\omega}\omega$ that obeys $f_{\Phi(s,n)}$ and satisfies $R(s) \Vdash \dot{\tau}(\check{n}) \in [\check{F}_{\tilde{n},n}(\check{s})]$.

Step 2: For each $s \in {}^{\tilde{n}}2$ and $n \in \{n_1, n_2\}$, strengthen $R(s)$ so that either $R(s) \Vdash \dot{\tau}(\check{n}) \notin \check{V}$ or $(\exists x \in I(n))\, R(s) \Vdash \dot{\tau}(\check{n}) = \check{x}$. If the latter case holds, shrink $F_{\tilde{n},n}(s)$ so that it has only one path.

Step 3: For this step, fix $n \in \{n_1, n_2\}$. For each pair of distinct $s_1, s_2 \in {}^{\tilde{n}}2$, strengthen each $R(s_1)$ and $R(s_2)$ and shrink each $F_{\tilde{n},n}(s_1)$ and $F_{\tilde{n},n}(s_2)$ so that one of the following holds:

i) $(\exists x \in I(n))\, [F_{\tilde{n},n}(s_1)] = [F_{\tilde{n},n}(s_2)] = \{x\}$;
ii) $\mathrm{Stem}(F_{\tilde{n},n}(s_1)) \perp \mathrm{Stem}(F_{\tilde{n},n}(s_2))$.

That is, if i) cannot be satisfied, then we may use Lemma 3.5 to satisfy ii).

Step 4: For each pair of distinct $s_1, s_2 \in {}^{\tilde{n}}2$ such that either $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \notin \check{V}$ or $R(s_2) \Vdash \dot{\tau}(\check{n}_2) \notin \check{V}$, use Lemma 3.5 to strengthen $R(s_1)$

and $R(s_2)$ and shrink $F_{\tilde{n},n_1}(s_1)$ and $F_{\tilde{n},n_1}(s_1)$ so that

$$\operatorname{Stem}(F_{\tilde{n},n_1}(s_1)) \perp \operatorname{Stem}(F_{\tilde{n},n_2}(s_2)).$$

Step 5: For each $s \in {}^{\tilde{n}}2$, do the following: If $R(s) \Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then replace both $F_{\tilde{n},n_1}(s)$ and $F_{\tilde{n},n_2}(s)$ with $F_{\tilde{n},n_1}(s) \cap F_{\tilde{n},n_2}(s)$. Otherwise, strengthen $R(s)$ and shrink $F_{\tilde{n},n_1}(s)$ and $F_{\tilde{n},n_2}(s)$ so that

$$\operatorname{Stem}(F_{\tilde{n},n_1}(s)) \perp \operatorname{Stem}(F_{\tilde{n},n_2}(s)).$$

This completes the construction of $\{R(s) : s \in {}^{\tilde{n}}2\}$, $F_{\tilde{n},n_1}$, and $F_{\tilde{n},n_2}$. We will now prove that it works. Fix $\tilde{n} \in \omega$ and $s_1, s_2 \in {}^{\tilde{n}}2$. Let $T_1 := F_{\tilde{n},n_1}(s_1)$ and $T_2 := F_{\tilde{n},n_2}(s_2)$. We must show that one of 3a′), 3b′), or 3c′) holds. The cleanest way to do this is to break into cases depending on whether or not $s_1 = s_2$.

Case $s_1 \neq s_2$: If either $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \notin \check{V}$ or $R(s_2) \Vdash \dot{\tau}(\check{n}_2) \notin \check{V}$, then by Step 4, we see that 3c′) holds. Otherwise, by Step 2, $(\exists x \in I(n_1))\,[T_1] = \{x\}$ and $(\exists x \in I(n_1))\,[T_2] = \{x\}$. Hence, either 3b′) or 3c′) holds.

