On $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{s}$

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The *cardinal invariants of the continuum* are uncountable cardinals whose size is at most the cardinality of the real numbers. We are mostly interested in cardinals with a nice topological or combinatorial definition.

1. By ω we denote the set (cardinal) of the natural numbers.
2. By $\mathfrak{c}$ we denote the cardinality of the real numbers.

1. The cardinal invariants of the continuum are cardinals $\mathfrak{j}$ such that:

$$\omega < \mathfrak{j} \leq \mathfrak{c}$$

2. The *Continuum Hypothesis* (*CH*) is the following statement:

$\mathfrak{c}$ is the first uncountable cardinal

3. All cardinal invariants are $\mathfrak{c}$ under *CH.*
4. *Martin's Axiom (MA)* implies that most cardinal invariants are $\mathfrak{c}$.

The point is that the value of $\mathfrak{c}$ does not determine many of the combinatorial and topological properties of the "reals" $(\wp(\omega), 2^{\omega}, \omega^{\omega}, \mathbb{R}...)$. Let's look at two models where $\mathfrak{c} = \omega_2$.

The Sacks model	**A model of** PFA
There is a non-meager set of size ω_1	Every set of size ω_1 is meager
There is a non-null set of size ω_1	Every set of size ω_1 has measure zero
ω^ω can be covered with ω_1-many meager sets	Union of ω_1-many meager sets is meager
$\mathbb{R}$ can be covered with ω_1-many null sets	Union of ω_1-many null sets has measure zero

In both models we have that $\mathfrak{c} = \omega_2$, however, the structure and properties of the reals are very different in those models. The value of the cardinal invariants in a model provide us a lot of information regarding the reals in such model.

Many of the cardinal invariants can be seen as the first moment where a "diagonalization argument fails". With this knowledge, we can carry some of the previous known constructions using CH to a different model.

Let $f, g \in \omega^\omega$, define $f \leq^* g$ if and only if $f(n) \leq g(n)$ holds for all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$. We say that $\mathcal{D} \subseteq \omega^\omega$ is *dominating* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leq^* g$.

Definition

The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family.

Definition

The *dominating number* $\mathfrak{d}$ is the size of the smallest of a dominating family.

Clearly, we have that $\mathfrak{b} \leq \mathfrak{d}$.

Lemma

$\mathfrak{b}$ *is uncountable.*

Proof.

We need to show that every countable subset of ω^ω is bounded. Let $\mathcal{B} = \{f_n \mid n \in \omega\}$, define $g \in \omega^\omega$ given by $g(n) = f_0(n) + ... + f_n(n)$. It is easy to see that g bounds $\mathcal{B}$. $\square$

Obviously, the whole ω^ω is unbounded, so we get:

Corollary

$\omega < \mathfrak{b} \leq \mathfrak{c}$.

Definition

An infinite family $\mathcal{A} \subseteq [\omega]^{\omega}$ is *almost disjoint (AD)* if the intersection of any two different elements of $\mathcal{A}$ is finite. A MAD *family* is a maximal almost disjoint family.

Note that MAD families exists under the Axiom of Choice (in fact, every AD family can be extended to a MAD family). There are models of ZF where there is no MADness.

Lemma (Sierpiński)

There is a MAD family of size $\mathfrak{c}$.

Proof.

For every $f \in 2^\omega$, let $\widehat{f} = \{f \upharpoonright n \mid n \in \omega\}$. Note that $\mathcal{A} = \{\widehat{f} \mid f \in 2^\omega\}$ is an AD family of size $\mathfrak{c}$. We just need to take any MAD family extending $\mathcal{A}$. □

A major (vague) question regarding the study of MAD families is the following:

Problem (Simon)

Is there an "essentially different" construction of a MAD family?

Definition

The *almost disjointness number* $\mathfrak{a}$ is the smallest size of a MAD family.

Proposition

$\mathfrak{b} \leq \mathfrak{a}$.

Proof.

Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \kappa\}$ be a MAD family, we will construct an unbounded family of size κ. We may assume that $\{A_n \mid n \in \omega\}$ are disjoint and $\omega = \bigcup_{n \in \omega} A_n$. The idea is to view the $\{A_n \mid n \in \omega\}$ as the "columns" of $\omega \times \omega$. For every $\omega \leq \alpha < \kappa$ define $f_\alpha : \omega \longrightarrow \omega$ such that $f_\alpha(n) = max(A_n \cap A_\alpha) + 1$ ($f_\alpha(n) = 0$ if $A_n \cap A_\alpha$ are disjoint). It follows that $\mathcal{B} = \{f_\alpha \mid \omega \leq \alpha < \kappa\}$ is an unbounded family. $\square$

In particular, we get that $\mathfrak{a}$ is another example of a cardinal invariant of the continuum.

We say that S *splits* X if $S \cap X$ and $X \setminus S$ are both infinite. A family $\mathcal{S} \subseteq [\omega]^{\omega}$ is a *splitting family* if for every $X \in [\omega]^{\omega}$ there is $S \in \mathcal{S}$ such that S splits X.

Definition

The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family.

Note that $[\omega]^{\omega}$ is a splitting family.

Fact

$\omega < \mathfrak{s} \leq \mathfrak{c}$.

We now have the invariants $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{s}$. We know that $\mathfrak{b} \leq \mathfrak{a}$, is there any other relation? They are all the same under CH, is it possible for them to be different?

Definition

Let W a forcing extension of V.

1. Let $f : \omega \longrightarrow \omega$ with $f \in W$. We say that f is a *dominating real over* V if $g \leq^* f$ for every $g \in V$.
2. Let $S \in [\omega]^\omega$ with $S \in W$. We say that S is an *unsplitted real over* V if for every $A \in [\omega]^\omega \cap V$ either $S \subseteq^* A$ or $S \subseteq^* \omega \setminus A$.

1. There is a dominating real over V in W if and only if $\omega^\omega \cap V$ is not unbounded in W.
2. There is an unsplitted real over V in W if and only if $[\omega]^\omega \cap V$ is not splitting in W.

Theorem (Dordal, Baumgartner)

It is consistent that $\mathfrak{s} < \mathfrak{b}$.

Such inequality holds in the Hechler model. Hechler forcing adds dominating reals. Dordal and Baumgartner showed that it does not add unsplitted reals. Alternatively, we can use the following result:

Theorem (Judah,Shelah)

Let $\mathbb{P}$ *be a Suslin ccc forcing.* $\mathbb{P}$ *does not add an unsplitted real (even in the iteration).*

Definition

We say that a forcing $\mathbb{P} = (P, \leq, \perp)$ is *Suslin ccc* if $\mathbb{P}$ is ccc, $P \subseteq \omega^\omega$ is analytic and $\leq, \perp$ are analytic relations.

