

THE n -RAMSEY PROPERTIES MAY BE DIFFERENT

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ABSTRACT. We give examples of n -Ramsey spaces that are not $(n+1)$ -Ramsey under several assumptions. We improve results from Kubis and Szeptycki by building such examples from $\mathfrak{b} = \mathfrak{c}$ and $\diamond(\mathfrak{b}) + \mathfrak{d} = \omega_1$. We also introduce a new splitting-like cardinal invariant and then show that the same holds under $\mathfrak{s} = \mathfrak{b}$.

1. INTRODUCTION

A widely studied topological property is that of sequential compactness. Recall that a space X is *sequentially compact* if for every sequence $\{x_n : n \in \omega\} \subseteq X$ has a convergent subsequence.

An important fact about this property is that it characterizes compactness in metric spaces, that is, a metric space is sequentially compact if and only if it is compact. Nevertheless, this fact is not true in general, since ω_1 is a sequentially compact non compact space and $\beta\omega$, the Stone-Cěch compactification of a countable discrete space, is compact but fails to be sequentially compact (see [54]).

In [37], Kubiś and Szeptycki introduced higher dimensional versions of sequential compactness and showed that these properties also hold in metric spaces. The 2-dimensional case had been previously introduced and proved to hold in compact metric spaces in [6] by Bojańczyk, Kopczyński and Toruńczyk. There the authors used this to obtain idempotents in compact semigroups.

A similar notion was studied by Helmut Knaust in [35] and [36], where the author defined Ramsey-uniformly convergence and a space to have the *Ramsey property* if any double sequence $\{x(i, j) : i, j \in \omega\}$ such that $\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} x(i, j) = x$ also converges Ramsey-uniformly to x . The notion of Ramsey-uniformly convergence coincides with the notion of convergence introduced in [37] in the 2-dimensional case. It was shown by Knaust that any pointwise compact subset of the Baire-1 functions on a Polish space has the Ramsey property [36], that any angelic space so that $\chi(x, A) < \mathfrak{p}$ for all countable relatively compact subsets A in X and all $x \in X$ has the Ramsey property and that so does all spaces of continuous functions $C(Y)$ with the topology of pointwise convergence where Y is a quasi Suslin space [35]. This result in the particular case of first countable spaces (and then for the real numbers with their usual topology) was previously showed by Rosenthal (see [48]).

We follow the convention that $[B]^n$ denotes the family of all subsets of B of cardinality n . Analogously, $[B]^\omega$ denotes the family of all of its countable infinite

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subsets and $[B]^{<\omega} = \bigcup_{n \in \omega} [B]^n$. Recall that if X is a topological space, $p \in X$, B is a countable set and $f : B \rightarrow X$, then we say that f *converges to* p if for all $U \subseteq X$ open such that $p \in U$, there exists a finite set $F \subseteq B$ such that $f''(B \setminus F) \subseteq U$. It is natural to generalize this definition to higher dimensions. Following [37], if $n \geq 1$ and $f : [B]^n \rightarrow X$, then f *converges to* p if for every neighborhood U of p there is a finite set $F \subseteq B$ such that $f''[B \setminus F]^n \subseteq U$ and in this case we say that the function is *f convergent*¹.

Definition 1. [37] *Let X be a topological space and $n \geq 1$. We say that X is an n -Ramsey space² if for every function $f : [\omega]^n \rightarrow X$ there exists $B \in [\omega]^\omega$ such that $f \upharpoonright [B]^n$ is convergent. We say that $f \upharpoonright [B]^n$ is a convergent subsequence of f .*

Observe that being 1-Ramsey is equivalent to being sequentially compact. It is also not hard to see that Ramsey's Theorem³ is equivalent to the assertion that every finite space X is n -Ramsey for all $n \geq 1$.

In [37], Kubiś and Szeptycki proved the following result:

Theorem 2. [37] *Every compact metric space is n -Ramsey for every $n \geq 1$.*

Thus we can not differentiate the class of n -Ramsey spaces from the class of m -Ramsey spaces with $n \neq m$ in the realm of metric spaces. So, the interesting question is whether we can distinguish these properties in the general category of topological spaces.

It is easy to see that every $(n + 1)$ -Ramsey space is also n -Ramsey (Proposition 2.2 in [37]) and Kubiś and Szeptycki proved the following:

Theorem 3. [37]

- (1) *There is a compact sequentially compact space (that is, 1-Ramsey) that is not 2-Ramsey.*
- (2) *Under CH, for every $n \in \omega$, there exists a n -Ramsey space that is not $(n + 1)$ -Ramsey.*

In this work we will improve item (2) of Theorem 3 by constructing n -Ramsey that are not $(n + 1)$ -Ramsey under several cardinal invariant assumptions. We also show that there is Fréchet example for item (1) answering a question posed by the authors in [37]. This example was sketched in that paper under the assumption of a completely separable mad family, which follows under, e.g., $\mathfrak{c} \leq \aleph_\omega$. We point out that the construction does not need a maximal almost disjoint family and since the existence of completely separable almost disjoint families follows from ZFC, an easy modification of the result in [37] can be carried out without extra assumptions. Although we will be quoting theorems from [37], it is not needed that the reader is familiar with it, since we will be recalling the needed notions and results.

The theory of n -Ramsey spaces is just in its beginnings and we expect it to be richer and more interesting than the one of sequentially compact spaces. This is our humble contribution to this new theory.

¹Remind that $f''A = \{f(x) : x \in A\}$.

²This notion must not be confused with the notion of Ramsey spaces defined and studied by Todorćević in [51].

³Ramsey's theorem states that if $n, m \geq 1$ and $f : [\mathbb{N}]^n \rightarrow m$, then there exists $Y \subseteq \mathbb{N}$ infinite such that $|f''[Y]^n| = 1$.

2. PRELIMINARIES

All spaces considered here are Hausdorff. Denote by ω the first infinite ordinal, by ω_1 the first uncountable ordinal and by \mathfrak{c} the cardinality of \mathbb{R} . Given two infinite sets $A, B \subseteq \omega$, we write $A \subseteq^* B$ if $A \setminus B$ is finite. Two sets A, B are *almost disjoint* if $A \cap B$ is finite. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint (ad)* if its elements are pairwise almost disjoint and it is *maximal almost disjoint (mad)* if it is ad and is not properly contained in any other almost disjoint family.

We say that a set $S \in [\omega]^\omega$ *splits* $A \subseteq \omega$ if $|S \cap A| = \omega = |A \setminus S|$. A family $\mathcal{S} \subseteq [\omega]^\omega$ is a *splitting family* if for every $X \in \omega$ there exists $S \in \mathcal{S}$ such that S splits X .

The set of functions from X to Y will be denoted by X^Y . For two functions $f, g \in \omega^\omega$, we write $f \leq^* g$ if $\{n \in \omega : f(n) > g(n)\}$ is finite. A family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if for every $f \in \omega^\omega$ there exists $b \in \mathcal{B}$ such that $b \not\leq^* f$ and \mathcal{B} is *dominating* if for every $f \in \omega^\omega$ there exists $d \in \mathcal{B}$ such that $f \leq^* d$. We also say that \mathcal{B} is a *scale* if \mathcal{B} is a dominating family and it is well ordered by \leq^* .

We define below the cardinal invariants used in this work for the ease of the reader,

- \mathfrak{a} is the minimum size of a mad family,
- \mathfrak{s} is the minimum size of a splitting family,
- \mathfrak{b} is the minimum size of an unbounded family,
- \mathfrak{d} is the minimum size of a dominating family.

Each of these cardinals is uncountable and smaller or equal to \mathfrak{c} . It is also well known that $\mathfrak{b} \leq \mathfrak{d}$, $\mathfrak{b} \leq \mathfrak{a}$ and $\mathfrak{s} \leq \mathfrak{d}$ and there are no other provable relations among them in ZFC. It is worth noting that scales exist if and only if $\mathfrak{b} = \mathfrak{d}$. A nice survey about cardinal characteristics and their properties is [5].

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an *ideal* if it is closed under finite unions and subsets, i.e, if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$ and if $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$. An ideal \mathcal{I} is *tall* if for every $X \in [\omega]^\omega$ there exists $I \in \mathcal{I}$ such that $|X \cap I| = \omega$. Given a tall ideal \mathcal{I} , the cardinal $\text{cov}^*(\mathcal{I})$ is defined as follows:

$$\text{cov}^*(\mathcal{I}) = \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{I} (\forall A \in [\omega]^\omega \exists B \in \mathcal{H} (|A \cap B| = \omega))\}.$$

We can also define $\text{cof}(\mathcal{I})$, the cofinality of an ideal \mathcal{I} , as the minimum size of a subfamily of \mathcal{I} that generates it, that is:

$$\text{cof}(\mathcal{I}) = \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \wedge \forall I \in \mathcal{I} \exists J \in \mathcal{J} (I \subseteq J)\}.$$

For more on ideals on countable sets and their cardinal invariants the reader may consult [30], [26], [32], [44] and [45].

Now, we summarize some definition and results of almost disjoint families that will be helpful in our constructions.

Definition 4. *If \mathcal{A} is an almost disjoint family, we denote the ideal generated by \mathcal{A} as $\mathcal{I}(\mathcal{A})$. We always assume that our ideals contain the finite sets, then $\mathcal{I}(\mathcal{A}) = \langle \mathcal{A} \cup [\omega]^{<\omega} \rangle$.*

Here, $\langle \mathcal{B} \rangle$ denotes the ideal generated by \mathcal{B} . Clearly an element of $\mathcal{I}(\mathcal{A})$ is a set that can be almost covered by a finite subfamily of \mathcal{A} .

Definition 5. *For an almost disjoint family \mathcal{A} we define $\mathcal{I}^{++}(\mathcal{A}) := \{X \subseteq \omega \mid \{A \in \mathcal{A} \mid |X \cap A| = \omega\} \geq \omega\}$.*

Note that $\mathcal{I}^{++}(\mathcal{A}) \subseteq \mathcal{I}^+(\mathcal{A})$ and \mathcal{A} is mad if and only if, $\mathcal{I}^{++}(\mathcal{A}) = \mathcal{I}^+(\mathcal{A})$.

If \mathcal{A} is an almost disjoint family on a countable set N , then its *Isbell-Mrówka space*, $\Psi(\mathcal{A})$, is $N \cup \mathcal{A}$, where N is the set of isolated points and the basic neighborhoods of each $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus F$ for $F \in [N]^{<\omega}$. The space $\Psi(\mathcal{A})$ has interesting properties, it is Hausdorff, zero-dimensional, separable and locally compact. The Franklin space of \mathcal{A} is then the one point compactification of $\Psi(\mathcal{A})$, i.e, $\mathcal{F}(\mathcal{A}) = \Psi(\mathcal{A}) \cup \{\infty\}$. The basic neighborhoods of ∞ are of the form:

$$\{\infty\} \cup (\mathcal{A} \setminus \{A_0, \dots, A_n\}) \cup \left(N \setminus \bigcup_{m \leq n} (A_m \cup F) \right),$$

where $n \in \omega$, $A_0, \dots, A_n \in \mathcal{A}$ and $F \in [N]^{<\omega}$. To learn more on Isbell-Mrówka spaces the reader may consult the survey [24] or the articles [25], [31], [47], [18], [16] and [9].

Since their early beginnings the Mrówka-Isbell spaces $\Psi(\mathcal{A})$ have turned out to be a fruitful source of examples and counterexamples in topology, since combinatorial properties of \mathcal{A} translate in topological properties of $\Psi(\mathcal{A})$. Plenty of applications of Mrówka-Isbell spaces to topology and functional analysis can be found in the literature. However, constructing mad families with special combinatorial properties is usually a difficult task without extra assumptions beyond ZFC. An example of ad families with special combinatorial properties are the completely separable ones.

Definition 6. *An almost disjoint family \mathcal{A} is completely separable if for all $X \in \mathcal{I}^{++}(\mathcal{A})$ there exists $A \in \mathcal{A}$ such that $A \subseteq X$.*

These families were introduced by Hechler in [23] and in [11]. Erdős and Shelah asked if there exist completely separable families that are mad. Most of the early work on this question was done by Balcar and Simon in [2], who proved that completely separable mad families exist under many assumptions. The next major advance on this was done by Shelah in [49], who introduced a technique that allowed him to prove that there are completely separable families under $\mathfrak{s} < \mathfrak{a}$ and also under $\mathfrak{s} \geq \mathfrak{a} +$ a certain ‘‘PCF hypothesis’’. The last big progress in the search of these families was done by Mildenerger, Raghavan and Steprāns in [40] and [46] who improved and extended the Shelah’s technique in order to get the existence of a completely separable mad family under $\mathfrak{s} \leq \mathfrak{a}$. This technique has turned out to be very powerful not only in the construction of completely separable mad families but also for other special mad families, e.g, weakly tight mad families (under $\mathfrak{s} \leq \mathfrak{b}$) and +-Ramsey mad families (see [46] and [20] respectively). This technique will play a crucial role in Section 8. The interested reader may consult [27] and [19].

Finally, Simon proved the following remarkable theorem:

Theorem 7. [15] *There is a completely separable nowhere mad family.*

So we can find a completely separable almost disjoint family in ZFC if we do not demand maximality on it. As we will see in the Section 4, this family will lead to a sequentially compact space not 2-Ramsey. The recursive construction can be carried on thanks to the following result:

Lemma 8. *See [15] and [19]. If \mathcal{A} is a completely separable almost disjoint family and $X \in \mathcal{I}^{++}(\mathcal{A})$, then the set $\{A \in \mathcal{A} \mid A \subseteq X\}$ has cardinality \mathfrak{c} .*

Recall that a space X is Fréchet if whenever $x \in \overline{A}$, with $A \subseteq X$, there exists $\{x_n : n \in \omega\} \subseteq A$ that converges to x . Obviously, every first countable space is

Fréchet, but there are many Fréchet spaces that are not first countable. An almost disjoint family \mathcal{A} is nowhere mad if for every $X \in \mathcal{I}^+(\mathcal{A})$ there exists $B \in [X]^\omega$ that is almost disjoint with every element in \mathcal{A} .

Theorem 9. [27] *Let \mathcal{A} be an almost disjoint family, then \mathcal{A} is nowhere mad if and only if $\mathcal{F}(\mathcal{A})$ is a Fréchet space.*

We now remind some important results concerning the n -Ramsey property proved in [37].

Proposition 10. [37] *The Franklin space of a mad family is not 2-Ramsey.*

Let \mathcal{A} be an almost disjoint family on a countable set N , $f : [\omega]^n \rightarrow \mathcal{F}(\mathcal{A})$ and suppose we want to see if we can find a convergent subsequence of f . By passing to an infinite subset of ω if needed (this can be done by applying Ramsey's Theorem), we may assume that one of the following cases holds:

- (1) f is constant ∞ ,
- (2) $f''[\omega]^n \subseteq \mathcal{A}$,
- (3) $f''[\omega]^n \subseteq N$.

The following lemma tells us that we only need to worry in the last case:

Lemma 11. [37] *If \mathcal{A} is an almost disjoint family on a countable set N of isolated points and $n \geq 1$, $\mathcal{F}(\mathcal{A})$ is n -Ramsey if and only if for all $f : [\omega]^n \rightarrow N$, there exists $B \in [\omega]^\omega$ such that $f \upharpoonright [B]^n$ converges.*

Recall that the *character* of $x \in X$ is the minimum cardinality of a local base of x and it is denoted by $\chi(x, X)$. The character of X is $\chi(X) = \sup\{\chi(x, X) : x \in X\}$.