Case $s_1 = s_2$: If $R(s_1) \not\Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then by Step 5, we see that 3c′) holds. Otherwise, we are in the case that

$$R(s_1) \Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2).$$

By Step 5, $T_1 = T_2$. Now, if $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \in \check{V}$, then of course also $R(s_1) \Vdash \dot{\tau}(\check{n}_2) \in \check{V}$, and by Step 2) we see that 3b′) holds. Otherwise, $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \notin \check{V}$. Hence, $[T_1]$ is not a singleton. We will show that 3a′) holds. Consider any $s \in {}^{\tilde{n}}2$. We must show

$$R(s) \Vdash (\dot{\tau}(\check{n}_1) \in [\check{T}_1] \vee \dot{\tau}(\check{n}_2) \in [\check{T}_1]) \rightarrow \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2).$$

If $s = s_1$, we are done. Now suppose $s \neq s_1$. It suffices to show

$$R(s) \Vdash \neg(\dot{\tau}(\check{n}_1) \in [\check{T}_1] \vee \dot{\tau}(\check{n}_2) \in [\check{T}_1]).$$

That is, it suffices to show $R(s) \Vdash \dot{\tau}(\check{n}_1) \notin [\check{T}_1]$ and $R(s) \Vdash \dot{\tau}(\check{n}_2) \notin [\check{T}_1]$. Since $s \neq s_1$ and $[T_1]$ is not a singleton, by Step 3, $\operatorname{Stem}(F_{\tilde{n},n}(s)) \perp \operatorname{Stem}(T_1)$. Recall that

$$R(s) \Vdash \dot{\tau}(\check{n}_1) \in [\check{F}_{\tilde{n},n}(\check{s})].$$

Hence, since $[\check{F}_{\tilde{n},n}(\check{s})] \cap [T_1] = \emptyset$, $R(s) \Vdash \dot{\tau}(\check{n}_1) \notin [\check{T}_1]$. By a similar argument, $R(s) \Vdash \dot{\tau}(\check{n}_2) \notin [\check{T}_1]$. This completes the proof. □

4. Silver Forcing

In this section, we will show that the shrink wrapping property does not hold between the ground model and any Silver forcing extension.

Definition 4.1. A tree $T \subseteq {}^{<\omega}2$ is a *Silver* tree iff it is leafless and the following are satisfied. There is an infinite set of levels $L \subseteq \omega$ such that for each $t \in T$, if $\mathrm{Dom}(t) \in L$, then both $t^\frown 0$ and $t^\frown 1$ are in T, and if $\mathrm{Dom}(t) \notin L$, then exactly one of $t^\frown 0$ or $t^\frown 1$ is in T. Also, if $x_1, x_2 \in [T]$ are two paths through T and $l \notin L$, then $x_1(l) = x_2(l)$. The poset of all Silver trees ordered by inclusion is called Silver forcing $\mathbb{V}$.

Fact 4.2. *Suppose G is $\mathbb{V}$-generic over V. Let $g = \bigcap G$. Then*

$$\{T \in \mathbb{V} : g \in [T]\} = G.$$

For this reason, we will sometimes say that g is $\mathbb{V}$-generic over V.

Definition 4.3. Let $p \subseteq {}^{<\omega}2$ be a tree. Let $t, s \in p$ be such that $\mathrm{Dom}(t) = \mathrm{Dom}(s)$. When we say "replace p below t with p below s", we mean replace p with

$$\{u \in p : u \not\supseteq t\} \cup \{t^\frown w \in p : s^\frown w \in p\}.$$

That is, the subtree of p below s is replacing the subtree of p below t.

In the following we will talk about elementary submodels of V, but we might as well be talking about elementary submodels of $V_\Theta \subseteq V$ for some large enough ordinal Θ.