(Note that if T is an Aronszajn tree with no uncountable antichain, then T is Suslin, ccc, but it is not Suslin ccc).

Iterating any Suslin ccc forcing that adds a dominating real, will give us a model of $\mathfrak{s} < \mathfrak{b}$. This inequality also holds in the Laver model (this was proved by Alan Dow).

The consistency of $\omega_1 = \mathfrak{b} < \mathfrak{s}$ and $\omega_1 = \mathfrak{b} < \mathfrak{a}$ are much harder. In this tutorial we will explain how to force them. The consistency of both of them was first proved by Shelah.

There is a huge work regarding these cardinal invariants. We would like to make some historic comments in here. As was mentioned before, the story began when Shelah constructed models of $\omega_1 = \mathfrak{b} < \mathfrak{s}$ and $\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s}$. Shelah used a countable support iteration of creature forcings. Dow constructed a model where $\mathfrak{b} = \omega_1$ and every compact countably tight space of weight ω_1 is Fréchet (which implies that $\mathfrak{b} < \mathfrak{a}$). Brendle used ccc forcings for constructing models of $\kappa = \mathfrak{b} < \mathfrak{a} = \kappa^+$ where κ is any uncountable regular cardinal. Fischer and Steprāns constructed models of $\kappa = \mathfrak{b} < \mathfrak{s} = \kappa^+$ where κ is any uncountable regular cardinal.

After that, Brendle and Fischer used matrix iterations to prove that for any regular cardinals $\kappa < \lambda$, it is consistent that $\kappa = \mathfrak{b} = \mathfrak{a} < \mathfrak{s} = \lambda$ and if κ is bigger than a measurable cardinal, then it is consistent that $\kappa = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \lambda$. Brendle and Raghavan find a decomposition of the original forcing of Shelah, which we will use in the tutorial. The consistency of $\omega_2 = \mathfrak{d} < \mathfrak{a} = \omega_3$ and $\omega_1 < \mathfrak{u} < \mathfrak{a}$ was obtained by Shelah when he developed the technique of forcing along a template. Much later, Fischer and Mejía proved that it is consistent that $\omega_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{a}$. The consistency of $\omega_1 = \mathfrak{u} < \mathfrak{a}$ was recently proved by Guzmán and Kalajdzievski.

There are still many interesting open questions remaining:

Problem (Roitman)

Does $\mathfrak{d} = \omega_1$ *imply that* $\mathfrak{a} = \omega_1$*?*

Problem (Brendle and Raghavan)

Does $\mathfrak{b} = \mathfrak{s} = \omega_1$ *imply that* $\mathfrak{a} = \omega_1$*?*

Note that a positive solution to the question of Brendle and Raghavan would provide a positive solution to the problem of Roitman.

We say that a family $\mathcal{F} \subseteq \wp(\omega)$ is a *filter* if the following conditions hold:

1. $\omega \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
3. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.
4. $\mathcal{F} \cap [\omega]^{<\omega} = \emptyset$.

The concept of a filter formalizes a kind of "largeness" notion, the elements which belong to the filter are regarded as large, while its complements are regarded as small. An *ultrafilter* is a maximal filter.

By $\mathcal{F}^+$ we will denote the collection of $\mathcal{F}$*-positive sets*: $A \in \mathcal{F}^+$ if $|A \cap B| = \omega$ for every $B \in \mathcal{F}$.

We say that $\mathcal{I} \subseteq \wp(X)$ is an *ideal on* X if the following conditions hold:

Definition

1. $X \notin \mathcal{I}$ and $\emptyset \in \mathcal{I}$.
2. If $A \in \mathcal{I}$ and $B \subseteq^* A$ then $B \in \mathcal{I}$.
3. If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

If $\mathcal{I}$ is an ideal, define the *dual filter* $\mathcal{I}^* = \{\omega \setminus A \mid A \in \mathcal{I}\}$.

Definition

Let $\mathbb{P}$ be a partial order and $\mathcal{F}$ a filter (on ω). We say that $\mathbb{P}$ *diagonalizes* $\mathcal{F}$ if $\mathbb{P}$ adds an infinite set almost contained in every element of $\mathcal{F}$ (such set is called a *pseudointersection of* $\mathcal{F}$).

We want a model of $\mathfrak{b} < \mathfrak{s}$. In order to do that, we must find a forcing that adds an unsplitted real and does not add dominating reals.

Lemma

Let $\mathbb{P}$ be a partial order and $\mathcal{U}$ an ultrafilter. If $\mathbb{P}$ diagonalizes $\mathcal{U}$, then $\mathbb{P}$ adds an unsplitted real.

We need to find a way to diagonalize a ultrafilter without adding dominating reals.

Problem

Let $\mathcal{F}$ be a filter. How can we diagonalize it?

There are several ways to do that. There is currently a lot of research on finding different forcing for diagonalizing filters. We will now introduce the Mathias forcing, which is a very natural way to do it.

Many of the proofs of the results in this tutorial, can be consulted in the papers "Canjar Filters II" by Hrušák, Guzmán and Martínez, or "The ultrafilter and almost disjointness numbers" by Guzmán and Kalajdzievski.

Definition

If $\mathcal{F}$ is a filter on ω (or on any countable set) we define the *Mathias forcing* $\mathbb{M}(\mathcal{F})$ *with respect to* $\mathcal{F}$ as the set of all pairs (s, A) where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{F}$. If $(s, A), (t, B) \in \mathbb{M}(\mathcal{F})$ then $(s, A) \leq (t, B)$ if the following conditions hold:

1. t is an initial segment of s.
2. $A \subseteq B$.
3. $(s \setminus t) \subseteq B$.

Lemma

Let $\mathcal{F}$ be a filter. $\mathbb{M}(\mathcal{F})$ is a ccc forcing that diagonalizes $\mathcal{F}$.

In this way, if $\mathcal{U}$ is an ultrafilter, then $\mathbb{M}(\mathcal{U})$ will add an unsplitted real. In order to build a model of $\mathfrak{b} < \mathfrak{s}$, we want to find an ultrafilter whose Mathias does not add dominating reals (the situation is more subtle that this, we will come back to this later).