Theorem 12. [37] *Every sequentially compact space with character less than \mathfrak{b} is n -Ramsey for all $n \geq 1$.*

In Theorem 57 we will see that \mathfrak{b} is optimal for the Theorem above. A consequence of Theorem 12 and Lemma 11 is:

Corollary 13. [37] *Let \mathcal{A} be an almost disjoint family on a countable set N , such that for every $f : [\omega]^n \rightarrow N$ there exists $X \in [\omega]^\omega$ so that $f''[X]^n$ has closure of size less than \mathfrak{b} in $\Psi(\mathcal{A})$. Then $\Psi(\mathcal{A})$ is n -Ramsey.*

We also get:

Corollary 14. [37] *Let \mathcal{A} be an almost disjoint family on a countable set N of isolated points and $n \geq 1$. If for all $f : [\omega]^n \rightarrow N$, there exists $B \in [\omega]^\omega$ such that*

$$\{A \in \mathcal{A} \mid A \cap f \upharpoonright [B]^n\}$$

has size less than \mathfrak{b} , then $\mathcal{F}(\mathcal{A})$ is n -Ramsey.

One of the key features of the counterexamples given in [37] is the structure of the FIN^n ideal on ω^n . These ideals were introduced by Katetov in [34]. Recall that $\text{Fin} = [\omega]^{<\omega}$ denotes the ideal of finite sets of ω . The ideal FIN^n is defined recursively as the Fubini product of $\text{FIN} \otimes \text{FIN}^{n-1}$, where $\text{FIN} = [\omega]^{<\omega}$. That is, $X \in \text{FIN}^n$ if

$$\{x \in \omega^{n-1} : |\{n \in \omega : x \hat{\ } n \in X\}| = \omega\} \in \text{FIN}^{n-1}.$$

To learn more about these ideals, the reader may consult [3], [22], [38] and [39]. The following combinatorial lemma will be very useful for us in the future.

Lemma 15. [37] *For every $f : [\omega]^n \rightarrow \omega^{n+1}$, there exists $X \in [\omega]^\omega$ such that $f''[X]^n \in \text{FIN}^{n+1}$.*

To deal with these kind of functions, we will also need an useful tool known as the canonical Ramsey theorem. A very deep and powerful fact due to Erdős and Rado is that there is a finite, very easy to describe "base" for the equivalence relations on $[\omega]^n$.

For the rest of this paper, given $a \in [\omega]^n$, we write it as $\{a_i : i < n\}$ where $a_0 < \dots < a_{n-1}$. If $I \subseteq n$, let $a \upharpoonright I := \{a_i : i \in I\}$. For every $I \subseteq n$, we can define the equivalence relation E_I on $[\omega]^n$ given as follows:

$$aE_I b \iff a \upharpoonright I = b \upharpoonright I.$$

Note that:

- $aE_\emptyset b$ for every $a, b \in [\omega]^n$ and
- $aE_n b$ if and only if $a = b$.

Of course not every equivalence relation is of this form. However, the surprising fact is that every equivalence relation is "locally" one of the ones we just described. Formally, we have the following:

Theorem 16. [12] *For every equivalence relation E on $[\omega]^k$ there are $M \in [\omega]^\omega$ and $I \subseteq k$ such that aEb if and only if $a \upharpoonright I = b \upharpoonright I$.*

The reader can learn more in this topic from Todorćević's book [51].

Applying Theorem 16 several times, we get the following:

Corollary 17. *Let E_1, \dots, E_l equivalence relations on $[\omega]^n$. There exist $M \in [\omega]^\omega$ and $I_1, \dots, I_l \subseteq n$ such that for all $a, b \in [M]^n$ and $i \leq l$, we have that*

$$aE_i b \iff a \upharpoonright I_i = b \upharpoonright I_i.$$

3. EXAMPLES OF n -RAMSEY SPACES

In this section we will give some examples of n -Ramsey spaces. By Theorem 12, we know that there are sequentially compact spaces that fail to even satisfy the 2-Ramsey property and that, in general, the properties n -Ramsey and $(n+1)$ -Ramsey may be consistently different. To see how subtle is to stratify the classes of n -Ramsey spaces, we will analyze some classical examples of sequentially compact non first countable spaces. These are necessary assumptions due to the following results:

Proposition 18. [37]

- (1) *If $n \geq 1$ and X is $(n+1)$ -Ramsey, then X is n -Ramsey. In particular if X is n -Ramsey for some $n \geq 1$, then X is sequentially compact.*
- (2) *If X is sequentially compact and first countable, then X is n -Ramsey for all $n \geq 1$.*

The following five examples are all the sequentially compact, non first countable spaces that appear in the book [50] (which can also be conveniently accessed in the website [43]):

- (1) The Fort space.
- (2) The closed long ray.
- (3) The altered long ray.
- (4) The Alexandroff square.

(5) $\omega_1 + 1$.

We will describe them in detail below and see that indeed all of them are n -Ramsey for every $n \geq 1$. Note that as an easy consequence of Proposition 18 item (2) we have that:

Corollary 19. [37] *Let X be sequentially compact space in which the clousure of every countable set is first countable. Then X is n -Ramsey for all $n \geq 1$.*

From the previous corollary, we have that the one-point compactification of every discrete space is n -Ramsey for all $n \geq 1$, in particular, the *Fort space*, that is, the one-point compactification of the discrete space of size \mathfrak{c} , is n -Ramsey for all $n \geq 1$.

Example 20. (Closed long ray) *Consider $\mathbb{L} = \omega_1 \times [0, 1)$ with the lexicographic order and $\tilde{\mathbb{L}} = \mathbb{L} \cup \{\infty\}$ where ∞ is the maximum point. Then the closed long ray $\tilde{\mathbb{L}}$ with the order topology is n -Ramsey for all $n \geq 1$.*

Proof. Let $f : [\omega]^n \rightarrow \tilde{\mathbb{L}}$, applying Ramsey's theorem we know that there exist $B \in [\omega]^\omega$ such that either, $f''[B]^n \subseteq \mathbb{L}$ or $f''[B]^n = \{\infty\}$. In later case we are done and in the former note that $f''[B]^n \subseteq \mathbb{L}$ and \mathbb{L} is first countable and sequentially compact so we can apply Corollary 19. \square

Note that $\tilde{\mathbb{L}}$ in Example 20 is sequentially compact but it is not first countable, since $\infty \in \tilde{\mathbb{L}}$ has no countable base.

Example 21. (Altered long ray) *Consider $\hat{\mathbb{L}} = \omega_1 \times [0, 1) \cup \{\infty\}$ where $\omega_1 \times [0, 1)$ has the lexicographic order and the basic neighborhoods of ∞ are of the form:*

$$N(\infty, \beta) = \{\infty\} \cup \bigcup_{\alpha \geq \beta} (\{\alpha\} \times (0, 1)) = \{\infty\} \cup ((\omega_1 \setminus \beta) \times (0, 1)).$$

Proof. Let $f : [\omega]^n \rightarrow \hat{\mathbb{L}}$, applying Ramsey's theorem we know that there exist $B \in [\omega]^\omega$ such that either, $f''[B]^n \subseteq \mathbb{L}$ or $f''[B]^n = \{\infty\}$. But note that the topology of \mathbb{L} as a subspace of $\tilde{\mathbb{L}}$ coincides with the topology of \mathbb{L} as a subspace of $\hat{\mathbb{L}}$. \square

Let us denote by Δ the set of points of the form (x, x) for $x \in [0, 1]$. If $x \in [0, 1]$ then denote by V_x the vertical line given by x , i.e, $V_x = \{(x, y) \mid y \in [0, 1]\}$.

The following example illustrates the usual way in which we will be applying Theorem 16 to derive n -Ramsey properties of certain spaces.

Example 22. (Alexandroff square) *Let $\mathbb{A} = [0, 1] \times [0, 1]$ with the following topology: if $p = (x, x) \in \Delta$ then a basic neighborhood of p is of the form:*

$$N(p, x_0, \dots, x_{n-1}, \epsilon) = ([0, 1] \times (x - \epsilon, x + \epsilon)) \setminus \left(\bigcup_{i \in n} V_{x_i} \right),$$

where $n \in \omega$, $\epsilon > 0$ and $x \notin \{x_i \mid i \in n\}$. If $p = (x, y) \notin \Delta$ then a basic neighborhood of p is of the form $N(p, \epsilon) = \{(x, t) \mid |y - t| < \epsilon\}$. Then \mathbb{A} is n -Ramsey for all $n \geq 1$.

Proof. Let $f : [\omega]^n \rightarrow \mathbb{A}$. Now for every $i \in 2$ let $f_i : [\omega]^n \rightarrow [0, 1]$ such that for all $s \in [\omega]^n$ we have that $f(s) = (f_0(s), f_1(s))$. Note that:

- (i) If there is $B \in [\omega]^\omega$ such that $f_0 \upharpoonright [B]^n$ is constant, then $f \upharpoonright [B]^n$ has a convergent subsequence⁴.

Applying Theorem 16 to the function f_0 , we can get $M \in [\omega]^\omega$ and $I \subseteq n$ such that for all $s, t \in [M]^n$, $f_0(s) = f_0(t)$ if and only if $s \upharpoonright I = t \upharpoonright I$.

If $I = \emptyset$ then we are done by (i). So assume that $I \neq \emptyset$ and let us consider $f_1 \upharpoonright [M]^n$. As $[0, 1]$ is n -Ramsey (with the usual topology), there exists $N \in [M]^\omega$ and $x \in [0, 1]$ such that $f_1 \upharpoonright [N]^n$ converges to x . We claim that $f \upharpoonright [N]^n$ converges to $p = (x, x)$. So let $N(p, x_0, \dots, x_{n-1}, \epsilon)$ be a basic neighborhood of p . As $f_1 \upharpoonright [N]^n$ converges to x , then there exists $F \in [N]^{<\omega}$ such that $f_1''[N \setminus F]^n \subseteq (x - \epsilon, x + \epsilon)$.

Now, as $I \neq \emptyset$, we claim that there is $G \in [N]^{<\omega}$ such that $f_0''[N \setminus G]^n \cap \{x_i \mid i \in n\} = \emptyset$. Indeed, for every $i \in n$ there is $s \in [\omega]^{<\omega}$ not empty such that $f_0(t) = x_i$ if and only if $t \upharpoonright I = s_i$. Now, $G = \bigcup_{i \in n} s_i$ is as desired.

This shows that $f''[N \setminus (F \cup G)]^n \subseteq N(p, x_0, \dots, x_{n-1}, \epsilon)$ and then $f \upharpoonright [N]^n$ converges to p . \square

Finally, we will prove that in the realm of limit ordinals, being sequentially compact, having uncountable cofinality and being n -Ramsey for all $n \geq 1$ are equivalent. We will first need the following:

Proposition 23. *If γ is an ordinal and $X \subseteq \gamma$ is countable, then \overline{X} is countable.*

Proof. Suppose that \overline{X} is not countable and let Y of size ω_1 such that $X \subseteq Y \subseteq \overline{X}$. As Y is a set of ordinals, by going to a subset if necessary, we can enumerate Y in order type ω_1 , i.e., $Y = \{x_\alpha \mid \alpha \in \omega_1\}$ where $x_\alpha < x_\beta$ if $\alpha < \beta$. Note that as X is countable, there exists $\alpha \in \omega_1$ such that $\{x_\beta \mid \alpha < \beta < \omega_1\} \cap X = \emptyset$, so in particular $\{x_\alpha, x_\beta\} \cap X = \emptyset$ for any $\beta > \alpha$, which shows that $x_\beta \notin \overline{X}$, a contradiction. \square

Note that by the last proposition, if γ is a limit ordinal with uncountable cofinality and $X \subseteq \gamma$ is countable, then the closure of X in γ is sequentially compact and first countable, so by Corollary 19 we have that:

Corollary 24. *Let γ be a limit ordinal. Then $\text{cof}(\gamma) > \omega$ if and only if γ with the order topology is n -Ramsey for all $n \geq 1$.*

Finally, we get a full characterization of those ordinals that are n -Ramsey for all $n \in \omega$.

Corollary 25. *An ordinal γ is n -Ramsey for all $n \geq 1$ if and only if $\text{cof}(\gamma) \neq \omega$.*

Proof. Let $f : [\omega]^n \rightarrow \gamma$ and assume that γ has uncountable cofinality. Let $X = f''[\omega]^n$ and define $\alpha = \sup(X)$. Since γ does not have countable cofinality, $\alpha \in \gamma$. It follows then that $\alpha = \max(\overline{X})$. Take \overline{X} with the inherit order, since the maximum of \overline{X} exists, $\beta = \text{ot}(\overline{X})$ is a countable successor ordinal. Therefore it is first countable and sequentially compact, which implies that β is n -Ramsey by Proposition 18 item (2). In consequence, f has a convergent subsequence in $\overline{X} \subseteq \gamma$. \square

With this we now know that all sequentially compact examples in [50] (and only those) are n -Ramsey for all $n \geq 1$.

⁴In fact, more can be said here: if there is $p \in \mathbb{A} \setminus \Delta$ and $B \in [\omega]^\omega$ such that $f \upharpoonright [B]^n$ converges to p , then f_0 is constant on $[B]^n$.

4. SEQUENTIALLY COMPACT FRÉCHET SPACES THAT ARE NOT 2-RAMSEY

It is well known that in the case of Franklin spaces sequentially compact does not imply being Fréchet, however, Paul Szeptycki pointed out that a 2-Ramsey Franklin space must be Fréchet.

Proposition 26. *If \mathcal{A} is an almost disjoint family such that $\mathcal{F}(\mathcal{A})$ is 2-Ramsey, then $\mathcal{F}(\mathcal{A})$ is Fréchet.*

Proof. Suppose that $\mathcal{F}(\mathcal{A})$ is not Fréchet, this means, by Theorem 9, that \mathcal{A} is somewhere mad, i.e, there exists $X \in \mathcal{I}^+(\mathcal{A})$ such that $\mathcal{A} \upharpoonright X := \{A \cap X \mid A \in \mathcal{A}(|A \cap X| = \omega)\}$ is an infinite mad family on X and then $\mathcal{F}(\mathcal{A} \upharpoonright X)$ is not 2-Ramsey by Proposition 10.

Now let $Y := X \cup \{A \mid A \in \mathcal{A} \upharpoonright X\} \cup \{\infty\}$ with the topology inherited by $\mathcal{F}(\mathcal{A})$. It is easy to see that Y is a closed subset of $\mathcal{F}(\mathcal{A})$ and it is homeomorphic to $\mathcal{F}(\mathcal{A} \upharpoonright X)$. As $\mathcal{F}(\mathcal{A})$ is 2-Ramsey and $Y \subseteq \mathcal{F}(\mathcal{A})$ is closed, Y is 2-Ramsey and consequently $\mathcal{F}(\mathcal{A} \upharpoonright X)$ is 2-Ramsey, which is a contradiction. \square

In [37], the authors asked whether there is a ZFC example of a Fréchet sequentially compact space that is not 2-Ramsey. They showed that a completely separable mad family suffices for the construction of such example. As discussed earlier, the existence of a completely separable mad family holds under very weak assumptions like $\mathfrak{c} < \aleph_\omega$, however, as we stated in Theorem 7, completely separable almost disjoint families, possibly not maximal, do exist in ZFC. The existence of one of these families in ZFC is good enough for the construction sketched in [37], we give a full proof below.

If N is countable set, $\mathcal{A} \subseteq \mathcal{P}(N)$ and $X \in [N]^\omega$, then $X \perp \mathcal{A}$ means that $X \cap A$ is finite for all $A \in \mathcal{A}$. The family $\{X \in [N]^\omega \mid X \perp \mathcal{A}\}$ is denoted by \mathcal{A}^\perp .