Lemma 4.4. *Let M be a countable elementary submodel of V and let $p \in \mathbb{V}$ be in M. Then there is some $p' \leq p$ (not in M) such that each branch through p' is $\mathbb{V}$-generic over M.*

Proof. Let $\langle U_n : n \in \omega \rangle$ be an enumeration of the dense subsets of $\mathbb{V}$ that are in M. We will define a decreasing sequence of conditions $p = p_{-1} \geq p_0 \geq p_1 \geq \ldots$ in M. Now fix $n \geq 0$ and suppose we have defined this sequence for $p_{-1} \geq \ldots \geq p_{n-1}$. We will define p_n. Let $\langle t_n^i : i < 2^n \rangle$ be the nodes on the n-th splitting level of p_{n-1}. First shrink $p_{n-1}|t_n^0$ to be within U_n, calling the resulting condition p_{n-1}^0. This shrinking is possible because p_{n-1} is in M. Then for each $i \neq 0$, replace p_{n-1}^0 below t_n^i with p_{n-1}^0 below t_n^0. Call the resulting condition $\tilde{p}_{n-1}^0$. Then shrink $\tilde{p}_{n-1}^0|t_n^1$ to be within U_n, calling the resulting condition p_{n-1}^1. Then for each $i \neq 1$, replace p_{n-1}^1 below t_n^i with p_{n-1}^1 below t_n^1. Call the resulting condition $\tilde{p}_{n-1}^1$. Continue this for all $i < 2^n$. After all this shrinking, let $p_n := \tilde{p}_{n-1}^{2^n-1}$. Now unfix n. Note that $p_n \in \mathbb{V}$.

We have now constructed the sequence $p = p_{-1} \geq p_0 \geq p_1 \geq \ldots$ with the property that for each $n \in \omega$, each branch through p_n is a path through some element of U_n. Let $p' = \bigcap_{n \in \omega} p_n$. Then each branch through p' is a branch through an element of each U_n. Hence, each branch through p' is $\mathbb{V}$-generic over M. ☐

Theorem 4.5. *Consider Silver forcing $\mathbb{V}$. There is some $\dot{\mathcal{X}}$ such that there is no p and $\mathcal{W}$ such that $p \Vdash \check{\mathcal{W}}$ is a shrink wrapper for $\dot{\mathcal{X}}$.*

Proof. Given a function $r : \omega \to 2$ and $n \in \omega$, let $\text{Flatten}(r, n) : \omega \to 2$ be the function

$$\text{Flatten}(r,n)(i) := \begin{cases} 0 & \text{if } i \leq n, \\ r(i) & \text{otherwise.} \end{cases}$$

Let $\dot{r}$ be the canonical name for the generic real. We have $1 \Vdash \dot{r} : \omega \to 2$. Let $\vec{0} \in {}^{\omega}2$ be the constant zero function. Let $\langle \dot{x}_n \in {}^{\omega}2 : n \in \omega \rangle$ be a sequence of names such that for each $n \in \omega$,

$$1 \Vdash \dot{x}_{2n} = \begin{cases} \text{Flatten}(\dot{r}, n) & \text{if } \dot{r}(n) = 0, \\ \vec{0} & \text{if } \dot{r}(n) = 1, \end{cases}$$

and

$$1 \Vdash \dot{x}_{2n+1} = \begin{cases} \vec{0} & \text{if } \dot{r}(n) = 0, \\ \text{Flatten}(\dot{r}, n) & \text{if } \dot{r}(n) = 1. \end{cases}$$

That is, one of $\dot{x}_{2n}$ and $\dot{x}_{2n+1}$ will be a final segment of the generic real with initial zeros, and the other will the constant zero function. Define $\dot{\mathcal{X}}$ such that

$$1 \Vdash \dot{\mathcal{X}} = \langle \dot{x}_n : n \in \omega \rangle.$$

Suppose there is some condition p and some $\mathcal{W} = \langle \mathcal{F}, I \rangle$ such that

$$p \Vdash \check{\mathcal{W}} \text{ is a shrink wrapper for } \dot{\mathcal{X}}.$$

We will find a contradiction.