Exercise

If $\mathcal{U}$ is not a P-point, then $\mathbb{M}(\mathcal{U})$ adds a dominating real.

Exercise

If $\mathcal{U}$ is a Q-point, then $\mathbb{M}(\mathcal{U})$ adds a dominating real.

Is it possible for $\mathbb{M}(\mathcal{F})$ to not add a dominating real?

Yes! it is possible. If $\mathcal{F}$ is the cofinite filter, then $\mathbb{M}(\mathcal{F})$ is countable, so it is equivalent to Cohen forcing, hence it does not add a dominating real. A more interesting example is the following:

Theorem (Canjar)

$\mathfrak{d} = \mathfrak{c}$ implies that there is a P-point $\mathcal{U}$ such that $\mathbb{M}(\mathcal{U})$ does not add dominating reals.

Definition

We say that a filter $\mathcal{F}$ is *Canjar* if $\mathbb{M}(\mathcal{F})$ does not add a dominating real.

It will be convenient to find a combinatorial characterization of the previous notion. We will need the following notion:

Definition

Let $\mathcal{F}$ be a filter on ω. Define the filter $\mathcal{F}^{<\omega}$ in $[\omega]^{<\omega} \setminus \{\emptyset\}$ as the filter generated by $\left\{[A]^{<\omega} \setminus \{\emptyset\} \mid A \in \mathcal{F}\right\}$.

However, we will only care about the positive sets for $\mathcal{F}^{<\omega}$:

Fact

If $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$, then $X \in (\mathcal{F}^{<\omega})^{+}$ if and only if for every $A \in \mathcal{F}$, there is $s \in X$ such that $s \subseteq A$.

Theorem

Let $\mathcal{F}$ be a filter on ω. The following are equivalent:

1. *$\mathcal{F}$ is Canjar.*
2. *(Hrušák, Minami) For every $\{X_n \mid n \in \omega\} \subseteq \left(\mathcal{F}^{<\omega}\right)^+$ there are $Y_n \in [X_n]^{<\omega}$ such that $\bigcup\limits_{n\in\omega} Y_n \in \left(\mathcal{F}^{<\omega}\right)^+$.*
3. *(Chodounský, Repovš and Zdomskyy) $\mathcal{F}$ is Menger (as a subspace of $\wp(\omega) \simeq 2^\omega$).*

By the theorem of Canjar, under CH there is a Canjar ultrafilter. A natural attempt to build a model of $\mathfrak{b} < \mathfrak{s}$ would be the following:

We start with a model of CH. Now, we perform a finite or countable support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \omega_2 \rangle$ where $\mathbb{P}_\alpha \Vdash \mathbb{Q}_\alpha = \mathbb{M}(\dot{\mathcal{U}}_\alpha)$ where $\dot{\mathcal{U}}_\alpha$ is a $\mathbb{P}_\alpha$-name of a Canjar ultrafilter (which exists since at every intermediate model we have CH).

Unfortunately, this simple approach may not work. Let see why:

1. Since $V \models \mathrm{CH}$, we can find a Canjar ultrafilter $\mathcal{U}_0$. Let $G_0 \subseteq \mathbb{M}(\mathcal{U}_0)$ be a $(V, \mathbb{M}(\mathcal{U}_0))$-generic filter and let $V_1 = V[G_0]$. In V_1, there are no dominating reals over V.
2. Since $V_1 \models \mathrm{CH}$, we can find a Canjar ultrafilter $\mathcal{U}_1$. Let $G_1 \subseteq \mathbb{M}(\mathcal{U}_1)$ be a $(V_1, \mathbb{M}(\mathcal{U}_1))$-generic filter and let $V_2 = V_1[G_1]$. In V_2, there are no dominating reals over V_1 but... we do not know if there are dominating reals over V!!

The fact that $\mathcal{U}_1$ is Canjar only allows us to conclude that we do not add dominating reals to V_1, but we do not know anything about V. We need to be more careful.

The previous situation is not surprising at all, since the following is false:

If $\mathbb{P}$ does not add dominating reals and $\mathbb{P}$ forces that "$\mathbb{Q}$ does not add dominating reals", then $\mathbb{P} * \mathbb{Q}$ does not add dominating reals.

(Take $\mathbb{P}$ the forcing for adding ω_1-Cohen reals and $\mathbb{Q}$ to be the Hechler forcing of V, the proof of this fact is int eh Handbook article by Uri Abraham).

We will say that a family of functions $\mathcal{B} \subseteq \omega^{\omega}$ is a $\mathfrak{b}$-*family* if the following holds:

1. Every element of $\mathcal{B}$ is an increasing function.
2. Given $\{f_n \mid n \in \omega\} \subseteq \mathcal{B}$ there is $g \in \mathcal{B}$ such that $f_n \leq^* g$ for every $n \in \omega$.
3. $\mathcal{B}$ is unbounded.

An example of a $\mathfrak{b}$-family would be a well-ordered unbounded family, another example is the set of all increasing functions.

Definition

If $\mathcal{B}$ is a $\mathfrak{b}$-family and $\mathbb{P}$ is a partial order, we say that $\mathbb{P}$ *preserves* $\mathcal{B}$ if $\mathcal{B}$ is still unbounded after forcing with $\mathbb{P}$.

Note that if $\mathbb{P}$ is a proper forcing that preserves $\mathcal{B}$, then $\mathcal{B}$ is still a $\mathfrak{b}$-family in the extension.

Definition

Let $\mathcal{B}$ be a $\mathfrak{b}$-family and $\mathcal{F}$ a filter. We say that $\mathcal{F}$ is *$\mathcal{B}$-Canjar* if $\mathbb{M}(\mathcal{F})$ preserves $\mathcal{B}$.

Note that if $\mathcal{F}$ is $\mathcal{B}$-Canjar (for some $\mathfrak{b}$-family $\mathcal{B}$), then $\mathcal{F}$ is Canjar. As expected, $\mathcal{B}$-Canjar filters have a similar characterization as the one of Canjar.

Given a sequence $\overline{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ and $f \in \omega^\omega$, we define the set $\overline{X}_f = \bigcup_{n\in\omega} (X_n \cap \wp(f(n)))$. Note that $\mathcal{F}$ is *Canjar* if if for every sequence $\overline{X} = \{X_n \mid n \in \omega\} \subseteq (\mathcal{F}^{<\omega})^+$ there is $f \in \omega^\omega$ such that $\overline{X}_f \in (\mathcal{F}^{<\omega})^+$.