Lemma 27. *Let \mathcal{A} an almost disjoint family on a countable set N of isolated points. If there exists $\mathcal{A}_0 = \{A_m \mid m \in \omega\} \subseteq [\omega]^\omega$ and $f : [\omega]^n \rightarrow N$ such that:*

- (i) *Either, $\mathcal{A}_0 \subseteq \mathcal{A}$ or $\mathcal{A}_0 \subseteq \mathcal{A}^\perp$.*
- (ii) *\mathcal{A}_0 is a pairwise disjoint family.*
- (iii) *f is one-to-one.*
- (iv) *If $\min(s) = m$ then $f(s) \in A_m$.*
- (v) *For all $B \in [\omega]^\omega$ there exists $A \in \mathcal{A} \setminus \mathcal{A}_0$ such that $|f''[B]^n \cap A| = \omega$.*

Then $\mathcal{F}(\mathcal{A})$ is not n -Ramsey.

Proof. We will show that $f \upharpoonright [B]^n$ does not converge to any point in $\mathcal{F}(\mathcal{A})$ for all $B \in [\omega]^\omega$. Fix $B \in [\omega]^\omega$, then as $f \upharpoonright [B]^n$ is one-to-one, it does not converge to any isolated point. Let $F \in [B]^{<\omega}$ and $m \in \omega$. There are $m' > m$ and $a \in [B \setminus F]^n$ such that $\min(a) = m'$. Thus $f(a) \in A_{m'}$. This shows that $f''[B \setminus F]^n \not\subseteq \{A_m\} \cup A_m$ for any F and thus $f \upharpoonright [B]^n$ does not converge to A_m in the case where $\mathcal{A}_0 \subseteq \mathcal{A}$.

Now let us see that $f \upharpoonright [B]^n$ does not converge to any point in $\mathcal{A} \setminus \mathcal{A}_0$ either. Let $F \in [B]^{<\omega}$ and fix $A \in \mathcal{A} \setminus \mathcal{A}_0$. Take $m = \min(B \setminus F)$. Thus by condition (iv), $f''[B \setminus F]^n \cap A_m$ is infinite and since $A \cap A_m$ is finite, it follows that $f''[B \setminus F]^n \not\subseteq \{A\} \cup A$.

The only thing that remains to be seen is that $f \upharpoonright [B]^n$ does not converge to ∞ . Let $A \in \mathcal{A} \setminus \mathcal{A}_0$ given by item (v). We will show that $f''[B \setminus F]^n \not\subseteq U := \mathcal{F}(\mathcal{A}) \setminus (\{A\} \cup A)$ for all $F \in [B]^{<\omega}$. Since U is a basic neighborhood of ∞ in $\mathcal{F}(\mathcal{A})$, this will suffice. Let $m = \max F$. Since $A \cap A_i$ is finite for every $i \leq m$ and $|f''[B]^n \cap A| = \omega$, there exists $a \in [B \setminus F]^n$ such that $f(a) \in A$ and we are done.

This shows that $\mathcal{F}(\mathcal{A})$ is not n -Ramsey as witnessed by f . \square

We can now prove the following, answering a question in [37].

Theorem 28. *There exists (in ZFC) a space which is Fréchet, sequentially compact and it is not 2-Ramsey.*

Proof. Let \mathcal{A} be an almost disjoint family on $\omega \times \omega$ which is completely separable, infinite and nowhere mad. We can assume that $A_n = \{n\} \times \omega \in \mathcal{A}$ for all $n \in \omega$. Now let $G : [\omega]^2 \rightarrow \mathcal{F}(\mathcal{A})$ given by $G(\{m, n\}) = (m, n)$, where $m < n$. We will show that \mathcal{A} , $\mathcal{A}_0 = \{A_n \mid n \in \omega\}$ and $f = G$ satisfy the hypothesis of Lemma 27 for $n = 2$.

Conditions (ii), (iii) and (iv) are clear. For (v) note that (iii) and (iv) imply that $|G''[B] \cap A_m| = \omega$ for every $m \in B$, in particular $G''[B]^2 \in \mathcal{I}^{++}(\mathcal{A})$. Since \mathcal{A} is completely separable, using Lemma 8, there exists $A \in \mathcal{A} \setminus \mathcal{A}_0$ as required in (v). \square

5. n -RAMSEY SPACES FROM $\mathfrak{b} = \mathfrak{c}$.

The relevance of the cardinals \mathfrak{b} and \mathfrak{d} in our examples is evident due to the next few facts regarding $\text{cov}^*(\text{FIN}^n)$ and $\text{cof}(\text{FIN}^n)$. We will first introduce the concept of Hechler trees in order to make the analysis of these results cleaner.

Given $n \in \omega$, we will say that $T \subseteq \omega^{\leq n+1}$ is a *Hechler tree*, if for every $s \in T \cap \omega^{\leq n}$, the set of successors of s is *coinitial* on ω , i.e., there exists $k_s \in \omega$ such that $\text{succ}_T(s) = \{i \in \omega \mid s \hat{\ } i \in T\} = \omega \setminus k_s$. We can naturally associate a function $h_T : T \cap \omega^{\leq n} \rightarrow \omega$ to each Hechler tree $T \subseteq \omega^{\leq n+1}$ by defining $h(s) = k_s$.

For $s, t \in \omega^{\leq n}$, we say that $s \prec t$ if either $s \sqsubset t$ or $s = r \hat{\ } l$, $t = r \hat{\ } m$ for some $r \in \omega^{\leq n}$ and $l < m$, where $s \sqsubset t$ means that s is a proper initial segment of t . Enumerate $\omega^{\leq n} = \{s_i \mid i \in \omega\}$ such that $s_i \prec s_j$ implies $i < j$. We fix this enumeration once and for all.

Definition 29. *Let $f \in \omega^\omega$ and let $T \subseteq \omega^{n+1}$ be a Hechler tree. We define*

- $T_f \subseteq \omega^{\leq n+1}$ where $\emptyset \in T_f$ and $\text{succ}(s_i) = \omega \setminus f(i)$ for every $s_i \in T_f \cap [\omega]^{\leq n}$.
- $f_T \in \omega^\omega$ where $f_T(n) = k$ if and only if $s_n \in T$ and $\text{succ}_T(s_n) = \omega \setminus k$ and $f_T(n) = 0$ otherwise.

Let $C_s = \{t \in \omega^{n+1} \mid s \sqsubseteq t\}$ for every $s \in \omega^{\leq n}$ and let us write C_k instead of $C_{\{k\}}$. Also, for a tree $T \subseteq \omega^{\leq n}$ we denote its set of branches by $[T] = T \cap \omega^n$. It follows directly from the definition that for every $X \in \text{FIN}^n$ there exists a Hechler tree $T \subseteq \omega^n$ such that $X \cap T = \emptyset$.

Lemma 30. *For every $n \geq 1$, if $\mathcal{H} \subseteq \text{FIN}^{n+1}$ is such that $|\mathcal{H}| < \mathfrak{b}$ and $B \in [\omega]^\omega$, then there exists a block sequence⁵ $A \subseteq B^{n+1}$ such that $|A \cap X| < \omega$ for every $X \in \mathcal{H}$.*

Proof. For every $X \in \mathcal{H}$ let T_X be a Hechler tree disjoint from X . We write f_X instead of f_{T_X} .

Since $|\mathcal{H}| < \mathfrak{b}$, there is a function $f \in \omega^\omega$ such that $f >^* f_X$ for all $X \in \mathcal{H}$. We will recursively define a block sequence $(a_m \mid m \in \omega) \subseteq \omega^{n+1}$. Let $(a_m^i : i \leq n)$

⁵Recall that $(a_m \mid m \in \omega) \subseteq \omega^n$ is a block sequence if $\max a_m < \min a_{m+1}$ for all $m \in \omega$ and each a_m is increasing.

denote the sequence a_m . Assume we have constructed a_k for every $k < m$. For a_m define $a_m^0 \in B \setminus (f(\emptyset) \cup \max(a_{m-1}))$ (where $a_{-1} = 0$). Now for every $i < n$ pick

$$a_m^{i+1} \in B \setminus f((a_m^0, \dots, a_m^i))$$

bigger than a_m^i . These choices are possible since B is infinite. This finishes the construction of $A = (a_m \mid m \in \omega) \subseteq B^{n+1}$.

To see that it works fix $X \in \mathcal{H}$. Since $f \succ^* f_X$ there exists $k \in \omega$ such that if $f_X(n) > f(n)$ then $s_n \in \bigcup_{i < k} C_i$. As $(a_m \mid m \in \omega)$ is a block sequence, $\{a_m^0 \mid m \in \omega\}$ is increasing and thus $a_m^0 > \max\{k, f_X(\emptyset)\}$ for all but finitely many $m \in \omega$. Given any a_m^0 as above, $a_m^0 > f_X(\emptyset)$ and as (a_m^0, \dots, a_m^i) extends a_m^0 in ω^{n+1} for any $i \leq n$, it follows from the assumption over k that $f((a_m^0, \dots, a_m^i)) > f_X((a_m^0, \dots, a_m^i))$. In particular $(a_m^0, \dots, a_m^n) \in T_X$ which implies that $X \cap A \subseteq \{a_m \mid a_m^0 < k\}$, with the latter set being finite. \square

The previous lemma gives in particular that $\mathfrak{b} \leq \text{cov}^*(\text{FIN}^{n+1})$ for every $n \geq 1$. The other inequality can also be proved using similar ideas.

Proposition 31. $\text{cov}^*(\text{FIN}^{n+1}) = \mathfrak{b}$ for every $n \geq 1$.

Proof. We only need to prove that $\text{cov}^*(\text{FIN}^{n+1}) \leq \mathfrak{b}$. Suppose that $\kappa < \text{cov}^*(\text{FIN}^{n+1})$ and let $\{h_\beta \in \omega^\omega \mid \beta \in \kappa\}$. Recall that \mathfrak{b} is the minimum size of a family $\mathcal{B} \subseteq \omega^\omega$ that is unbounded on each infinite subset of ω (see [52]), so it is enough to see $\{h_\beta \in \omega^\omega \mid \beta \in \kappa\}$ is bounded on some $B \in [\omega]^\omega$. Enumerate ω^n as $\{t_l \mid l \in \omega\}$ such that if there is $r \in \omega^{n-1}$ such that $t_l = r \hat{\ } i$ and $t_{l'} = r \hat{\ } j$ and $i < j$ then $l < l'$.

For each $\beta \in \kappa$ consider the set $Y_\beta = \{t_l \hat{\ } m \mid l \in \omega \wedge m \leq h_\beta(l)\}$. Clearly each Y_β is in FIN^{n+1} . Now consider $\mathcal{H} = \{Y_\beta \mid \beta \in \kappa\} \cup \{C_{t_l} \mid l \in \omega\}$. It is clear that $|\mathcal{H}| < \text{cov}^*(\text{FIN}^{n+1})$, so there exists $A \in [\omega^{n+1}]^\omega$ such that $A \cap C_{t_l} =^* \emptyset$ for all $l \in \omega$ and $A \cap Y_\beta =^* \emptyset$ for every $\beta \in \kappa$.

Now define $f : \omega \rightarrow \omega$ as follows:

$$f(l) = \begin{cases} \max\{i \in \omega \mid t_l \hat{\ } i \in A\} & \text{if } A \cap C_{t_l} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Let $B = \{l \in \omega : A \cap C_{t_l} \neq \emptyset\} \in [\omega]^\omega$. Now, as $A \cap Y_\beta =^* \emptyset$, there exists $l \in \omega$ such that if $m > l$ then $C_{t_m} \cap (A \cap Y_\beta) = \emptyset$, which means that $f(m) > h_\beta(m)$ whenever $m \in B \setminus (l+1)$, i.e. f dominates h_β on B . \square

Now we are ready to prove that under $\mathfrak{b} = \mathfrak{c}$ we can distinguish the classes of n -Ramsey and $(n+1)$ -Ramsey spaces for every $n \geq 1$.

Theorem 32. ($\mathfrak{b} = \mathfrak{c}$) For all $n \geq 2$ there exists a n -Ramsey space which is not $(n+1)$ -Ramsey.

Proof. We will construct an almost disjoint family \mathcal{A} on ω^{n+1} such that its Franklin space is n -Ramsey but not $(n+1)$ -Ramsey. Let $G : [\omega]^{n+1} \rightarrow \mathcal{F}(\mathcal{A})$ given by $G(\{m_0, \dots, m_n\}) = (m_0, \dots, m_n)$, where $m_0 < \dots < m_n$.

Let us enumerate $(\omega^{n+1})^{[\omega]^n} = \{f_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$ and $[\omega]^\omega = \{B_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$. We will construct $\{A_\alpha \mid \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega^{n+1})$ and $\{X_\alpha \mid \omega \leq \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ such that the following hold:

- (1) For all $n \in \omega$, $A_n = C_n$ and A_α is a block sequence for every $\alpha \geq \omega$ (hence $A_\alpha \in \text{FIN}^{n+1}$).
- (2) $\forall \beta, \alpha \in \mathfrak{c}$ ($\beta \neq \alpha \rightarrow A_\beta \cap A_\alpha =^* \emptyset$).

- (3) $f''_\alpha[X_\alpha]^n \in \text{FIN}^{n+1}$.
- (4) $\forall \beta, \alpha \in \mathfrak{c} (\omega \leq \beta < \alpha \rightarrow f''_\beta[X_\beta]^n \cap A_\alpha =^* \emptyset)$.
- (5) $\forall \alpha \in \mathfrak{c} (|A_\alpha \cap G''[B_\alpha]^{n+1}| = \omega)$.

Assume that we have already constructed $\{A_\beta \mid \beta < \alpha\}$ and $\{X_\beta \mid \beta < \alpha\}$ for some $\alpha \geq \omega$. We can apply Lemma 30 to the family \mathcal{H} defined as follows:

$$\mathcal{H} = \{f''_\beta[X_\beta]^n \mid \omega \leq \beta < \alpha\} \cup \{A_\beta \mid \beta < \alpha\}$$

and the infinite set B_α . In this way we get a block sequence $A_\alpha \subseteq B_\alpha^{n+1}$ such that:

- (a) $\forall \beta \in \alpha (|A_\beta \cap A_\alpha| < \omega)$.
- (b) $\forall \beta \in \alpha (\omega \leq \beta < \alpha \rightarrow |f''_\beta[X_\beta]^n \cap A_\alpha| < \omega)$.

This ensures that conditions (2) and (4) hold, also, as A_α is a block sequence of elements of B_α , condition (5) holds. Now, find $X_\alpha \in [\omega]^\omega$ such that $f''_\alpha[X_\alpha]^n \in \text{FIN}^{n+1}$ (and this one exists by Lemma 15). This finishes the construction.

Consider $\mathcal{A} = \{A_\alpha \mid \alpha \in \mathfrak{c}\}$. Condition (4) implies that $f''_\beta[X_\beta]^n$ has closure of size less than \mathfrak{c} in $\mathcal{F}(\mathcal{A})$ and, as $\mathfrak{b} = \mathfrak{c}$, it has closure of size less than \mathfrak{b} , so by Corollary 14, it has a convergent subsequence. On the other hand, condition (5) implies that condition (v) of Lemma 27 holds and the other three conditions when $f = G$ are clear, so $\mathcal{F}(\mathcal{A})$ is not $(n+1)$ -Ramsey. \square

6. SMALL n -RAMSEY SPACES.

Parametrized diamonds were defined by Dzamonja, Hrušák and Moore in [41]. These principles are weak versions of the well known Jensen's diamond principle \diamond but they have the advantage that are consistent with $\neg\text{CH}$. Moreover, many constructions that follow from \diamond can be carried with an appropriated parametrized diamond. These weak diamond principles can be defined from abstract cardinal invariants. Following [53], we say that triple (A, B, E) is an *invariant* if A and B are sets of cardinality at most continuum and E is a relation between A and B , such that every element in A is E -related with some element in B and for every $b \in B$ there is an $a \in A$ such that $(a, b) \notin E$.