Let M be a countable elementary substructure of V such that $p, \mathcal{W}, \dot{\mathcal{X}} \in M$. By Lemma 4.4, let $p' \leq p$ be such that all branches through p' are $\mathbb{V}$-generic over M. Let $n := |\text{Stem}(p')|$. Let $\tilde{n} \in \omega$ be such that $\{2n, 2n+1\} = \eta(\tilde{n})$. That is, $\{2n, 2n+1\}$ is the $\tilde{n}$-th pair.

Let r_0 be the leftmost branch through $p'|(\text{Stem}(p')^\frown 0)$ and let r_1 be the leftmost branch through $p'|(\text{Stem}(p')^\frown 1)$. Hence, $r_0(l) = r_1(l)$ for all $l \neq n$. Let $u : \omega \to 2$ be such that

$$u = \text{Flatten}(r_0, n) = \text{Flatten}(r_1, n).$$

Note that $u \notin M$.

Now,

$$M \models p \Vdash \check{W} \text{ is a shrink wrapper for } \dot{\mathcal{X}}.$$

Given a name $\dot{\tau}$ and a generic filter G, let $\dot{\tau}_G$ refer to the valuation of $\dot{\tau}$ with respect to G. Since r_0 and r_1 are both paths through p', they are generic over M. Thus, we have

$$M[r_0] \models \mathcal{W} \text{ is a shrink wrapper for } \dot{\mathcal{X}}_{r_0}.$$

By part 2) of Definition 1.7, we have

$$M[r_0] \models (\exists s_1 \in {}^{\tilde{n}}2)(\dot{x}_{2n})_{r_0} \in [F_{\tilde{n},2n}(s_1)].$$

Fix this s_1. Let $T_1 := F_{\tilde{n},2n}(s_1)$. We have

$$(\dot{x}_{2n})_{r_0}^{M[r_0]} \in [T_1].$$

Similarly, we have

$$M[r_1] \models (\exists s_2 \in {}^{\tilde{n}}2)(\dot{x}_{2n+1})_{r_1} \in [F_{\tilde{n},2n+1}(s_2)].$$

Fix this s_2. Let $T_2 = F_{\tilde{n},2n+1}(s_2)$. We have

$$(\dot{x}_{2n+1})_{r_1}^{M[r_1]} \in [T_2].$$

Here is the crucial part: by the definition of r_0, r_1, and $\dot{\mathcal{X}}$, since $r_0(n) = 0$ and $r_1(n) = 1$, we have

$$(\dot{x}_{2n})_{r_0}^{M[r_0]} = \text{Flatten}(r_0, n) = u = \text{Flatten}(r_1, n) = (\dot{x}_{2n+1})_{r_1}^{M[r_1]}.$$

Thus, we have

$$[T_1] \cap [T_2] \neq \emptyset.$$

Since $\mathcal{W} \in M[r_0]$, by absoluteness we have

$$M[r_0] \models [T_1] \cap [T_2] \neq \emptyset.$$

Working in $M[r_0]$, by part 3) of Definition 1.7 applied to $C_1 := [T_1]$ and $C_2 := [T_2]$, it must be that either 3a) or 3b) holds. Note that

$$(\dot{x}_{2n})_{r_0}^{M[r_0]} = u \notin M,$$

which implies that 3b) cannot hold. Since $(\dot{x}_{2n})_{r_0}^{M[r_0]} \in [T_1]$, using 3a) we have that

$$M[r_0] \models (\dot{x}_{2n})_{r_0} = (\dot{x}_{2n+1})_{r_0}.$$

Thus,

$$(\dot{x}_{2n})_{r_0}^{M[r_0]} = (\dot{x}_{2n+1})_{r_0}^{M[r_0]}.$$

We already know that the left hand side of this equation is u. On the other hand, by definition, the right hand side must be $\vec{0}$. This is a contradiction. □

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