Lemma (G., Hrušák, Martínez)

Let $\mathcal{B} \subseteq \omega^\omega$ be a $\mathfrak{b}$-family and $\mathcal{F}$ a filter. $\mathcal{F}$ is $\mathcal{B}$-Canjar if and only if for every sequence $\overline{X} = \{X_n \mid n \in \omega\} \subseteq (\mathcal{F}^{<\omega})^+$ there is $f \in \mathcal{B}$ such that $\overline{X}_f \in (\mathcal{F}^{<\omega})^+$.

Let W be a forcing extension of V. We say that a filter $\mathcal{F} \in W$ is *V-Canjar* if $\mathcal{F}$ is $(\omega^\omega \cap V)$-Canjar.

In order to build a model of $\mathfrak{b} < \mathfrak{s}$, we can perform an iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \omega_2 \rangle$ such that $\mathbb{P}_\alpha \Vdash \mathbb{Q}_\alpha = \mathbb{M}(\dot{\mathcal{U}}_\alpha)$ where $\dot{\mathcal{U}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a V-Canjar ultrafilter (for the moment, do not worry about limit steps).

In order to do this, we need to find a way to construct V-Canjar ultrafilters. Can even find a V-Canjar filter?

Yes, the cofinite filter is V-Canjar since its Mathias is Cohen forcing. Is there anything more interesting?

Theorem (Brendle)

If $\mathcal{F}$ is an F_σ-filter, then $\mathcal{F}$ is $\mathcal{B}$-Canjar for every $\mathfrak{b}$-family $\mathcal{B}$.

We will sketch a proof of the theorem of Brendle. But first,we need some preliminary remarks.

Given X a collection of finite non empty-subsets of ω, we define $\mathcal{C}(X) = \{A \subseteq \omega \mid \forall s \in X\,(s \cap A \neq \emptyset)\}$.

Lemma

Let $\mathcal{F}$ be a filter, $\mathcal{D} \subseteq \mathcal{F}$ be a compact set and $X \in (\mathcal{F}^{<\omega})^{+}$.

1. *$\mathcal{C}(X)$ is a compact set.*
2. *There is $Y \in [X]^{<\omega}$ such that for every $A \in \mathcal{D}$ there is $s \in Y$ such that $s \subseteq A$.*

We will now prove the theorem of Brendle. Let $\mathcal{F}$ be an F_σ-filter, $\mathcal{B} \subseteq \omega^\omega$ a $\mathfrak{b}$-family. Let $\mathcal{F} = \bigcup\limits_{n\in\omega} C_n$ such that each C_n is compact and $C_n \subseteq C_{n+1}$ for every $n \in \omega$ (recall that $\wp(\omega)$ is compact, so closed sets are compact).

Let $\overline{X} = \{X_n \mid n \in \omega\} \subseteq \left(\mathcal{F}^{<\omega}\right)^+$. We can now define a function $g : \omega \longrightarrow \omega$ such that for every $n \in \omega$, the following holds:

$$\text{Every } A \in C_n \text{ contains an element of } X_n \cap \wp(g(n))$$

Since $\mathcal{B}$ is unbounded, there is $f \in \mathcal{B}$ such that $f \nleq^* g$. It follows that $\overline{X}_f \in \left(\mathcal{F}^{<\omega}\right)^+$.

In this way, F_σ-filters are Canjar. It turns out that in the definable world, there is nothing more:

Theorem (Chodounský, Repovš, Zdomskyy and G., Hrušák, Martínez)

Let $\mathcal{F}$ be an analytic filter. $\mathcal{F}$ is Canjar if and only if $\mathcal{F}$ is F_σ.

Great!!! We just need an F_σ-ultrafilter... but there are no such things. However, we can build an *ultrafilter using F_σ-pieces.*

The following forcing was introduced by Laflamme:

Definition

Let $\mathbb{F}_\sigma$ be the collection of all F_σ-filters on ω. If $\mathcal{F}, \mathcal{G} \in \mathbb{F}_\sigma$, define $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{G} \subseteq \mathcal{F}$.

1. $\mathbb{F}_\sigma$ is a σ-closed forcing.
2. If $G \subseteq \mathbb{F}_\sigma$ is a generic filter, then $\mathcal{U}_{gen} = \bigcup G$ is an ultrafilter on ω.

We will prove that if $\mathcal{B}$ is a $\mathfrak{b}$-family, then $\mathbb{F}_\sigma$ forces that $\dot{\mathcal{U}}_{gen}$ is $\mathcal{B}$-Canjar. In this way, if we iterate ω_2-many times the forcing $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$ over a model of CH, we will get a model of $\mathfrak{b} < \mathfrak{s}$!

(Note: instead of forcing with $\mathbb{F}_\sigma$, it is possible to build a $\mathcal{B}$-Canjar ultrafilter by hand using CH).

This was not the original approach to get $\mathfrak{b} < \mathfrak{s}$. The original forcing of Shelah was a creature forcing using logarithmic measures. Later, Brendle and Raghavan proved that the forcing of Shelah is forcing equivalent to $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$. Laflamme was the first to note that $\dot{\mathcal{U}}_{gen}$ is forced to be a Canjar ultrafilter.

Lemma

If $\mathcal{U}$ is an ultrafilter and $Y \subseteq [\omega]^{<\omega}$ then $Y \in (\mathcal{U}^{<\omega})^+$ if and only if $\mathcal{C}(Y) \subseteq \mathcal{U}$.

Lemma

Let $\mathcal{F} \in \mathbb{F}_\sigma$ and $X \subseteq [\omega]^{<\omega}$, then $\mathcal{F} \Vdash X \in (\dot{\mathcal{U}}_{gen}{}^{<\omega})^+$ if and only if $\mathcal{C}(X) \subseteq \mathcal{F}$.

Lemma

Let $\mathcal{F}$ *be a filter,* $\mathcal{D} \subseteq \mathcal{F}$ *be a compact set and* $X \in \left(\mathcal{F}^{<\omega}\right)^{+}$ *such that* $\mathcal{C}\left(X\right) \subseteq \mathcal{F}$. *For every* $n \in \omega$ *there is* $S \in [X]^{<\omega}$ *such that if* $A_0, ..., A_n \in \mathcal{C}\left(S\right)$ *and* $F \in \mathcal{D}$ *then* $A_0 \cap ... \cap A_n \cap F \neq \emptyset$.