The evaluation $\langle A, B, E \rangle$ of an invariant (A, B, E) is defined as follows:

$$\langle A, B, E \rangle = \min\{|X| : X \subseteq B \text{ and } (\forall a \in A \exists b \in B (aEb))\}.$$

An invariant (A, B, E) is called a *Borel invariant* [4] if A , B and E are Borel subsets on some Polish space. Given a Borel invariant (A, B, E) , its parametrized diamond principle is defined as follows:

$$\begin{aligned} \diamond(A, B, E) \equiv & \forall F : 2^{<\omega_1} \rightarrow A \text{ Borel } \exists g : \omega_1 \rightarrow B \forall f \in 2^{\omega_1} \\ & \{\alpha \in \omega_1 : F(f \upharpoonright \alpha)Eg(\alpha)\} \text{ is stationary.} \end{aligned}$$

In this context, $F : 2^{<\omega_1} \rightarrow A$ is *Borel*, if for every $\delta < \omega_1$ the restriction of F to 2^δ is a Borel map. There exists also a notion of parametrized diamond where the invariant and the function F above do not have the constraint of being Borel. This notion is denoted by $\Phi(A, B, E)$, nevertheless, most of the applications of these diamond principles have been shown to follow from its definable counterpart, and unlike them, the definable ones hold in many models in which the associated cardinal invariant is small.

For well known cardinal invariants, we write its evaluation instead of its triple. For example, $\diamond(\mathfrak{b}) \equiv \diamond(\omega^\omega, \omega^\omega, \geq^*)$. Some examples of the use of parametrized diamonds in topology are the following:

- (1) [41] $\diamond(\mathfrak{b}) \implies \mathfrak{a} = \omega_1$,
- (2) [41] $\diamond(\text{non}(\mathcal{M}))$ implies that there is a Souslin line,
- (3) [41] $\diamond(\mathfrak{s}^\omega)$ implies that there is a perfectly normal countably compact non-compact space,
- (4) [41] $\diamond(\mathbb{R}, \neq)$ implies that there are no Q -sets,
- (5) [41] $\diamond(\mathfrak{r})$ implies that there is a P -point of character ω_1 ,
- (6) [8] $\diamond(\mathfrak{r}_{\text{scat}})$ implies that there is a countable T_3 irresolvable space of weight ω_1 ,
- (7) [17] $\diamond(\mathfrak{s})$ implies that there is a family of sequentially compact spaces $\{X_\alpha : \alpha < \kappa\}$ such that $\prod_{\alpha < \kappa} X_\alpha$ is not countably compact,
- (8) [17] $\diamond(\mathfrak{b}^*)$ implies the existence of a separable Jakovlev space of size ω_1 ,
- (9) [17] $\diamond(\mathfrak{b}^*)$ implies that for every $\eta \in \omega_1$ there exists a separable compact sequential scattered space of sequential order $\eta + 1$,
- (10) [13] $\diamond(\text{hom}_H)$ implies the existence of stable ordered-union ultrafilters of character ω_1 .
- (11) [10] $\diamond(\mathfrak{b})$ implies that there is a Fréchet α_3 space defined from an almost disjoint family that is not bisequential,
- (12) [42] $\diamond(\omega, <)$ implies that every separable, locally compact space with property (a) has countable extent,
- (13) [14] $\diamond(\mathfrak{r}_P)$ implies the existence of gruff ultrafilters.
- (14) [29] $\diamond(2, =)$ implies that there exists a Fréchet and countable topological group that is not metrizable.

We will consider partial functions in ω^ω and they will be denoted by $f; \omega \rightarrow \omega$. Let $\mathbb{P} = \{f; \omega \rightarrow \omega : |\text{dom}(f)| = \omega\}$. Hence, we can look at the cardinal invariant defined by $(\mathbb{P}, \omega^\omega, \not\leq)$, (where we say that $f \not\leq g$ if $\{n \in \omega : n \in \text{dom}(f) \wedge f(n) < g(n)\}$ is infinite). It is known that its evaluation $\langle \mathbb{P}, \omega^\omega, \not\leq \rangle$ turns out to be \mathfrak{b} (see [52]). However, in order to prove that its parametrized \diamond -principles are also equivalents, we need to show that both invariants are *Borel Tukey equivalents*.

Definition 33. [4] *Given two Borel invariants (A, B, E) and (A', B', E') , we say that $(A, B, E) \leq_T^B (A', B', E')$ if there are Borel maps $\phi : A \rightarrow A'$ and $\psi : B \rightarrow B'$ such that $(\phi(a), \psi(b)) \in E'$ implies $(a, b) \in E$.*

We say that two invariants (A, B, E) and (A', B', E') are Borel Tukey equivalents (denoted by $(A, B, E) \equiv_T (A', B', E')$) if $(A, B, E) \geq_T^B (A', B', E')$ and $(A, B, E) \leq_T^B (A', B', E')$. The general notion of Tukey reductions not restricted to Borel sets (often denoted by \leq_T) was defined in [53]. Since we are only dealing with Borel invariants, we will use \leq_T and \equiv_T for the Borel versions.

The importance of Tukey relations between invariants in our context relies in the following result:

Proposition 34. [41] *If $(A, B, E) \leq_T^B (A', B', E')$, then $\diamond(A', B', E')$ implies $\diamond(A, B, E)$.*

We will prove that $\diamond(\mathfrak{b})$ is equivalent to $\diamond(\mathbb{P}, \omega^\omega, \not\leq)$ by showing that $(\omega^\omega, \omega^\omega, \not\leq^*) \equiv_T (\mathbb{P}, \omega^\omega, \not\leq)$.

Proposition 35. $\diamond(\mathfrak{b})$ is equivalent to $\diamond(\mathbb{P}, \omega^\omega, \not\leq)$.

Proof. Let us begin by noting that $(\mathbb{P}, \omega^\omega, \not\leq)$ is a Borel invariant. We can identify \mathbb{P} as a subset of ω^ω if we add $\{-1\}$ to the range of the functions and assume that

$f(n) = -1$ whenever f is undefined on n . Formally, let $W = \omega \cup \{-1\}$. Clearly $W^\omega \simeq \omega^\omega$ and

$$\mathbb{P} = \left\{ f \in W^\omega : \forall m \in \omega \exists n > m (f(n) \neq -1) \right\}$$

which is a Borel subset as $f(n) \neq -1$ is an open condition. Similarly we can check that

$$f \succeq g \Leftrightarrow \forall m \in \omega \exists n > m \left((f(n) \neq -1) \wedge (f(n) < g(n)) \right)$$

and then \succeq is a Borel relation.

To see that $(\omega^\omega, \omega^\omega, \not\prec^*) \leq_T (\mathbb{P}, \omega^\omega, \not\prec)$, just take $\phi : (\omega^\omega) \rightarrow \mathbb{P}$ as the inclusion, that is, $\phi(f)$ is the function f with its codomain replaced by W , and $\psi : \omega^\omega \rightarrow \omega^\omega$ is the identity. It is easy to see that if $\phi(f) \not\prec g$ then $f \not\prec^* \psi(g)$.

For the other inequality define $\phi : \mathbb{P} \rightarrow \omega^\omega$ such that $\phi(f)(n) = f(\hat{n})$ where $\hat{n} = \min\{m > n : f(m) \neq -1\}$. Let us also define $\psi : \omega^\omega \rightarrow \omega^\omega$ by $\psi(g)(n) = \max\{g(i) : i \leq n\}$. Clearly ψ is Borel. Before showing that ϕ is Borel, let us prove that ϕ and ψ work.

Assume $\phi(f) \not\prec^* g$. Let $A = \{n \in \omega : g(n) > \phi(f)(n)\}$ which is an infinite set. Given $n \in A$ and \hat{n} defined as above:

$$\psi(g)(\hat{n}) \geq g(n) > \phi(f)(n) = f(\hat{n}).$$

Thus $\hat{A} = \{\hat{n} : n \in A\}$ is infinite and shows that $f \not\prec \psi(g)(n)$.

It remains to show that ϕ is Borel. For this notice that $\{S_{n,m} : n, m \in \omega\}$, where $S_{n,m} = \{f \in \omega^\omega : f(n) = m\}$, is a subbase for the topology on ω^ω . It then suffices to see that $\phi^{-1}[S_{n,m}]$ is Borel for every $n, m \in \omega$. But this is clear since

$$f \in \phi^{-1}[S_{n,m}] \Leftrightarrow \exists k \geq m (f(k) = m \wedge (\forall j \in \omega (m \leq j < k \rightarrow f(j) = -1))).$$

By Proposition 34 the corresponding diamond principles are equivalent. \square

We are now ready to state the parametrized diamond we will use, in the right representation for our purposes:

$$\begin{aligned} \diamond(\mathfrak{b}) \equiv & \forall F : 2^{<\omega_1} \rightarrow \mathbb{P} \text{ Borel } \exists g : \omega_1 \rightarrow \omega^\omega \forall f \in 2^{\omega_1} \\ & \{\alpha \in \omega_1 : F(f \upharpoonright \alpha) \not\prec^* g(\alpha)\} \text{ is stationary.} \end{aligned}$$

We call the function g in the previous statement, a $\diamond(\mathfrak{b})$ -sequence, and we will say that g guesses f at α via F if $F(f \upharpoonright \alpha) \not\prec^* g(\alpha)$. We will often omit mentioning the function F when no confusion arises.

The existence of n -Ramsey spaces that fail to be $(n+1)$ -Ramsey under $\mathfrak{b} = \mathfrak{c}$ can be adapted to obtain the same counterexamples with size ω_1 under $\diamond(\mathfrak{b})$. However, even though we can achieve the recursion that guarantees that the space is not $(n+1)$ -Ramsey using the guessing property of the $\diamond(\mathfrak{b})$ -sequence, we also have to ensure that for every sequence $f : [\omega]^\omega \rightarrow \omega^{n+1}$, there is a subset $X \in [\omega]^\omega$ such that the closure of $f''[X]^n$ has size less than \mathfrak{b} (hence it has countable closure since $\diamond(\mathfrak{b})$ implies $\mathfrak{b} = \omega_1$). We can not achieve this by using the guessing property since we will need every of these functions being guessed cofinally. To avoid this trammel, we will make use of the fact that \mathfrak{d} is the cofinality of the ideal FIN^n . Moreover, under the assumption $\mathfrak{b} = \mathfrak{d}$ we can even get a cofinal family in FIN^n with stronger properties.

We will use the notation preceding Lemma 30 for the following result. In particular, we will use again Hechler trees and the same enumeration $\{s_n : n \in \omega\}$ for ω^{n+1} .

Lemma 36. *If $\mathfrak{b} = \mathfrak{d}$, there is a family $\{T(\alpha) : \alpha < \mathfrak{d}\}$ of Hechler trees in $\omega^{\leq n+1}$ such that*

- (1) $\forall \alpha < \beta \exists N \in \omega (T(\beta) \subseteq T(\alpha) \cup \bigcup_{i < N} C_i)$ and
- (2) $\forall X \in \text{FIN}^{n+1} \exists \alpha < \mathfrak{d} \forall \beta > \alpha \exists N \in \omega (X \cap T(\alpha) \subseteq \bigcup_{i < N} C_i)$

In particular there is a base for FIN^{n+1} of size \mathfrak{d} .

Proof. Let $\{d_\alpha : \alpha < \mathfrak{d}\}$ be a scale. As mentioned in the introduction, this family exists since $\mathfrak{b} = \mathfrak{d}$.

For every $\alpha < \mathfrak{d}$ define $T(\alpha) = T_{d_\alpha}$. Let us see that this family works. Let $\alpha < \beta$. Since $d_\alpha <^* d_\beta$, there exists $k \in \omega$ such that $d_\alpha(m) < d_\beta(m)$ for every $m \geq k$. Let $N \in \omega$ such that $\{s_j : j < k\} \subseteq \bigcup_{i < N} C_i$. Thus $T(\beta) \subseteq T(\alpha) \cup \bigcup_{i < N} C_i$ since $s_m \notin \bigcup_{i < N} C_i$ implies $\text{succ}_{T(\beta)}(s_m) \subseteq \text{succ}_{T(\alpha)}$.

For (2) fix $X \in \text{FIN}^{n+1}$ and let T a Hechler tree disjoint from X . Define $h : \omega \rightarrow \omega$ by $h \upharpoonright T = h_T$ and $h(s_j) = 0$ for $s_j \notin T$. There is $\alpha < \mathfrak{d}$ such that $h <^* d_\alpha$. Pick $k \in \omega$ such that $h(m) < d_\alpha(m)$ for every $m \in \omega$ with $s_m \notin \bigcup_{i < k} C_i$. Then $T(\alpha) \subseteq T \cup \bigcup_{i < k} C_i$ and in consequence $T(\alpha) \cap X \subseteq \bigcup_{i < k} C_i$. Let $\beta > \alpha$. By (1), $T(\beta) \subseteq T(\alpha) \cup \bigcup_{i < N'} C_i$. Thus $X \cap T(\beta) \subseteq \bigcup_{i < N'} C_i$ for $N = \max\{k, N'\}$.

For the last observation let $H(\alpha) = \omega^{n+1} \setminus T(\alpha)$ and for every $N \in \omega$ let $H(\alpha, N) = H(\alpha) \cup \bigcup_{i < N} C_i$. Therefore $\{H(\alpha, N) : \alpha < \mathfrak{d} \wedge N \in \omega\}$ is a cofinal family in FIN^{n+1} . \square

For the last part of the previous lemma we do not need $\mathfrak{b} = \mathfrak{d}$, in general we have the following:

Corollary 37. $\text{cof}(\text{FIN}^{n+1}) = \mathfrak{d}$.

Proof. By the proof of part 2 in the previous Lemma, we have a cofinal family in FIN^{n+1} of size \mathfrak{d} . Thus $\text{cof}(\text{FIN}^{n+1}) \leq \mathfrak{d}$.

For the other inequality let $\kappa < \mathfrak{d}$ and $\{X_\alpha : \alpha \in \kappa\} \subseteq \text{FIN}^{n+1}$. For each X_α , find a Hechler tree $T_\alpha \subseteq \omega^{n+1}$ such that $X_\alpha \cap T_\alpha = \emptyset$. We can find a function $f \in \omega^\omega$ that dominates the family $\{f_{T_\alpha} : \alpha < \kappa\}$. Thus $X = \omega^{n+1} \setminus T_f \in \text{FIN}^{n+1}$ satisfies that $X \not\subseteq X_\alpha$ for all $\alpha < \kappa$ and the family $\{X_\alpha : \alpha < \kappa\}$ is not cofinal. \square

Recall that we identify ω^{n+1} with the set of increasing sequences of length $n+1$ and given $B \in [\omega]^\omega$, define $B_{n+1}^\uparrow = \{(a_0, \dots, a_n) : \forall i \leq n (a_i \in B \wedge (a_i < a_{i+1}))\}$. For $i, k \in \omega$ define $C(i, k) = \{(a_0, \dots, a_n) \in C_i : a_n < k\}$.

Theorem 38. $(\diamond(\mathfrak{b}) + \mathfrak{d} = \omega_1)$. *For every $n \in \omega$, there exists an n -Ramsey non $(n+1)$ -Ramsey space of size ω_1 .*

Proof. Fix $n \in \omega$ and let $\{T(\alpha) : \omega \leq \alpha < \omega_1\}$ as in Lemma 36. We will recursively construct an almost disjoint family $\{A_\alpha : \alpha < \omega_1\} \subseteq \omega^{n+1}$. For every infinite $\alpha < \omega_1$ let $\alpha = \{\alpha_m : m \in \omega\}$. We will define a Borel function $F : 2^{< \omega_1} \rightarrow \mathbb{P}$. By a suitable coding we can assume that the domain of F is the set of pairs $(B, \{A_\beta : \beta < \alpha\})$ where

- $B \in [\omega]^\omega$,
- α is indecomposable,

- $A_\alpha \subseteq T(\alpha)$ for $\alpha \geq \omega$,
- $\omega \leq \alpha < \omega_1$,
- $A_n = C_n$ for every $n \in \omega$ and
- $\{A_\beta : \beta < \alpha\} \subseteq [\omega]^\omega$ is an almost disjoint family.