Theorem

If $\mathcal{B} \in V$ is a $\mathfrak{b}$-family, then $\mathbb{F}_\sigma$ forces that $\dot{\mathcal{U}}_{gen}$ is $\mathcal{B}$-Canjar.

Proof.

By the previous observation and since $\mathbb{F}_\sigma$ is σ-closed, it is enough to show that if $\mathcal{F} \Vdash \overline{X} = \langle X_n \rangle_{n \in \omega} \subseteq (\dot{\mathcal{U}}_{gen}^{<\omega})^+$ then there is $\mathcal{G} \leq \mathcal{F}$ and $f \in \mathcal{B}$ such that $\mathcal{C}\left(\overline{X}_f\right) \subseteq \mathcal{G}$.

Let $\mathcal{F} = \bigcup \mathcal{C}_n$ where each $\mathcal{C}_n$ is compact and they form an increasing chain. By a previous lemma, there is $g : \omega \longrightarrow \omega$ such that the following holds for every $n \in \omega$:

$$\text{If } F \in \mathcal{C}_n \text{ and } A_0, \ldots, A_n \in \mathcal{C}\left(X_n \cap \wp\left(g\left(n\right)\right)\right)$$
$$\text{then } A_0 \cap \ldots . \cap A_n \cap F \neq \emptyset.$$

$\square$

Proof.

Since $\mathcal{B}$ is unbounded, then there is $f \in \mathcal{B}$ such that $f \nleq^* g$. We claim that $\mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)$ generates a filter. Let $F \in \mathcal{C}_n$ and $A_0, \ldots, A_m \in \mathcal{C}\left(\overline{X}_f\right)$. We must show that $A_0 \cap \ldots . \cap A_m \cap F \neq \emptyset$. Since f is not bounded by g, we may find $r > n, m$ such that $f(r) > g(r)$. In this way, $A_0, \ldots, A_n \in \mathcal{C}\left(X_n \cap \wp\left(g\left(n\right)\right)\right)$ and then $A_0 \cap \ldots . \cap A_m \cap F \neq \emptyset$. Finally, we can define $\mathcal{G}$ as the filter generated by $\mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)$. □

Theorem (Shelah)

It is consistent that $\mathfrak{b} < \mathfrak{s}$.

Proof.

Start with $V \models \mathsf{CH}$ and let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \omega_2 \rangle$ be the countable support iteration where $\mathbb{P}_\alpha \Vdash \mathbb{Q}_\alpha = \mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$. By the previous result, we preserve the unboundedness of $V \cap \omega^\omega$ at every successor step. The limit steps are taken care of by an iteration theorem of Shelah. It follows that in the final model,we will have $\mathfrak{b} = \omega_1$ and $\mathfrak{s} = \omega_2$. □

We will now construct a model of $\mathfrak{b} < \mathfrak{a}$ (it can be proved that $\mathfrak{a} = \mathfrak{b}$ holds in the last model). In fact, in our model, we will have $\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \omega_2$. As was mentioned earlier, it is an open problem if $\omega_1 = \mathfrak{b} = \mathfrak{s} < \mathfrak{a}$ is consistent.

Definition

Let $\mathcal{A}$ be a MAD family and $\mathbb{P}$ a partial order. We say that $\mathbb{P}$ *destroys* $\mathcal{A}$ if $\mathcal{A}$ is no longer maximal after forcing with $\mathbb{P}$.

To get a model of $\mathfrak{b} < \mathfrak{a}$, we need to find a way to destroy MAD families without adding dominating reals.

Definition

Let $\mathcal{A}$ be an AD family. By $\mathcal{I}(\mathcal{A})$ we will denote the ideal generated by $\mathcal{A}$ and by $\mathcal{F}(\mathcal{A})$ its dual filter.

1. $X \in \mathcal{I}(\mathcal{A})$ if and only if there are $A_0, \ldots, A_n \in \mathcal{A}$ such that $X \subseteq^* A_0 \cup \ldots \cup A_n$.
2. $Y \in \mathcal{F}(\mathcal{A})$ if and only if there are $A_0, \ldots, A_n \in \mathcal{A}$ such that $(\omega \setminus A_0) \cap \ldots \cap (\omega \setminus A_n) \subseteq^* Y$.

Fact

Let $\mathcal{A}$ be a MAD family and $\mathbb{P}$ a partial order. The following are equivalent:

1. $\mathbb{P}$ *destroys* $\mathcal{A}$.
2. $\mathbb{P}$ *diagonalizes* $\mathcal{F}(\mathcal{A})$.

In order to get a model of $\mathfrak{b} < \mathfrak{a}$, we could try to iterate the forcings $\mathbb{M}\left(\mathcal{F}\left(\mathcal{A}\right)\right)$. Unfortunately, this simple approach can not work:

Theorem (Brendle)

$\mathfrak{b} = \mathfrak{c}$ *implies that there is a MAD family* $\mathcal{A}$ *such that* $\mathcal{F}\left(\mathcal{A}\right)$ *is not Canjar.*

Theorem (Chodounský, Repovš, Zdomskyy and G., Hrušák, Martínez)

There is (in ZFC) a MAD family $\mathcal{A}$ *such that* $\mathcal{F}\left(\mathcal{A}\right)$ *is not Canjar.*

So... What do we do?

A key observation is that if $\mathcal{F}$ and $\mathcal{G}$ are two filters with $\mathcal{F} \subseteq \mathcal{G}$ and a forcing $\mathbb{P}$ diagonalizes $\mathcal{G}$, then it will also diagonalize $\mathcal{F}$.

In this way, it could be possible that even if $\mathcal{F}(\mathcal{A})$ is not Canjar, there is a Canjar filter $\mathcal{G}$ such that $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{G}$. It turns out that this is true! (under CH). We take a similar approach as last time.

Definition

Let $\mathcal{A}$ be a MAD family. Define $\mathbb{F}_\sigma(\mathcal{A}) = \{\mathcal{F} \in \mathbb{F}_\sigma \mid \mathcal{F} \cap \mathcal{I}(\mathcal{A}) = \emptyset\}$. We order $\mathbb{F}_\sigma(\mathcal{A})$ by inclusion.

Lemma

Let $\mathcal{A}$ be a MAD family.