Given $(B, \{A_\beta : \beta < \alpha\})$, for every $i \in B \cap \text{succ}_{T(\alpha)}(\emptyset)$ define

$$F(B, \{A_\beta : \beta < \alpha\})(i) = \min \left\{ k \in \omega : C(i, k) \cap T(\alpha) \cap B_{n+1}^\uparrow \setminus \left(\bigcup_{m < i} A_{\alpha_m} \right) \right\}.$$

It is not hard to see that F is a Borel function. Let $g : \omega_1 \rightarrow \omega^\omega$ be a $\diamond(\mathfrak{b})$ -sequence with the incarnation of \mathfrak{b} mentioned earlier. Now for every $n \in \omega$ let $A_n = C_n$ and for every infinite $\alpha < \omega_1$ define

$$A_\alpha = \bigcup_{i \in \omega} \left(C(i, g(\alpha)(i)) \setminus \bigcup_{m < i} A_{\alpha_m} \right) \cap T(\alpha).$$

We can assume that every A_α is infinite by increasing $g(\alpha)$ if necessary. It is clear from the definition that $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is an almost disjoint family. We will now see that $\mathcal{F}(\mathcal{A})$ is n -Ramsey. If $f : [\omega]^n \rightarrow \omega^{n+1}$, there exists $X \in [\omega]^\omega$ such that $f''[X]^n \in \text{FIN}^{n+1}$ by Lemma 15 and by Lemma 36 part 2, there exists $\alpha < \omega_1$ such that for every $\beta > \alpha$ there is $N \in \omega$ such that $f''[X]^n \cap A_\beta \subseteq f''[X]^n \cap T(\beta) \subseteq \bigcup_{i < N} C_i$. Since $A_\beta \cap C_i \subseteq C(i, g(\beta)(i))$ for every infinite β , this set is finite and $f''[X]^n$ is almost disjoint with all but countable many A_β . By Corollary 13, $\mathcal{F}(\mathcal{A})$ is n -Ramsey.

To see that it is not $(n+1)$ -Ramsey let $G : [\omega]^{n+1} \rightarrow \omega^{n+1}$ where G maps each set s to its increasing enumeration. Thus $G''[B]^{n+1} = B_{n+1}^\uparrow$ for every $B \in [\omega]^\omega$. We will omit the subindex $n+1$. Let $B \in [\omega]^\omega$, we have to show that B is not a convergent subsequence for G . This is equivalent to show that there exists $\alpha < \omega_1$ such that $A_\alpha \cap B^\uparrow$ is infinite by Lemma 27.

Given $(B, \{A_\alpha : \alpha < \omega_1\})$ coded by $f \in 2^{\omega_1}$, we can assume that $\{B, \{A_\beta : \beta < \alpha\}\}$ is coded by $f \upharpoonright \alpha$ by passing to the club of idecomposable ordinals in ω_1 . Since $(B, \{A_\alpha : \alpha < \omega_1\})$ is coded by some $f \in 2^{\omega_1}$, there exists $\alpha < \omega_1$ infinite such that

$$g(\alpha) \not\leq^* F(B, \{A_\beta : \beta < \alpha\}).$$

We use h as a shorthand for $F((B, \{A_\beta : \beta < \alpha\}))$. Hence there are infinitely many $i \in B$ such that $g(\alpha)(i) > h(i)$. Let B' be this set and thus $C(i, h(i)) \subseteq C(i, g(\alpha)(i))$ for every $i \in B'$. By the definition of h , there exists

$$\begin{aligned} b_i &\in C(i, h(i)) \cap T(\alpha) \cap B_{n+1}^\uparrow \setminus \left(\bigcup_{m < i} A_{\alpha_m} \right) \\ &\subseteq C(i, g(\alpha)(i)) \cap T(\alpha) \setminus \left(\bigcup_{m < i} A_{\alpha_m} \right) \subseteq A_\alpha. \end{aligned}$$

As the family $\{C(i, g(\alpha)(i)) : i \in B'\}$ is pairwise disjoint, $\{b_i : i \in B'\}$ is infinite and contained in $B^\uparrow \cap A_\alpha$. \square

Another property of the parametrized diamonds principles is that they turn the corresponding cardinal invariant small. For instance, $\diamond(\mathfrak{d}) \Rightarrow \mathfrak{d} = \omega_1$. Thus we can state the previous theorem under $\diamond(\mathfrak{d})$. In Section 8, we will produce the same examples under $\mathfrak{s} = \mathfrak{b}$. Although $\diamond(\mathfrak{d})$ (or even $\mathfrak{d} = \omega_1$) implies $\mathfrak{s} = \mathfrak{b}$, a feature

of the space constructed in this section is that it has size ω_1 which does not follow from Theorem 52.

Question 39. Does $\diamond(\mathfrak{b})$ imply that for every $n \geq 1$ there is an almost disjoint family \mathcal{A} such that $\mathcal{F}(\mathcal{A})$ is n -Ramsey but not $(n+1)$ -Ramsey?

7. HIGH DIMENSIONAL SPLITTING-LIKE CARDINAL INVARIANTS

We say that $\mathcal{P} = \{P_n : n \in \omega\} \subseteq [\omega]^{<\omega}$ is an *interval partition* if \mathcal{P} is a partition of ω consisting of consecutive intervals.

Definition 40. Let $\mathcal{P} = \{P_n : n \in \omega\}$ be an interval partition and $S \subseteq \omega$. We say that S *block-splits* \mathcal{P} if

$$|\{n \in \omega : P_n \subseteq S\}| = \omega = |\{n \in \omega : P_n \cap S = \emptyset\}|.$$

A family $\mathcal{S} \subseteq [\omega]^\omega$ is a *block-splitting family* if for every interval partition \mathcal{P} , there exists $S \in \mathcal{S}$ such that S block-splits \mathcal{P} .

We also define

$$\mathfrak{bs} = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a block splitting family}\}.$$

It is easy to see that every block-splitting family is a splitting family. It follows that $\mathfrak{s} \leq \mathfrak{bs}$. In fact, we have the following theorem due to Kamburelis and Węglorz.

Theorem 41. [33] $\mathfrak{bs} = \max\{\mathfrak{b}, \mathfrak{s}\}$

Recall that for any $s \in \omega^{\leq n}$, we denote by C_s the set of $t \in \omega^n$ such that $t \upharpoonright \text{dom}(s) = s$. If $n \geq 1$ let us denote by \mathcal{L}_n the set of functions $f : [\omega]^n \rightarrow \omega^{n+1}$ such that there is no $X \in [\omega]^\omega$ so that $f''[X]^n \subseteq C_s$ for some $s \in \omega^{<n+1} \setminus \{\emptyset\}$ (note that it is equivalent if we require this for $s \in \omega^1$).

Definition 42. Let $n \geq 1$, $f \in \mathcal{L}_n$ and $S \in [\omega]^\omega$. We say that S *splits* f (or more precisely \mathcal{L}_n -splits f) if there are disjoint $X, Y \in [\omega]^\omega$ such that:

- (1) $f''[X]^n \subseteq S^{n+1}$ and
- (2) $f''[Y]^n \subseteq (\omega \setminus S)^{n+1}$

We define μ_n as the minimum size of an \mathcal{L}_n -splitting family, i.e. the minimum size of a family $\mathcal{S} \subseteq [\omega]^\omega$ such that every $f \in \mathcal{L}_n$ is \mathcal{L}_n -split by an element in \mathcal{S} .

At the moment, it is not even clear that there are \mathcal{L}_n -splitting families, but we will show later in this section that they do exist and find an upper bound for μ_n .

We can consider ω^n as the set of increasing sequences of ω of length n through the computable bijection φ that maps (i_0, \dots, i_{n-1}) into

$$\left(i_0, i_0 + i_1 + 1, \dots, \sum_{j < n} i_j + (n-1) \right).$$

Notation 43. For a set $X \subseteq \omega$ and $n \in \omega$ let

- $nX = \{nx : x \in X\}$ and
- $\frac{X}{n} = \{\frac{x}{n} : x \in X\}$ if $n \neq 0$.

We start by finding a lower bound for this new invariants.

Proposition 44. $\mathfrak{s} \leq \mu_1$.

Proof. Let \mathcal{S} be an \mathcal{L}_1 -splitting family of size μ_1 . With this, we will define a splitting family of size at most μ_1 . For every $X \in [\omega]^\omega$, define $f_X : \omega \rightarrow \omega^2$ such that if $X = \{x_n : n \in \omega\}$ is the increasing enumeration of X , then $f(n) = (2x_n, 2x_n + 1)$. Clearly, $f_X \in \mathcal{L}_1$.

There exists $S \in \mathcal{S}$ that \mathcal{L}_1 -splits f_X . Hence there are disjoint $Y, Z \in [\omega]^\omega$ such that $f''Y \subseteq S^2$ and $f''Z \subseteq (\omega \setminus S)^2$. In particular, for every $n \in Y$, we know that $f(n) \in S^2$, which implies that $2x_n \in S$. Similarly, $2x_m \in \omega \setminus S$ for every $m \in Z$.

Thus, we can define $S' = S \cap 2\omega$ and $\widehat{S} = \frac{S'}{2}$ for every $S \in \mathcal{S}$. It is now easy to see that $\{\widehat{S} : S \in \mathcal{S}\}$ is a splitting family of size at most μ_1 . \square

Given $t \in \omega^n$, we write $t = (t_0, \dots, t_{n-1})$ and denote $im^*(t) = \{t_0, \dots, t_{n-1}\}$. Now we will see that the cardinals μ_n form a non-decreasing sequence.

Proposition 45. *Let $n \geq 1$. Every \mathcal{L}_{n+1} -splitting family is also an \mathcal{L}_n -splitting family. Hence $\mu_n \leq \mu_{n+1}$ for every $n \geq 1$.*

Proof. Let \mathcal{S} be an \mathcal{L}_{n+1} -splitting family and let $f \in \mathcal{L}_n$. Define $f' \in \mathcal{L}_{n+1}$ such that f' maps the cone C_s into $f(s) \frown k$ where k is the minimum i that can be concatenated to the sequence $f(s)$. Formally,

$$f'(s) = f(s \setminus \max(s)) \frown (f(s \setminus \max(s))_n + 1)$$

for every $s \in [\omega]^{n+1}$. Let $S \in \mathcal{S}$ that \mathcal{L}_{n+1} -splits f' and let $X, Y \in [\omega]^\omega$ witnessing this. Let $a \in [X]^n$ and pick $x \in X$ such that $x > \max(a)$. Since $f'[X]^{n+1} \subseteq S^{n+2}$, and $im^*(f(a)) \subseteq im^*(f'(a \cup \{x\}))$, it is now easy to see that $f(a) \in S^{n+1}$. Therefore $f''[X]^n \subseteq S^{n+1}$. Similarly $f''[Y]^n \subseteq (\omega \setminus S)^{n+1}$. \square

We now give an upper bound for μ_n and implicitly show that \mathcal{L}_n -splitting families exists. In order to achieve this, we need to work with equivalence relations on $[\omega]^n$ and apply Erdős-Rado theorem.

Theorem 46. *For every $n \in \omega$, $\mu_n \leq \mathfrak{b}_s$.*

Proof. Fix $n \in \omega$. We will show that every block splitting family is \mathcal{L}_n -splitting. Given $f \in \mathcal{L}_n$ and $a \in [\omega]^n$, we write $f(a)_i = b_i$ if $f(a) = (b_0, b_1, \dots, b_n)$. Using Corollary 17, we find $M \in [\omega]^\omega$ and $I_j \subseteq n$ for every $j \leq n$, such that for every $a, a' \in [M]^n$, $f(a)_j = f(a')_j$ iff $a \upharpoonright I_j = a' \upharpoonright I_j$.

We claim that $I_j \neq \emptyset$ for every $j \leq n$. Indeed, if $I_j = \emptyset$ for some j , there exists $k \in \omega$ such that $f(a)_j = k$ for every $a \in [M]^n$. Since $f(a)$ is increasing for every $a \in [\omega]^n$, there is a finite set F such that $f(a)_0 \in F$ for every $a \in [M]^n$, thus applying the usual Ramsey theorem, we find $M' \in [M]^\omega$ and $l < k$ such that $f(a)_0 = l$ for every $a \in [M']^n$. That is, $f''[M']^n \subseteq C_l$, contradicting that $f \in \mathcal{L}_n$.

For every $j \leq n$, let $m_j = \max(I_j)$. As none of the I_j is empty, the set $\{f(a)_j : a \in [M]^n\}$ is infinite. Moreover,

- (*) for every $j \leq n$ and $k \in \omega$, there exists $z \in M$ such that $f(a)_j > k$ for every $a \in [M]^n$ with $z \in a = \{a_0, \dots, a_{n-1}\}$ and $z = a_i$ for some $i \leq m_j$.

To see this, fix $j \leq n$ and $k \in \omega$. For every $k' \leq k$ there exists $a(k') \in [M]^n \cup \{\emptyset\}$ such that $f(a)_j = k'$ if and only if $a \upharpoonright I_j = a(k') \upharpoonright I_j$ for every $a \in [M]^n$ (possibly, $a(k') = \emptyset$ if $f(a)_j \neq k'$ for every $a \in [M]^n$). Now let $F = \bigcup_{k' \leq k} a(k')$. Thus any $z > \max(F)$ works.

It is also worth noting that given $j \leq n$, $z \in M$ and $t = (t_0, \dots, t_{m_j-1}) \in (M \cap z)^{m_j}$, there is $r(j, z, t) \in \omega$ such that

$$(**) f(t \widehat{\ } \{z\} \widehat{\ } s)_j = r(j, z, t)$$

for every $s \in [M \setminus (z+1)]^{n-m_j}$ (where $t \widehat{\ } s$ is the concatenation of the sequences t and s).

We are going to define an interval partition $\mathcal{P} = \{P_l : l \in \omega\}$ such that any S , that block splits \mathcal{P} , will also \mathcal{L}_n -split f . It is clear that if we achieve this, then $\mu_n \leq \mathfrak{b}\mathfrak{s}$.

To define this partition \mathcal{P} , we will also define an increasing sequence $\{x_l : l \in \omega\} \subseteq M$ and an increasing sequence $\{k_l : l \in \omega\}$.

Let $x_0 = \min(M)$ and $k_0 = 0$. For $l > 0$, by $(*)$, we can find $x_l > \max\{x_i : i < l\}$ in M such that $f(a)_j > k_{l-1}$ for any $j \leq n$ and any $a \in [M]^n$ such that $a_i = x_l$ for some $i \leq m_j$. Given $j \leq n$ and $t = \{t_0, \dots, t_{m_j-1}\} \subseteq \{x_i : i < l\}$ let $r(j, x_l, t)$ as in $(**)$ and define

$$k_l > \max(\{r(j, x_l, t) : j \leq n \wedge t \in [\{x_i : i < l\}]^{m_j}\} \cup \{k_i : i < l\}).$$

Define $P_l = (k_{l-1}, k_l]$ for every $l \in \omega$ and let $\mathcal{P} = \{P_l : l \in \omega\}$.

We proceed to show that \mathcal{P} is as required. Let $S \in [\omega]^\omega$ that block-splits \mathcal{P} and define $X = \{x_l : P_l \subseteq S\}$ and $Y = \{x_l : P_l \cap S = \emptyset\}$. Let $a = \{x_{l_i} : i < n\} \subseteq X$ where $x_{l_0} < x_{l_1} < \dots < x_{l_{n-1}}$. We will prove that $f(a) \in S^{n+1}$. Fix $j \leq n$ and let $i = m_j < n$. By the choice of l_i , it follows from the definition of x_{l_i} that $f(a)_j > k_{l_i-1}$. On the other hand, we have that

$$f(a)_j = f(t \widehat{\ } x_{l_i} \widehat{\ } s)_j = r(j, x_{l_i}, t) < k_{l_i}$$

with $t = \{x_{l_0}, \dots, x_{l_{m_j-1}}\}$ and $s = \{x_{l_{m_j+1}}, \dots, x_{l_{n-1}}\}$. Thus $f(a)_j \in P_{l_i}$ and $P_{l_i} \subseteq S$ since $x_{l_i} \in X$. As this is true for every $j \leq n$, we have that $f(a) \subseteq S^{n+1}$ and therefore $f''([X]^n) \subseteq S^{n+1}$. A similar argument shows that $f''([Y]^n) \subseteq (\omega \setminus S)^{n+1}$. \square

It follows from the results in this section that $\mathfrak{s} = \mu_n = \mathfrak{b}$ is consistent for every $n \geq 1$. We do not know if in ZFC one of these equalities hold.

We ask the following natural questions:

Question 47. *Given $n \geq 1$, is it consistent that $\mu_n < \mu_{n+1}$?*

Question 48. *It is true that $\mu_n = \mathfrak{b}\mathfrak{s}$ for all $n \geq 1$?*

Question 49. *Is it consistent that $\mathfrak{s} < \mu_0 < \mu_1 < \dots < \mu_n < \mu_{n+1} < \dots < \mathfrak{b}$?*

8. n -RAMSEY SPACES FROM $\mathfrak{s} = \mathfrak{b}$.

In this section we construct special almost disjoint families following the ideas and techniques developed by Shelah in [49] and improved by Mildenerger, Raghavan and Steprāns in [40] and [46]. For a survey of this method see [28], [19]. We first need to prove that every \mathcal{L}_n -splitting family is *everywhere \mathcal{L}_n -splitting*.

Lemma 50. *Let \mathcal{S} be a \mathcal{L}_n -splitting family, $A \in [\omega]^\omega$ and $f : [A]^n \rightarrow \omega^{n+1}$ such that there is no $X \in [A]^\omega$ so that $f''([X]^n) \subseteq C_s$ for some $s \in \omega^{<n+1} \setminus \{\emptyset\}$. Then there are $X, Y \in [A]^\omega$ and $S \in \mathcal{S}$ such that $f''([X]^n) \subseteq S^{n+1}$ and $f''([Y]^n) \subseteq (\omega \setminus S)^{n+1}$.*

Proof. Let $g : \omega \rightarrow A$ a bijection and consider the function $\bar{f} : [\omega]^n \rightarrow \omega^{n+1}$ given by $\bar{f}(\{n, m\}) = f(\{g(n), g(m)\})$. It is easy to see that $\bar{f} \in \mathcal{L}_n$, so there is $S \in \mathcal{S}$ and disjoint $X, Y \in [\omega]^\omega$ such that $\bar{f}''([X]^n) \subseteq S^{n+1}$ and $\bar{f}''([Y]^n) \subseteq (\omega \setminus S)^{n+1}$. Taking $\bar{X} = g''X$ and $\bar{Y} = g''Y$ we get that $f''[X]^n = f''[\bar{X}]^n$ and $f''[Y]^n = f''[\bar{Y}]^n$ so we get the result. \square

Recall that a set $Y \subset \omega$ is almost monochromatic (or almost homogeneous) for a partition $f : [\omega]^n \rightarrow 2$ if there is a finite set $F \subseteq Y$ such that f is constant on $[Y \setminus F]^n$. The cardinal par_n denotes the smallest cardinal κ such that there is a family of partitions of $[\omega]^n$ of size κ such that no infinite set is almost monochromatic for all of them simultaneously.

Note that if we consider partitions into some finite number of pieces k , instead of 2 pieces, we obtain the same cardinal. Moreover, for all $n \geq 2$ we have that $\text{par}_n = \text{par}_2$, in fact:

Theorem 51. ([5]) *For each $n \geq 2$, $\text{par}_n = \min\{\mathfrak{b}, \mathfrak{s}\}$.*

Note that if $\mathfrak{s} = \mathfrak{b}$ then $\text{par}_2 = \text{par}_n = \mathfrak{s} = \mathfrak{b} = \mathfrak{bs} = \mu_n$ for every $n \geq 1$.

If $S \subseteq \omega$ then we will follow the notation $S^0 := S$ and $S^1 := \omega \setminus S$. We are now ready to prove the main theorem in this section. It is interesting that the combinatorial characterizations of both $\max\{\mathfrak{b}, \mathfrak{s}\}$ and $\min\{\mathfrak{b}, \mathfrak{s}\}$ are used in the proof of this theorem.

Theorem 52. ($\mathfrak{s} = \mathfrak{b}$) *For every $n \geq 1$ there exists an almost disjoint family \mathcal{A} on ω^{n+1} of size \mathfrak{c} such that $\mathcal{F}(\mathcal{A})$ is n -Ramsey but not $(n+1)$ -Ramsey.*

Proof. Let us fix $\mathcal{S} = \{S_\alpha \mid \alpha \in \mathfrak{b}\}$ that is splitting and \mathcal{L}_n -splitting and also let us fix the following enumerations:

- $[\omega]^\omega = \{B_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$.
- $\mathcal{L}_n = \{f_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$.

As usual let $G : [\omega]^{n+1} \rightarrow \omega^{n+1}$ be the increasing enumeration.

We want to construct the following families:

- $\{A_\alpha \mid \alpha \in \mathfrak{c}\}$,
- $\{X_\alpha \mid \omega \leq \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$,
- $\{\tau_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}, \{\sigma_\alpha \mid \omega \leq \alpha < \mathfrak{c}\} \subseteq 2^{<\mathfrak{b}}$,

with the following properties:

- (1) $A_i = C_i$ for all $i \in \omega$.
- (2) If $\beta < \alpha$ then $\tau_\alpha \not\subseteq \tau_\beta$, $\tau_\alpha \not\subseteq \sigma_\beta$, $\sigma_\alpha \not\subseteq \tau_\beta$ and $\sigma_\alpha \not\subseteq \sigma_\beta$.
- (3) If $\beta \in \text{dom}(\tau_\alpha)$, then $A_\alpha \subseteq^* \left(S_\beta^{\tau_\alpha(\beta)}\right)^{n+1}$.
- (4) $f_\alpha''[X_\alpha]^n \in \text{FIN}^{n+1}$.
- (5) If $\alpha \geq \omega$, then A_α is block sequence.
- (6) $\forall \alpha \in \mathfrak{c} \setminus \omega (A_\alpha \subseteq G''[B_\alpha]^{n+1})$.
- (7) $\forall \beta, \alpha \in \mathfrak{c} (\omega \leq \beta < \alpha \implies \exists m_{\beta, \alpha} \in \omega (f_\beta''[X_\beta \setminus m_{\beta, \alpha}]^n \cap A_\alpha = \emptyset))$.
- (8) $\beta \in \text{dom}(\sigma_\alpha) \implies \exists n_\beta \in \omega (f_\alpha''[X_\alpha \setminus n_\beta]^n \cap \left(S_\beta^{1-\sigma_\alpha(\beta)}\right)^{n+1} = \emptyset)$.
- (9) $\forall \beta, \alpha \in \mathfrak{c} (\beta \neq \alpha \implies A_\beta \cap A_\alpha =^* \emptyset)$.

Let us suppose that we have already constructed the desired objects until step α . We are going to define X_α and σ_α .

- X_α and σ_α construction:

We are going to construct $\{\alpha_s \mid s \in 2^{<\omega}\} \subseteq \mathfrak{b}$, $\{\eta_s \mid s \in 2^{<\omega}\} \subseteq 2^{<\mathfrak{b}}$, $\{Y_s \mid s \in 2^{<\omega}\} \subseteq [\omega]^\omega$, $\{f_s \mid s \in 2^{<\omega}\}$ such that for all $s \in 2^{<\omega}$:

- (a) $f_\emptyset = f_\alpha$, $Y_\emptyset = \omega$.
- (b) $f_s : [Y_s]^n \rightarrow \omega^{n+1}$.

- (c) $\alpha_s = \text{dom}(\eta_s)$.
- (d) $Y_{s \frown 0}, Y_{s \frown 1} \in [Y_s]^\omega$ are disjoint.
- (e) $f_s''[Y_{s \frown i}]^n \subseteq (S_{\alpha_s}^i)^{n+1}$ for every $i \in 2$.
- (f) If $\beta \in \text{dom}(\eta_s)$ and $i \in 2$, then $f_{s \frown i}''[Y_{s \frown i} \setminus m]^n \cap (S_\beta^{1-\eta_s(\beta)})^{n+1} = \emptyset$ for some $m \in \omega$.
- (g) $s \subseteq t \Rightarrow \eta_s \subseteq \eta_t$.
- (h) $s \perp t \Rightarrow \eta_s \perp \eta_t$.
- (i) $f_{s \frown i} = f_s \upharpoonright Y_{s \frown i}$.
- (j) $\alpha_s < \alpha_{s \frown i}$.

As \mathcal{S} is \mathcal{L}_n -splitting, then we can find $\alpha_\emptyset = \min\{\beta \in \mathfrak{b} \mid S_\beta \text{ splits } f_\emptyset\}$. Then there exist $Z_0, Z_1 \in [\omega]^\omega$ such that $f_\emptyset''[Z_i]^n \subseteq (S_{\alpha_\emptyset}^i)^{n+1}$ for $i \in 2$. Now consider the following family of colorings on $[Z_0 \cup Z_1]^n$: for every $\beta < \alpha_\emptyset$ let $g_\beta : [Z_0 \cup Z_1]^n \rightarrow 3$ as follows:

$$g_\beta(\{x_1, \dots, x_n\}) = \begin{cases} 0 & \text{if } f_\emptyset(\{x_1, \dots, x_n\}) \in (S_\beta^0)^{n+1} \\ 1 & \text{if } f_\emptyset(\{x_1, \dots, x_n\}) \in (S_\beta^1)^{n+1} \\ 2 & \text{otherwise} \end{cases}$$

Note that for $i \in 2$, $|\{g_\beta \upharpoonright [Z_i]^n \mid \beta < \alpha_\emptyset\}| \leq |\alpha_\emptyset| < \mathfrak{b} = \text{par}_n$, then there exists $Y_i \in [Z_i]^\omega$ such that Y_i is almost monochromatic for g_β for every $\beta < \alpha_\emptyset$. It is worth to point out that we are defining η_\emptyset in terms of $Y_0 = Y_{\emptyset \frown 0}$ and $Y_1 = Y_{\emptyset \frown 1}$.

Claim 53. For every $\beta < \alpha_\emptyset$ there exists $j \in 2$ such that for every $i \in 2$ we have that $f''[Y_i \setminus m]^n \cap (S_\beta^{1-j})^{n+1} = \emptyset$ for some $m \in \omega$.

Proof. Consider Y_0 and Y_1 and the coloring g_β . If Y_i is almost 0-monochromatic for any $i \in 2$ then Y_{1-i} can not be almost 1-monochromatic, since S_β does not split f . In this case take $j = 0$. Hence $1 - j = 1$ and no matter which color Y_{1-i} takes, we know that there exists $n_\beta \in \omega$ such that $f_\emptyset([Y_i \setminus n_\beta]^n) \cap (S_\beta^1)^{n+1} = \emptyset$ for both i and $i - 1$. Analogously, if there is an $i < 2$ such that Y_i is 1-monochromatic, $j = 1$ works. In case both sets are 2-monochromatic, any $j \in 2$ satisfies the claim. \square

Now define $\eta_\emptyset : \alpha_\emptyset \rightarrow 2$ such that for every $\beta < \alpha_\emptyset$ we have that $\eta_\emptyset(\beta) = j$, where j is given by the claim. For $f_{(0)}$ and $f_{(1)}$ we just take the restriction $f_{(i)} = f \upharpoonright [Y_i]^n$.

Now suppose that we have already constructed f_s, Y_s, α_t and $\eta_t : \alpha_t \rightarrow 2$ for $s = t \frown j$ where $j \in 2$ and $t \in 2^{<\omega}$. We are going to construct $Y_{s \frown 0}, Y_{s \frown 1}, \alpha_s$ and η_s .

As $f_s : [Y_s]^n \rightarrow \omega^{n+1}$, there exists a minimum $\alpha_s \in \mathfrak{b}$ such that S_{α_s} splits f_s . Note that $\alpha_s > \alpha_t$ since no $\beta \leq \alpha_t$ splits f_s . Thus there exist $Z_0, Z_1 \in [Y_s]^\omega$ disjoint with $f_s''[Z_i]^n \subseteq (S_{\alpha_s}^i)^{n+1}$ for every $i \in 2$.

As before, consider the following family of colorings of $[Z_0 \cup Z_1]^n$: for every $\beta \in [\alpha_t, \alpha_s)$ let $g_\beta : [Z_0 \cup Z_1]^n \rightarrow 3$ as follows:

$$g_\beta(\{x_1, \dots, x_n\}) = \begin{cases} 0 & \text{if } f_s(\{x_1, \dots, x_n\}) \in (S_\beta^0)^{n+1} \\ 1 & \text{if } f_s(\{x_1, \dots, x_n\}) \in (S_\beta^1)^{n+1} \\ 2 & \text{otherwise} \end{cases}$$

Note that $|\{g_\beta \mid \beta \in [\alpha_t, \alpha_s)\}| \leq |\alpha_s| < \mathfrak{b} = \text{par}_n$, so there exists $Y_{s \frown i} \in [Z_i]^\omega$ such that $Y_{s \frown i}$ is almost-homogeneous for every $\beta \in [\alpha_t, \alpha_s)$. At this point we can define $\eta'_s : [\alpha_t, \alpha_s) \rightarrow 2$ with the proper adaptation of Claim 53. Now let $\eta_s = \eta_t \cup \eta'_s$.

From the definition of η'_s , we know that for every $\beta \in [\alpha_t, \alpha_s)$,

$$f''[Y_{s \smallfrown i} \setminus m_\beta]^n \cap \left(S_\beta^{1-\eta_s(\beta)} \right)^{n+1} = \emptyset$$

for some $m_\beta \in \omega$ and every $i \in 2$.

On the other hand, we know that $Y_{s \smallfrown 0}, Y_{s \smallfrown 1} \subseteq Y_s$ and our induction hypothesis implies that if $\beta \in \text{dom}(\eta_t)$, then $f''[Y_s \setminus m]^n \cap \left(S_\beta^{1-\eta_t(\beta)} \right)^{n+1} = \emptyset$ for some $m \in \omega$, so this last one condition is also true for $Y_{s \smallfrown 0}, Y_{s \smallfrown 1}$. Finally, let $f_{s \smallfrown i} = f_s \upharpoonright [Y_{s \smallfrown i}]^2$.

This shows that condition (f) holds and all other conditions are clear from the the construction.

To finish the construction of X_α and σ_α , let $\eta_g = \bigcup_{n \in \omega} \eta_{g \upharpoonright n}$ for every $g \in 2^\omega$. Notice that $\eta_g \in 2^{< \mathfrak{b}}$ as \mathfrak{b} has uncountable cofinality (in fact, it is regular, see [5]). Furthermore, if $f \neq g$, then η_f and η_g are incompatible nodes of $2^{< \mathfrak{b}}$. Since $\alpha < \mathfrak{c}$, we can find $g \in 2^\omega$ that satisfies that there is no $\beta < \alpha$ such that σ_β or τ_β extends η_g . Let $\sigma_\alpha = \eta_g$. Let X'_α be any pseudointersection of $\{Y_{g \upharpoonright n} \mid n \in \omega\}$. It follows that X'_α satisfies (8). By Lemma 15, we can find $X_\alpha \in [X'_\alpha]^\omega$ that also works for (4).

- A_α and τ_α construction.