1. $\mathbb{F}_\sigma(\mathcal{A})$ *is a σ-closed forcing.*
2. *If $G \subseteq \mathbb{F}_\sigma$ is a generic filter, then $\mathcal{U}_{gen}(\mathcal{A}) = \bigcup G$ is an ultrafilter on ω.*
3. $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{U}_{gen}(\mathcal{A})$.
4. *If $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$, then $\mathcal{F} \Vdash X \in (\dot{\mathcal{U}}_{gen}(\mathcal{A})^{<\omega})^+$ if and only if $\mathcal{C}(X) \subseteq \langle \mathcal{F} \cup \mathcal{F}(\mathcal{A}) \rangle$ (where $\langle \mathcal{F} \cup \mathcal{F}(\mathcal{A}) \rangle$ is the filter generated by $\mathcal{F} \cup \mathcal{F}(\mathcal{A})$).*

In this way, we get that $\mathbb{F}_\sigma(\mathcal{A}) * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$ destroys $\mathcal{A}$. As expected, the hard part is proving that it does not add dominating reals.

As before, the original forcing of Shelah was a creature forcing. Brendle and Raghavan proved that the forcing of Shelah is equivalent to $\mathbb{C}_{\omega_1} * \mathbb{F}_\sigma(\mathcal{A}) * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$.

The following is an important property of MAD families that we will often use:

Lemma

Let $\mathcal{A}$ be a MAD family and $X \subseteq \omega$. The following are equivalent:

1. $X \in \mathcal{I}(\mathcal{A})^{+}$.
2. *There is $\mathcal{B} \in [\mathcal{A}]^{\omega}$ such that if $B \in \mathcal{B}$, then $|X \cap B| = \omega$.*

If $\mathcal{A}$ can be extended to an F_σ-ideal $\mathcal{J}$, then we can just use $\mathbb{M}(\mathcal{J})$ to destroy $\mathcal{A}$ without adding dominating reals. Thus, the hard part is destroying MAD families that can not be extended to an F_σ-ideal.

Definition

Let $\mathcal{A}$ be a MAD family. We say that $\mathcal{A}$ is *Laflamme* if $\mathcal{A}$ can not be extended to an F_σ-ideal.

Laflamme MAD families exist under $\mathfrak{p} = \mathfrak{c}$, it is unknown if they exist in ZFC.

Given $X \subseteq [\omega]^{<\omega}$ and $A \in [\omega]^{\omega}$, we define $Catch(X, A) = \{s \in X \mid s \subseteq A\}$.

Definition

Let $\mathcal{F}$ be an F_σ-filter, $X \subseteq [\omega]^{<\omega}$ and $A \in [\omega]^{\omega}$. We will say that $\bigstar(\mathcal{F}, X, A)$ holds, if the following conditions are satisfied:

1. $A \in \mathcal{F}^+$.
2. If $B \in [A]^{\omega} \cap \mathcal{F}^+$ then $Catch(X, B) \in (\mathcal{F}^{<\omega})^+$ (i.e. for every $F \in \mathcal{F}$ there is $s \in X$ such that $s \subseteq F \cap B$).

Lemma

Let $\mathcal{A}$ *be a Laflamme MAD family and* $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$. *For every family* $\{X_n \mid n \in \omega\}$ *such that* $\mathcal{C}(X_n) \subseteq \left\langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \right\rangle$, *there is a countable family* $\mathcal{D} \in [\mathcal{A}]^\omega$ *such that* $\bigstar(\mathcal{F}, A, X_n)$ *holds for every* $n \in \omega$ *and* $A \in \mathcal{D}$.

This is the hardest lemma. We will skip it for now and prove it later if there is time. You can read the proof in the paper "Canjar Filters II" by Hrušák, G. and Martínez.

Given $A \in [\omega]^{\omega}$ and $l \in \omega$ define $Part_l(A)$ as the set of all sequences $\langle B_1, ..., B_l \rangle$ such that $A = \bigcup\limits_{i \leq l} B_i$ and $B_i \cap B_j = \emptyset$ whenever $i \neq j$. Note that $Part_l(A_i)$ is a compact space with the natural topology. Also it is clear that if $A \in \mathcal{F}^+$ and $\langle B_1, ..., B_l \rangle \in Part_l(A)$ then there is $j \leq l$ such that $B_j \in \mathcal{F}^+$.

Lemma

Let $\mathcal{F}$ be a filter, $\mathcal{C} \subseteq \mathcal{F}$ a compact set and $X \in \left(\mathcal{F}^{<\omega}\right)^{+}$. Let A such that $\bigstar(A, \mathcal{F}, X)$ holds and let $l \in \omega$. There is $n \in \omega$ with the property that for all $\langle B_1, ..., B_l \rangle \in Part_l(A)$ there is $i \leq l$ such that for every $F \in \mathcal{C}$, then $X \cap \wp(B_i \cap n)$ contains a subset of F.

Proof.

Let U_n be the set of all $\langle B_1, ..., B_l \rangle \in Part_l(A)$ such that there is $i \leq l$ with the property that if $F \in \mathcal{C}$ then $X \cap \wp(B_i \cap n)$ contains a subset of F. Note that $\{U_n \mid n \in \omega\}$ is an open cover (recall that $\bigstar(A, \mathcal{F}, X)$ holds and if we split A into finitely many pieces, then one of the pieces must be in $\mathcal{F}^{+}$) and the result follows since $Part_l(A)$ is compact. $\square$

Lemma

Let $\mathcal{F}$ be a filter, $\mathcal{C} \subseteq \mathcal{F}$ a compact set, $X \in \left(\mathcal{F}^{<\omega}\right)^{+}$, $A \in [\omega]^{\omega}$ such that $\bigstar\left(A, \mathcal{F}, X\right)$ holds and $l \in \omega$. There is $Y \in [X]^{<\omega}$ such that if $C_1, ..., C_l \in \mathcal{C}\left(Y\right)$ and $F \in \mathcal{C}$ then there is $s \in Y \cap [A]^{<\omega}$ such that $s \subseteq C_1 \cap ... \cap C_l \cap F$.

Proof.

Let n such that for every $\langle B_1, ..., B_{2^l} \rangle \in Part_{2^l}\left(A\right)$ and for every $F \in \mathcal{C}$ there is $j \leq 2^l$ for which $X \cap \wp\left(B_j \cap n\right)$ contains a subset of F. Let $Y = X \cap \wp\left(l\right)$, we will see that Y has the desired properties. Let $C_1, ..., C_l \in \mathcal{C}\left(Y\right)$ and $F \in \mathcal{C}$. For every $s : l \longrightarrow 2$ define B_s as the set of all $a \in A$ such that $a \in C_i$ if and only if $s\left(i\right) = 1$. Clearly $\langle B_s \rangle_{s \in 2^l} \in Part_{2^l}\left(A_l\right)$ and we may conclude that there is s such that $Y \cap \wp\left(B_s \cap n\right)$ contains an element of F. Since $C_1, ..., C_l \in \mathcal{C}\left(Y\right)$ we conclude that s must be the constant 1 function and this entails the desired conclusion. $\square$

Theorem

If $\mathcal{A}$ is a Laflamme MAD family, then $\mathbb{F}_\sigma(\mathcal{A})$ forces that $\dot{\mathcal{U}}_{gen}(\mathcal{A})$ is $\mathcal{B}$-Canjar for every $\mathfrak{b}$-family $\mathcal{B}$ in the ground model.