We are going to construct a collection $\{B_s \mid s \in 2^{< \omega}\} \subseteq [\omega]^\omega$, $\{\eta_s \mid s \in 2^{< \omega}\} \subseteq 2^{< \mathfrak{b}}$ and $\{\alpha_s \mid s \in 2^{< \omega}\}$ such that:

- (i) $B_\emptyset = B_\alpha$.
- (ii) $s \subsetneq t \Rightarrow \eta_s \subsetneq \eta_t$.
- (iii) $s \perp t \Rightarrow \eta_s \perp \eta_t$.
- (iv) $\alpha_s = \text{dom}(\eta_s)$.
- (v) $B_{s \smallfrown 0}, B_{s \smallfrown 1} \in [B_s]^\omega$ are disjoint.
- (vi) If $\beta < \alpha_s$, then $B_s \subseteq^* S_\beta^{\eta_s(\beta)}$.

Suppose that we have already constructed η_t and α_t for every $t \subsetneq s$ and B_s . Let $\alpha_s \in \mathfrak{b}$ be the minimum such that $|B_s \cap S_{\alpha_s}^0| = \omega = |B_s \cap S_{\alpha_s}^1|$, this is possible since \mathcal{S} is splitting. Notice also that $\alpha_t < \alpha_s$ whenever $t \subsetneq s$ by (vi) and (v). Let $\eta_s : \alpha_s \rightarrow 2$ such that for all $\beta < \alpha_s$, $|B_s \cap S_\beta^{\eta_s(\beta)}| = \omega$. Finally, for every $i \in 2$ let $B_{s \smallfrown i} = B_s \cap S_{\alpha_s}^i$. Note that for every $\beta \in \alpha_s$ and for $i \in \{0, 1\}$, we have that $B_{s \smallfrown i}$ is almost contained in $S_\beta^{\eta_s(\beta)}$. The latter, the fact that S_{α_s} does not split $B_{s \smallfrown i}$ and the fact that $B_s \subseteq B_t$ for $t \subseteq s$, imply that $\eta_s \subsetneq \eta_{s \smallfrown i}$ for every $i \in 2$.

Now, for every $g \in 2^\omega$ let $\eta_g = \bigcup_{n \in \omega} \eta_{g \upharpoonright n}$. As before $\eta_g \in 2^{< \mathfrak{b}}$ and if $f \neq g$, η_f and η_g are incompatible nodes of $2^{< \mathfrak{b}}$. As $\alpha < \mathfrak{c}$, we can find $g \in 2^\omega$ such that there is no $\beta < \alpha$ such that σ_β or τ_β extends η_g , then let $\tau_\alpha = \eta_g$. Consider Z a pseudointersection of $\{B_{g \upharpoonright n} \mid n \in \omega\}$ that is contained in $B_\emptyset = B_\alpha$.

Note that if $\beta < \alpha$ and σ_α and τ_β are incompatibles, then there exists $m \in \omega$ such that if $\xi = \sigma_\alpha \Delta \tau_\beta$ ⁶ then $f''_\beta[X_\beta \setminus m]^n \cap \left(S_\xi^{\sigma_\alpha(\xi)} \right)^{n+1} = \emptyset$. In particular, as Z is almost contained in $S_\xi^{\sigma_\alpha(\xi)}$, then $f''_\beta[X_\beta \setminus m]^n \cap (Z)^{n+1} =^* \emptyset$.

Consider the following families of elements of FIN^{n+1} :

$$\begin{aligned} \mathcal{H}_0 &= \{f''_\beta[X_\beta]^n \mid \omega \leq \beta < \alpha \wedge (\tau_\alpha \supseteq \sigma_\beta)\}, \\ \mathcal{H}_1 &= \{A_\beta \mid \omega \leq \beta < \alpha (\tau_\alpha \supseteq \tau_\beta)\} \end{aligned}$$

⁶Recall that $\sigma \Delta \tau = \min\{\alpha \mid \sigma(\alpha) \neq \tau(\alpha)\}$

and

$$\mathcal{H}_2 = \{A_i \mid i \in \omega\}.$$

Define $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2$. Note that $|\mathcal{H}| < \mathfrak{b}$, so we can apply Lemma 30 to the family \mathcal{H} and the infinite set Z to obtain A_α a block sequence of elements of Z such that its intersection with every element of \mathcal{H} is finite.

Given $\beta \in \text{dom}(\tau_\alpha)$, there is $s \in 2^{<\omega}$ such that $\beta \in \text{dom}(\eta_s)$ and $s \subseteq g$. Thus $Z \subseteq^* B_s \subseteq^* S_\beta^{\eta_s(\beta)} = S_\beta^{\tau_\alpha(\beta)}$. Therefore $A_\alpha \subseteq^* \left(S_\beta^{\tau_\alpha(\beta)}\right)^{n+1}$ as $A_\alpha \subseteq Z^{n+1}$ is a block sequence.

Let us verify that this construction satisfies conditions (1) to (9). Conditions (1), (2), (4), (5) are true by definition. Condition (6) holds since A_α is a block sequence of elements of $Z \subseteq B_\alpha$. Condition (8) follows from the choice of σ_α . We have already checked condition (3) at the end of the construction of A_α . It remains to show that (7) and (9) hold.

Condition (7): Let β and α in \mathfrak{c} such that $\omega \leq \beta < \alpha$. If $\sigma_\beta \subseteq \tau_\alpha$, then for the construction of A_α we know that $f''_\beta[X_\beta]^n \cap A_\alpha =^* \emptyset$ and, as A_α is block sequence, then there exists $m_{\beta,\alpha} \in \omega$ such that $f''_\beta[X_\beta \setminus m_{\beta,\alpha}]^n \cap A_\alpha = \emptyset$.

On the other hand if σ_β and τ_α are incompatibles then exists $\xi \in \text{dom}(\tau_\alpha) \cap \text{dom}(\sigma_\beta)$ such that $\tau_\alpha(\xi) = i$ and $\sigma_\beta(\xi) = 1 - i$. By (3) and (8) applied to ξ we get the result.

Condition (9): It is clear that if $i \neq j$ are finite ordinals then $A_i \cap A_j = \emptyset$. Also, if $\omega \leq \alpha < \mathfrak{c}$ then A_α is block sequence, so $A_\alpha \cap A_i$ is finite for every $i \in \omega$. Now let $\omega \leq \beta < \alpha < \mathfrak{c}$. If $\tau_\alpha \supseteq \tau_\beta$ then $A_\alpha \cap A_\beta =^* \emptyset$ by the construction of A_α , since $A_\beta \in \mathcal{H}$. Otherwise τ_α and τ_β are incompatible nodes in $2^{<\mathfrak{b}}$, so there exists $\xi \in \text{dom}(\tau_\alpha) \cap \text{dom}(\tau_\beta)$ such that $A_\alpha \subseteq^* (S_\xi^{\tau_\alpha(\xi)})^{n+1}$ and $A_\beta \subseteq^* (S_\xi^{1-\tau_\alpha(\xi)})^{n+1}$, in particular $A_\alpha \cap A_\beta$ is finite. This finishes the construction.

Let us see that $\mathcal{F}(\mathcal{A})$ is the desired space. Conditions (3) and (6) imply that for every infinite $\alpha \in \mathfrak{c}$, the elements of the almost disjoint family in the closure of $f''_\alpha[X_\alpha]^n$, are at most those A_β such that $\tau_\beta \subseteq \sigma_\alpha$. In particular, the closure of $f''_\alpha[X_\alpha]^n$ has size less than \mathfrak{b} . By Corollary 14 we get that $\mathcal{F}(\mathcal{A})$ is n -Ramsey. On the other hand, condition (7) imply that condition (v) of Lemma 27 holds and the other three conditions when $f = G$ are clear, so $\mathcal{F}(\mathcal{A})$ is not $(n+1)$ -Ramsey. \square

Most modifications of the technique developed by Shelah used previously in the literature use assumptions of the form $\iota \leq \theta$ where ι is a kind of splitting cardinal invariant and θ is another cardinal invariant imposed by the particularities of the problem. We do not know if the previous result can be weakened to $\mathfrak{s} \leq \mathfrak{b}$.

Question 54. *Does it follow from $\mathfrak{s} \leq \mathfrak{b}$ that there are n -Ramsey spaces that fail to be $(n+1)$ -Ramsey?*

Or even better

Question 55. *Is there in ZFC, a space that is n -Ramsey but no $(n+1)$ -Ramsey for every $n \geq 2$?*

9. ON THE CHARACTER OF n -RAMSEY SPACES

One of the key facts in the constructions of our examples has been that any sequentially compact space of character less than \mathfrak{b} is n -Ramsey for every $n \geq 1$. It

is then natural to ask what is the smallest possible character of an n -Ramsey space that is not $(n + 1)$ -Ramsey:

Definition 56. *If $n \geq 2$ let λ_n be the minimum character of a space X such that X is $(n - 1)$ -Ramsey but not n -Ramsey.*

We do not know if these cardinals are well defined since we do not know if there are n -Ramsey spaces that are not $(n + 1)$ -Ramsey in ZFC (for $n > 1$). To avoid unnecessary complications, let us agree that $\lambda_n = \mathfrak{c}$ when it is not well defined by the previous definition. Theorem 12 gives us \mathfrak{b} as a lower bound for λ_n for every $n \geq 2$. Now, we will see that, using the following space, introduced by Burke and van Douwen in [7], we can actually compute λ_2 .

Let $\{f_\alpha \mid \alpha \in \mathfrak{b}\}$ consisting of strictly increasing functions, which is well ordered by \leq^* and for each $\alpha \in \mathfrak{b}$ let $B_\alpha = \{(n, m) \mid m \leq f_\alpha(n)\}$. Now consider $\mathcal{B} := \{B_\alpha \mid \alpha \in \mathfrak{b}\}$ and let $X = (\omega \times \omega) \cup \mathcal{B}$ be a topological space, where the elements of $\omega \times \omega$ are isolated and a basic neighborhood for $B_\beta \in \mathcal{B}$ has the form

$$N(B_\alpha, B_\beta, F) = (B_\alpha, B_\beta] \cup ((B_\beta \setminus B_\alpha) \setminus F),$$

where $\alpha \in \beta \cup \{-1\}$, $B_{-1} = \emptyset$ and $F \in [\omega \times \omega]^{<\omega}$. It is easy to check that X is Hausdorff, separable and since the basic neighborhoods of every point are compact, then X is also locally compact (see Example 7.3 of [52]). On the other hand, X is not sequentially compact since for all $n \in \omega$, the sequence $((n, m))_{m \in \omega}$ does not converge in X . So let $\bar{X} = X \cup \{\infty\}$ be the one-point compactification of X . Then the basic neighborhoods of ∞ have the form:

$$\{\infty\} \cup (X \setminus C)$$

where $C \subseteq X$ is compact.

Theorem 57. *The space \bar{X} is a compact and sequentially compact space of character \mathfrak{b} that is not 2-Ramsey.*

Proof. Let us see that \bar{X} is sequentially compact. Let $f : \omega \rightarrow \bar{X}$. If f takes the value ∞ infinitely many times, then we are done. Also if f takes values in \mathcal{B} infinitely many times, then we are done since \mathfrak{b} has uncountable cofinality (and then it is sequentially compact). So let us suppose that $f : \omega \rightarrow \omega \times \omega$. If $f''\omega \in \mathcal{A}$ then there exists $B_\beta \in \mathcal{B}$ with $\beta \in \mathfrak{b}$ minimum such that $f''\omega \cap B_\beta$ is infinite. Taking $B \in [\omega]^\omega$ such that $f''B \subseteq B_\beta$ we have that $f \upharpoonright B$ converges to B_β . If $f''\omega \notin \mathcal{A}$ then let $B \in [\omega]^\omega$ and $k \in \omega$ such that $f''B \subseteq A_k$. It follows that $f \upharpoonright B$ converges to ∞ since every basic neighborhood of ∞ contains all but finite elements of A_n for every $n \in \omega$.

We will now see that \bar{X} is not 2-Ramsey, so consider $G : [\omega]^2 \rightarrow \bar{X}$ given by $G(\{n, m\}) = (n, m)$ where $n < m$. We will see that for all $B \in [\omega]^\omega$, $G \upharpoonright [B]^2$ does not converge in \bar{X} . So let us fix $B \in [\omega]^\omega$ and $B_\beta \in \mathcal{B}$. Note that for every basic neighborhood N of B_β we have that $N \cap A_n$ is finite for every $n \in \omega$, but $G''[B \setminus F]^2$ has infinite intersection with every A_n such that $n \in B \setminus F$. This shows that there is no $F \in [\omega]^{<\omega}$ such that $G''[B \setminus F]^2 \subseteq N$. On the other hand, as G is injective, $G \upharpoonright [B]^2$ does not converge to any isolated point. So the only thing that remains to be seen is that $G \upharpoonright [B]^2$ does not converge to ∞ .

Let $(m_k)_{k \in \omega}$ be the increasing enumeration of B and let $A = \{(m_k, m_{k+1}) \mid k \in \omega\}$. Note that:

- (1) $A \in \mathcal{A}$.

(2) For all $F \in [\omega]^{<\omega}$ we have that $A \subseteq^* G''[B \setminus F]^2$.

Now let $B_\beta \in \mathcal{B}$ such that $|B_\beta \cap A| = \omega$ and β is the minimum with this property. Consider the following basic neighborhood of ∞ :

$$N = \{\infty\} \cup (X \setminus N(B_{-1}, B_\beta, \emptyset)) = \{\infty\} \cup (\mathcal{B} \setminus (B_{-1}, B_\beta)) \cup ((\omega \times \omega) \setminus B_\beta).$$

Remember that $B_{-1} = \emptyset$. If there exists $F \in [\omega]^{<\omega}$ such that $G''[B \setminus F]^2 \subseteq N$, then in particular we have that $G''[B \setminus F]^2 \cap B_\beta = \emptyset$, and as $A \subseteq^* G''[B \setminus F]^2$, then $A \cap B_\beta$ is finite, which is a contradiction. This implies that there is no such F and then $G \upharpoonright [B]^2$ does not converge, which proves that \bar{X} is not 2-Ramsey. \square

With this last result and Theorem 12, we get the following nice topological characterization of \mathfrak{b} :

Corollary 58. $\mathfrak{b} = \min\{\chi(X) \mid X \text{ is sequentially compact but not 2-Ramsey}\}$, that is, $\mathfrak{b} = \lambda_2$.

We can still ask for the minimal size of an almost disjoint family \mathcal{A} such that $\mathcal{F}(\mathcal{A})$ is not 2-Ramsey, i.e, the minimal character of a space of the form $\mathcal{F}(\mathcal{A})$ that is not 2-Ramsey. In general we can ask this for every $n \geq 2$:

Definition 59. If $n \geq 2$ let $\mathfrak{a}_n(\text{Ramsey})$ be the minimum size of an almost disjoint family such that $\mathcal{F}(\mathcal{A})$ is $(n-1)$ -Ramsey but not n -Ramsey.

It follows from the results in [37] that $\mathfrak{b} \leq \mathfrak{a}_2(\text{Ramsey}) \leq \mathfrak{a}$. Also, it should be clear that $\lambda_n \leq \mathfrak{a}_n(\text{Ramsey})$ for all $n \geq 2$ whenever they are well defined.

Definition 60. [49] Let \mathfrak{a}_* be the minimal size of an ad family \mathcal{A} , such that there are (disjoint) $C_0, C_1, C_2, \dots \in \mathcal{A}^\perp$ so that for every $D \in [\omega]^\omega$, if $|C_n \cap D| = \omega$ for infinitely many $n \in \omega$, then there is $A \in \mathcal{A}$ for which $|A \cap D| = \omega$.

The following notions were introduced by Arhangel'skii [1]:

Definition 61. Let X be a topological space and $x \in X$. Then x is an α_i -point ($i = 2, 3$) if for every family $\{S_n \mid n \in \omega\}$ of sequences from X converging to x , there exists a sequence S that converges to x such that:

- (α_1) $S_n \setminus S$ is finite for all $n \in \omega$,
- (α_2) $S \cap S_n \neq \emptyset$ for all $n \in \omega$.
- (α_3) $|S \cap S_n| = \omega$ for infinitely many $n \in \omega$.