Proof.

It is enough to show that if $\mathcal{F} \Vdash$ "$\overline{X} = \langle X_n \rangle_{n\in\omega} \subseteq (\dot{\mathcal{U}}_{gen}(\mathcal{A})^{<\omega})^+$" then there is $\mathcal{G} \leq \mathcal{F}$ and $f \in \mathcal{B}$ such that $\mathcal{C}(\overline{X}_f) \subseteq \mathcal{G}$. Let $\mathcal{F} = \bigcup \mathcal{C}_n$ where each $\mathcal{C}_n$ is compact and they form an increasing chain. By the previous results, we may find $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that $\bigstar(A_n, \mathcal{F}, X_m)$ holds for every $n, m \in \omega$. We can then find an increasing function $g : \omega \longrightarrow \omega$ such that the following holds:

*) For every $n \in \omega$ and for every $i \leq n$, if $Y = X \cap \wp(g(n))$ then for every $C_0, ..., C_n \in \mathcal{C}(Y)$ and $F \in \mathcal{C}_n$ there is $s \in Y \cap [A_i]^{<\omega}$ such that $s \subseteq C_0 \cap ... \cap C_l \cap F$.

Since $\mathcal{B}$ is unbounded, we can find $f \in \mathcal{B}$ that is not dominated by g. It is easy to see that $\mathcal{G} = \langle \mathcal{F} \cup \mathcal{C}(\overline{X}_f) \rangle$ is a condition in $\mathbb{F}_\sigma(\mathcal{A})$ and has the desired properties. □

Theorem (Shelah)

It is consistent that $\omega_1 = \mathfrak{b} < \mathfrak{a}$.

Proof.

Start with $V \models$ CH and let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \omega_2 \rangle$ be the countable support iteration such that at every step $\alpha < \omega_2$ we choose a MAD family $\mathcal{A}_\alpha$. In case $\mathcal{A}_\alpha$ is Laflamme, define $\mathbb{P}_\alpha \Vdash \mathbb{Q}_\alpha = \mathbb{F}_\sigma(\dot{\mathcal{A}}_\alpha) * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\dot{\mathcal{A}}_\alpha))$ and if not, then $\mathbb{P}_\alpha \Vdash \mathbb{Q}_\alpha = \mathbb{M}\left(\mathcal{J}_\alpha\right)$ where $\mathcal{J}_\alpha$ is an F_σ-filter extending $\mathcal{A}_\alpha$. The iteration does not add dominating reals (as before), so at the end we get $\mathfrak{b} = \omega_1$ and with a careful bookkeeping devise, we get $\mathfrak{a} = \omega_2$. $\square$

Lyubomir Zdomsky has an easier argument for all of these

We only need to prove the lemma we skipped, which is the following:

Lemma

Let $\mathcal{A}$ be a Laflamme MAD family and $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$. For every family $\{X_n \mid n \in \omega\}$ such that $\mathcal{C}(X_n) \subseteq \left\langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \right\rangle$, there is a countable family $\mathcal{D} \in [\mathcal{A}]^\omega$ such that $\bigstar(\mathcal{F}, A, X_n)$ holds for every $n \in \omega$ and $A \in \mathcal{D}$.

Before that, let's play some game:

Let $\mathcal{A}$ be a MAD family, $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$ and $X \subseteq [\omega]^{<\omega}$ such that $\mathcal{C}(X) \subseteq \left\langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \right\rangle$. Fix $\langle \mathcal{C}_n \rangle_{n\in\omega}$ an increasing family of compact sets such that $\mathcal{F} = \bigcup \mathcal{C}_n$. The *Brendle game*, $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$ is defined as follows,

I	Y_0		Y_1		Y_2		$\cdots$
II		s_0		s_1		s_2	$\cdots$

Where $Y_m \in \mathcal{I}(\mathcal{A})^*$, $s_m \in [Y_m]^{<\omega}$ intersects all the elements of $\mathcal{C}_m$ and $\max(s_m) < \min(s_{m+1})$. Player I wins the game if $\bigcup\limits_{n\in\omega} s_n$ contains an element of X.

This game was based on the rank arguments used by Brendle. A similar (but different) approach using games was used by Brendle and Brooke-Taylor.

Note that this is an open game for I, i.e., if she wins, then she won already in a finite number of steps. By the Gale-Stewart theorem, the Brendle game is determined.

By $V[C_\alpha]$ we denote an extension of V by adding α-Cohen reals.

Lemma

Let $\mathcal{A}$ be a Laflamme MAD family and $X \subseteq [\omega]^{<\omega}$ such that $\mathcal{C}(X) \subseteq \langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \rangle$. In $V[C_{\omega_1}]$, the player I *has a winning strategy for $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$*

Proof.

We will prove the claim by contradiction, since $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$ is determined, we assume that II has a winning strategy, call it π. $\square$

Proof.

We now build a tree T and $\{B_t \mid t \in T\}$ recursively as follows:

1. $\emptyset \in T$ and $B_\emptyset = \omega$.
2. T_1 is the set of all $\langle s \rangle$ such that $s \in [\omega]^{<\omega}$ and there is $B \in \mathcal{I}(\mathcal{A})^*$ for which $\langle B, s \rangle$ is a legal partial play of $\mathcal{BR}(\mathcal{A}, \mathcal{F}, X)$ in which Player II is using her strategy π.
3. For every s such that $\langle s \rangle \in T_1$, we choose $B_s \in \mathcal{I}(\mathcal{A})^*$ for which $\langle B_s, s \rangle$ is a legal partial play.
4. Given a node $t = \langle s_0, s_1, ..., s_n \rangle \in T$ (and we know that the sequence $\left\langle B_{\langle s_0 \rangle}, s_0, B_{\langle s_0, s_1 \rangle}, s_1, ..., B_{\langle s_0, s_1, \ldots, s_n \rangle}, s_n \right\rangle$ is a legal partial play) let $suc_T(t)$ be the set of all $z \in [\omega]^{<\omega}$ for which there is $B \in \mathcal{I}(\mathcal{A})^*$ such that $\left\langle B_{\langle s_0 \rangle}, s_0, B_{\langle s_0, s_1 \rangle}, s_1, ..., B_{\langle s_0, s_1, \ldots, s_n \rangle}, s_n, B, z \right\rangle$ is a legal partial play (in which Player II is using her strategy π). We fix $B_{t^\frown \langle z \rangle} \in \mathcal{I}(\mathcal{A})^*$ with this property.