A space X is an α_i -space if every point $x \in X$ is an α_i -point.

Arhangel'skii introduced the α_i properties for $i \in \{1, 2, 3, 4, 5\}$ in [1]. It is easy to see that all α_1 points are α_2 points and that all α_2 -points are α_3 -points. We refer the reader to [1] for the rest of the definitions and a deep study on their relationship.

Proposition 62. \mathfrak{a}_* is the minimum size of an ad family such that $\mathcal{F}(\mathcal{A})$ is not α_3 .

Proof. Let $\kappa < \mathfrak{a}_*$ and be \mathcal{A} ad family of cardinality κ . As we want to prove that $\mathcal{F}(\mathcal{A})$ is α_3 , it is enough to prove that ∞ is an α_3 point since any other point has countable character. Let $\{S_n \mid n \in \omega\}$ be a collection of convergent sequences to ∞ . We can assume that S_n only takes values in ω for every $n \in \omega$, since any sequence in \mathcal{A} converges to ∞ . As S_n is convergent to ∞ , $S_n \in \mathcal{A}^\perp$ for every $n \in \omega$. Since $|\mathcal{A}| < \mathfrak{a}_*$, there exists $B \in [\omega]^\omega$ such that $S_n \cap B$ is infinite for infinitely many S_n

and $A \cap B$ is finite for every $A \in \mathcal{A}$. Then B is the required convergent sequence to ∞ .

Conversely, let \mathcal{A} be an ad family such that $\mathcal{F}(\mathcal{A})$ is α_3 . We want to prove that \mathcal{A} is not a witness of \mathfrak{a}_* , so let $C_0, C_1, C_2, \dots \in \mathcal{A}^\perp$. Note that every C_n is a convergent sequence to ∞ and as ∞ is an α_3 -point, there exists a sequence $B \subseteq \Psi(\mathcal{A})$ that converges to ∞ and such that $|C_n \cap B| = \omega$ for infinitely many $n \in \omega$. We can assume that $B \subseteq \omega$ since otherwise $B' = B \cap (\bigcup_{i \in \omega} C_i)$ is also a witness for the α_3 -property for the family $\{C_i \mid i \in \omega\}$. As B converges to ∞ , we also have that $B \cap A =^* \emptyset$ for all $A \in \mathcal{A}$, so \mathcal{A} is not a witness for \mathfrak{a}_* as we wanted to show. \square

Following the idea of the previous proof, we can actually show that 2-Ramsey ad families are α_3 . Given a non α_3 ad family \mathcal{A} and a family $\{S_n : n \in \omega\} \subseteq \mathcal{A}^\perp$ that witnesses this, we can define $f : [\omega]^2 \rightarrow \omega$ by $f(\{n, m\}) = S_n(m)$, where $S_n(m)$ is the m^{th} element of S_n . Thus \mathcal{A} and $\mathcal{A}_0 = \{S_n \mid n \in \omega\}$ satisfy the hypothesis of Lemma 27 for $n = 2$ and $\mathcal{F}(\mathcal{A})$ is not 2-Ramsey.

In general, it is possible to prove that 2-Ramsey spaces are α_3 . This result was independently proved by Szeptycki, Almontashery and Memarpanahi. Moreover, they also proved that 2-Ramsey spaces do not need to be α_2 and that α_1 spaces need not to be 2-Ramsey.

Theorem 63. *If X is 2-Ramsey, then X is α_3 .*

Proof. Let $x \in X$ and for every $n \in \omega$ let $Y_n = \{x(n, m) : m \in \omega\}$ be a sequence converging to x . Define $f : [\omega]^2 \rightarrow X$ by $f(\{n, m\}) = x(n, m)$, where $n < m$. Since X is 2-Ramsey, there exists $B \in [\omega]^\omega$ such that $f \upharpoonright [B]^2$ converges.

Let $Y := f''[B]^2 \in [X]^\omega$. It is clear that $Y \cap f''_n \omega$ is infinite for every $n \in B$, so we are done if we prove that Y is convergent to x . Let U be an open set such that $x \in U$. Since $f \upharpoonright [B]^2$ converges to x , we know that there exists $m \in \omega$ such that $f''[B \setminus m]^2 \subseteq U$. It follows that $Y \setminus U \subseteq \bigcup_{n \in B \cap m} Y_n$. But $\bigcup_{n \in B \cap m} Y_n$ is a convergent sequence to x , so $\bigcup_{n \in B \cap m} Y_n \setminus U$ is finite, then

$$Y \setminus U \subseteq \left(\bigcup_{n \in B \cap m} Y_n \right) \setminus U$$

and the last is finite. We conclude that Y converges to x . \square

It is also worth to mention that the previous proof follows the same idea as the proof that the spaces with the Ramsey property considered by Knaust [35] are α_3 . Recall that a space X has the Ramsey property if whenever $\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} x(i, j) = x$, there exists $B \in [\omega]^\omega$ such that $f \upharpoonright [B]^2$ converges to x , where $f(\{i, j\}) = x(i, j)$.

Corollary 64. *If \mathcal{A} is an ad family such that $\mathcal{F}(\mathcal{A})$ is 2-Ramsey, then $\mathcal{F}(\mathcal{A})$ is α_3 .*

In [21] it is proved that $\mathfrak{b} \leq \mathfrak{a}_* \leq \mathfrak{a}$ and Corollary 64 implies that $\mathfrak{a}_2(\text{Ramsey}) \leq \mathfrak{a}_*$, so we have that $\mathfrak{b} = \lambda_2 \leq \mathfrak{a}_2(\text{Ramsey}) \leq \mathfrak{a}_* \leq \mathfrak{a}$. Of course, the inequality $\lambda_n \leq \mathfrak{a}_n(\text{Ramsey})$ is clear from the definition. This is pretty much what we know so far about these cardinals.

Question 65. *Are there any other relations between the cardinals λ_n and $\mathfrak{a}_n(\text{Ramsey})$ with \mathfrak{b} and \mathfrak{a} ?*

Question 66. *Is it consistent that $\mathfrak{a}_2(\text{Ramsey}) < \mathfrak{a}_*$?*

Question 67. *Are all the λ_n and $\mathfrak{a}_m(\text{Ramsey})$ consistently different? In particular, is $\mathfrak{a}_2(\text{Ramsey}) < \dots < \mathfrak{a}_n(\text{Ramsey}) < \dots < \mathfrak{a}$ consistent?*

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REFERENCES

- [1] A.V. Arhangel'skii. The frequency spectrum of a topological space and the product operation. *Trans. Moscow Math. Soc.*, 40(2):163–200, 1981.
- [2] Bohuslav Balcar and Petr Simon. Disjoint refinement. In *Handbook of Boolean algebras, Vol. 2*, pages 333–388. North-Holland, Amsterdam, 1989.
- [3] Paweł Barbarski, Rafał Filipów, Nikodem Mrozek, and Piotr Szuca. When does the Katětov order imply that one ideal extends the other? *Colloq. Math.*, 130(1):91–102, 2013.
- [4] Andreas Blass. Reductions between cardinal characteristics of the continuum. In *Set theory (Boise, ID, 1992–1994)*, volume 192 of *Contemp. Math.*, pages 31–49. Amer. Math. Soc., Providence, RI, 1996.
- [5] Andreas Blass. *Combinatorial Cardinal Characteristics of the Continuum*, pages 395–489. Springer Netherlands, Dordrecht, 2010.
- [6] Mikołaj Bojańczyk, Eryk Kopczyński, and Szymon Toruńczyk. Ramsey's theorem for colors from a metric space. *Semigroup Forum*, 85(1):182–184, 2012.
- [7] Dennis K. Burke and Eric K. van Douwen. On countably compact extensions of normal locally compact M -spaces. In *Set-theoretic topology (Papers, Inst. Medicine and Math., Ohio Univ., Athens, Ohio, 1975–1976)*, pages 81–89. Academic Press, New York, 1977.
- [8] Jonathan Cancino-Manríquez, Michael Hrušák, and David Meza-Alcántara. Countable irresolvable spaces and cardinal invariants. *Topology Proc.*, 44:189–196, 2014.
- [9] C. Corral. Madness and weak forms of normality. *Acta Math. Hungar.*, 165(2):291–307, 2021.
- [10] César Corral and Michael Hrušák. Fréchet-like properties and almost disjoint families. *Topology Appl.*, 277:107216, 11, 2020.
- [11] Paul Erdős and Saharon Shelah. Separability properties of almost-disjoint families of sets. *Israel J. Math.*, 12:207–214, 1972.
- [12] Paul Erdős and Richard Rado. A combinatorial theorem. *Journal of the London Mathematical Society*, 1(4):249–255, 1950.
- [13] David Fernández-Bretón and Michael Hrušák. A parametrized diamond principle and union ultrafilters. *Colloq. Math.*, 153(2):261–271, 2018.
- [14] David J. Fernández-Bretón and Michael Hrušák. Gruff ultrafilters. *Topology Appl.*, 210:355–365, 2016.
- [15] Fred Galvin and Petr Simon. A Čech function in ZFC. *Fundamenta Mathematicae*, 193:181–188, 2007.
- [16] Sergio A. Garcia-Balan and Paul J. Szeptycki. Weak normality properties in Ψ -spaces. *Fund. Math.*, 258(2):137–151, 2022.
- [17] Miguel Ángel Gaspar-Arreola, Fernando Hernández-Hernández, and Michael Hrušák. Scattered spaces from weak diamonds. *Israel J. Math.*, 225(1):427–449, 2018.
- [18] O. Guzmán, M. Hrušák, V. O. Rodrigues, S. Todorčević, and A. H. Tomita. Maximal almost disjoint families and pseudocompactness of hyperspaces. *Topology Appl.*, 305:Paper No. 107872, 24, 2022.
- [19] Osvaldo Guzmán. On completely separable mad families.
- [20] Osvaldo Guzmán. There is a $+$ -Ramsey mad family. *Israel J. Math.*, 229(1):393–414, 2019.
- [21] Osvaldo Guzmán-González, Michael Hrušák, Carlos Azarel Martínez-Ranero, and Ulises Ariet Ramos-García. Generic existence of MAD families. *J. Symb. Log.*, 82(1):303–316, 2017.
- [22] Osvaldo Guzmán-González and David Meza-Alcántara. Some structural aspects of the Katětov order on Borel ideals. *Order*, 33(2):189–194, 2016.
- [23] Stephen H. Hechler. Classifying almost-disjoint families with applications to $\beta N - N$. *Israel J. Math.*, 10:413–432, 1971.

- [24] F. Hernández-Hernández and M. Hrušák. Topology of Mrówka-Isbell spaces. In *Pseudocompact topological spaces*, volume 55 of *Dev. Math.*, pages 253–289. Springer, Cham, 2018.
- [25] Fernando Hernández-Hernández and Michael Hrušák. Q -sets and normality of Ψ -spaces. volume 29, pages 155–165. 2005. Spring Topology and Dynamical Systems Conference.
- [26] Fernando Hernández-Hernández and Michael Hrušák. Cardinal invariants of analytic P -ideals. *Canad. J. Math.*, 59(3):575–595, 2007.
- [27] Michael Hrušák. Almost disjoint families and topology. In *Recent Progress in General Topology III*, pages 601–638. Springer, 2014.
- [28] Michael Hrušák and Petr Simon. Completely separable mad families. In *Open problems in topology II*, pages 179–184. Elsevier, 2007.
- [29] M. Hrušák and U. A. Ramos-García. Malykhin’s problem. *Adv. Math.*, 262:193–212, 2014.
- [30] Michael Hrušák. Combinatorics of filters and ideals. In *Set theory and its applications*, volume 533 of *Contemp. Math.*, pages 29–69. Amer. Math. Soc., Providence, RI, 2011.
- [31] Michael Hrušák, Fernando Hernández-Hernández, and Iván Martínez-Ruiz. Pseudocompactness of hyperspaces. *Topology Appl.*, 154(17):3048–3055, 2007.
- [32] Michael Hrušák, David Meza-Alcántara, and Hiroaki Minami. Pair-splitting, pair-reaping and cardinal invariants of F_σ -ideals. *J. Symbolic Logic*, 75(2):661–677, 2010.
- [33] Anastasis Kamburelis and B. Węglorz. Splittings. *Archive for mathematical logic*, 35(4):263–277, 1996.
- [34] M. Katětov. On descriptive classification of functions. In *General topology and its relations to modern analysis and algebra, III (Proc. Third Prague Topological Sympos., 1971)*, pages 235–242. Academia, Prague, 1972.
- [35] H. Knaust. Angelic spaces with the Ramsey property. In *Interaction between functional analysis, harmonic analysis, and probability (Columbia, MO, 1994)*, volume 175 of *Lecture Notes in Pure and Appl. Math.*, pages 267–273. Dekker, New York, 1996.
- [36] Helmut Knaust. Array convergence of functions of the first Baire class. *Proc. Amer. Math. Soc.*, 112(2):529–532, 1991.
- [37] Wiesław Kubiś and Paul Szeptycki. On a topological Ramsey theorem. *Canadian Mathematical Bulletin*, pages 1–10, 2021.
- [38] Adam Kwela. On a conjecture of Debs and Saint Raymond. *Fund. Math.*, 260(1):67–76, 2023.
- [39] Miklós Laczkovich and Ireneusz Reclaw. Ideal limits of sequences of continuous functions. *Fund. Math.*, 203(1):39–46, 2009.
- [40] Heike Mildenberger, Dilip Raghavan, and Juris Steprāns. Splitting families and complete separability. *Canad. Math. Bull.*, 57(1):119–124, 2014.
- [41] Justin Moore, Michael Hrušák, and Mirna Džamonja. Parametrized \diamond principles. *Transactions of the American Mathematical Society*, 356(6):2281–2306, 2004.
- [42] C. Morgan and S. G. Da Silva. Covering properties which, under weak diamond principles, constrain the extents of separable spaces. *Acta Math. Hungar.*, 128(4):358–368, 2010.
- [43] The pi Base Community. pi-Base/data.
- [44] Dilip Raghavan. The density zero ideal and the splitting number. *Ann. Pure Appl. Logic*, 171(7):102807, 15, 2020.
- [45] Dilip Raghavan and Saharon Shelah. Two inequalities between cardinal invariants. *Fund. Math.*, 237(2):187–200, 2017.
- [46] Dilip Raghavan and Juris Steprāns. On weakly tight families. *Canad. J. Math.*, 64(6):1378–1394, 2012.
- [47] Vinicius de Oliveira Rodrigues and Artur Hideyuki Tomita. Small MAD families whose Isbell-Mrówka space has pseudocompact hyperspace. *Fund. Math.*, 247(1):99–108, 2019.
- [48] Haskell P. Rosenthal. Some remarks concerning unconditional basic sequences. In *Texas functional analysis seminar 1982–1983 (Austin, Tex.)*, Longhorn Notes, pages 15–47. Univ. Texas Press, Austin, TX, 1983.
- [49] Saharon Shelah. MAD saturated families and SANE player. *Canad. J. Math.*, 63(6):1416–1435, 2011.
- [50] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in topology*. Dover Publications, Inc., Mineola, NY, 1995. Reprint of the second (1978) edition.
- [51] Stevo Todorćević. *Introduction to Ramsey Spaces (AM-174)*. Princeton University Press, 2010.
- [52] Eric K. van Douwen. The integers and topology. In *Handbook of set-theoretic topology*, pages 111–167. North-Holland, Amsterdam, 1984.

- [53] Peter Vojtas. *Generalized Galois-Tukey-connections between explicit relations on classical objects of real analysis*. Freie Universität Berlin. Fachbereich Mathematik, 1991.
- [54] Stephen Willard. *General topology*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

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