$\square$

Proof.

Note that if $t = \langle s_0, s_1, ..., s_n \rangle \in T$, then $\bigcup_{i \leq n} s_i$ does not contain an element of X, this is because π is a winning strategy for player II. Clearly T is a countable tree with no isolated branches, so it is equivalent to Cohen forcing when viewed as a forcing notion. Since T is countable, it appears in an intermediate extension of $V\left[C_{\omega_1}\right]$. Let $\beta < \omega_1$ such that $T \in V\left[C_\beta\right]$. $\square$

Proof.

Let $G \in V\left[C_{\omega_1}\right]$ be a $\left(T, V\left[C_\beta\right]\right)$-generic branch through T. It is easy to see that G induces a legal play of the game in which II followed her strategy. Let $D = \bigcup G$, and since π is a winning strategy for II, we conclude that D does not contain an element of X. By genericity $D \in \left\langle \mathcal{I}\left(\mathcal{A}\right)^* \cup \mathcal{F} \right\rangle^+$ however, $\omega \setminus D \in \mathcal{C}\left(X\right) \subseteq \left\langle \mathcal{I}\left(\mathcal{A}\right)^* \cup \mathcal{F} \right\rangle$ which is obviously a contradiction. This finishes the proof of the lemma. $\square$

Lemma

Let $\mathcal{A}$ be a Laflamme MAD family and $\mathcal{F} \in \mathbb{F}_{\sigma}(\mathcal{A})$. For every family $\{X_n \mid n \in \omega\}$ such that $\mathcal{C}(X_n) \subseteq \left\langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^{*} \right\rangle$, there is a countable family $\mathcal{D} \in [\mathcal{A}]^{\omega}$ such that $\bigstar(\mathcal{F}, A, X_n)$ holds for every $n \in \omega$ and $A \in \mathcal{D}$.

Proof.

We work in $V\left[C_{\omega_1}\right]$, where player I has winning strategies for all of the games $\mathcal{BR}\left(\mathcal{A},\mathcal{F},X_n\right)$ with $n\in\omega$. Let π_n be the winning strategy for the game $\mathcal{BR}\left(\mathcal{A},\mathcal{F},X_n\right)$. Let $\mathcal{W}$ be the set of elements of $\mathcal{I}\left(\mathcal{A}\right)^*$ that may be played by I following her winning strategy in any of these games. It is not hard to see that $\mathcal{W}$ is countable. Note that if $W\in\mathcal{W}$ then W almost contains every element of $\mathcal{A}$ except for finitely many (this is because $W\in\mathcal{I}\left(\mathcal{A}\right)^*$). Let $\mathcal{A}'\subseteq\mathcal{A}$ be the set of all $A\in\mathcal{A}$ for which there is $W\in\mathcal{W}$ such that $A\not\subseteq^* W$. Note that $\mathcal{A}'$ is countable. Since $\mathcal{A}$ is Laflamme in V, it is not contained in $\langle\mathcal{F}\cup\{\omega\setminus B\mid B\in\mathcal{A}'\}\rangle$, so there is $A_0\in\mathcal{A}$ such that $\omega\setminus A_0\notin\langle\mathcal{F}\cup\{\omega\setminus B\mid B\in\mathcal{A}'\}\rangle$. This implies that $A_0\in\mathcal{F}^+$ and A_0 is almost contained in every member of $\mathcal{W}$. We claim that $\bigstar\left(\mathcal{F},A_0,X_n\right)$ holds for each $n\in\omega$. □

Proof.

Let $F \in \mathcal{F}$ and consider the following play in $\mathcal{BR}\left(\mathcal{A}, \mathcal{F}, X_n\right)$,

I	W_0		W_1		W_2		$\cdots$
II		s_0		s_1		s_2	$\cdots$

Where the W_i are played by I according to π_n, $s_i \in [B \cap F]^{<\omega}$ and intersects every element of $\mathcal{C}_i$. This is possible since $B \cap F$ is positive and is almost contained in every W_n. Since π_n is a winning strategy, this means that I wins the game,which entails that $\bigcup s_n \subseteq B \cap F$ contains an element of X_n. $\square$

Proof.

We can then obtain each A_{n+1} by repeating the same argument and using that $\mathcal{I}(\mathcal{A})^*$ it is not contained in $\langle \mathcal{F} \cup \{\omega \setminus B \mid B \in \mathcal{A}'\} \cup \{\omega \setminus A_0, .., \omega \setminus A_n\} \rangle$. Let $\mathcal{D}_1 = \{A_n \mid n \in \omega\}$.

We get that $\bigstar(\mathcal{F}, A, X_n)$ holds for every $n \in \omega$ and $A \in \mathcal{D}_1$.

But we are not done yet! We got the conclusion of the lemma in $V[C_{\omega_1}]$, but we may want it in V. However, we can get the result in V by a simple genericity argument: $\square$

Proof.

We know that $V[C_{\omega_1}] \models \bigstar(\mathcal{F}, A_n, X_m)$ for every $n, m \in \omega$. However, it is easy to see that the statement $\bigstar(\mathcal{F}, A_n, X_m)$ is absolute between models of ZFC (in fact, we only need that it is downwards absolute, which is easy). So $V \models \bigstar(\mathcal{F}, A_n, X_m)$ for every $n, m \in \omega$. Since $\mathbb{C}_{\omega_1}$ has the countable chain condition, there is $\mathcal{D} \in [\mathcal{A}]^{\omega}$ such that $\mathbb{C}_{\omega_1} \Vdash$ "$\mathcal{D}_1 \subseteq \mathcal{D}$J. We may assume that that $\bigstar(\mathcal{F}, A, X_n)$ holds for every $n \in \omega$ and $A \in \mathcal{D}$.

This finishes the proof of the lemma and of the theorem. $\square$

Thank you!