COMPLEX ANALYSIS – Semester 2022-2

Contents

1 Holomorphic functions

1.1 The complex derivative

The basic objects of complex analysis are the holomorphic functions. These are functions that posses a complex derivative. As we will see this is quite a strong requirement and will allow us to make far reaching statements about this type of functions. To properly understand the concept of a complex derivative, let us recall first the concept of derivative in $\mathbb{R}^n$.

Definition 1.1. Let U be an open set in $\mathbb{R}^n$ and $A : U \to \mathbb{R}^m$ a function. Given $x \in U$ we say that A is *(totally) differentiable* at x iff there exists an $m \times n$-matrix A' such that,

$$A(x + \xi) = A(x) + A'\xi + \mathrm{o}(\|\xi\|)$$

for $\xi \in \mathbb{R}^n$ sufficiently small. Then, A' is called the *derivative* of A at x.

Recall that the matrix elements of A' are the partial derivatives

$$A'_{ij} = \frac{\partial A_i}{\partial x_j}.$$

Going from the real to the complex numbers, we can simply use the decomposition $z = x+\mathrm{i}y$ of a complex number z into a pair of real numbers (x, y) to define a concept of derivative. Thus, let U be an open set in $\mathbb{C}$ and consider a function $f : U \to \mathbb{C}$. We view U as an open set in $\mathbb{R}^2$ with coordinates (x, y) and $f = u + \mathrm{i}v$ as a function with values in $\mathbb{R}^2$ with coordinates (u, v). The total derivative of f, if it exists, is then a 2×2-matrix f' given by

$$f' = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

So far we have only recited concepts from real analysis and not made use of the fact that the complex numbers do not merely form a 2-dimensional real vector space, but a field. Indeed, this implies that there are special 2×2-matrices, namely those that correspond to multiplication by a complex number. As is easy to see, multiplication by $a+\mathrm{i}b$ corresponds to the matrix,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The crucial step that leads us from real to complex analysis is now the additional requirement that the derivative f' take this form. It is then more useful to think of f' as the complex number $a + \mathrm{i}b$, rather than this 2×2-matrix.

Definition 1.2. Let U be an open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. Given $z \in U$ we say that f is *complex differentiable* at z iff there exists $f'(z) \in \mathbb{C}$ such that,

$$f(z+\zeta) = f(z) + f'(z)\zeta + \mathrm{o}(|\zeta|)$$

for $\zeta \in \mathbb{C}$ sufficiently small. Then, $f'(z)$ is called the *complex derivative* of f at z. f is called *holomorphic* at z iff f is complex differentiable in an open neighborhood of z.

Proposition 1.3. *Let U be an open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. f is complex differentiable at $z \in U$ iff f is totally differentiable at z and its partial derivatives at z satisfy the* Cauchy-Riemann equations*,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If $U \subseteq \mathbb{C}$ is open we say that $f : U \to \mathbb{C}$ is holomorphic on U if it is holomorphic at all $z \in U$. We denote the space of functions that are holomorphic on U by $\mathcal{O}(U)$. In the following, non-empty connected open subsets of the complex plane will be of particular importance. We will refer to such open sets as *regions*. Since any non-empty open set in the complex plane is a disjoint union of regions it is sufficient to consider the spaces of holomorphic functions of the type $\mathcal{O}(D)$, where $D \subseteq \mathbb{C}$ is a region. The elements of $\mathcal{O}(\mathbb{C})$ are called *entire* functions.

Exercise 1. Let U be an open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. Given $z \in U$ we say that f is *complex conjugate differentiable* at z iff there exists $f_{\overline{z}}(z) \in \mathbb{C}$ such that,

$$f(z+\zeta) = f(z) + f_{\overline{z}}(z)\overline{\zeta} + \mathrm{o}(|\zeta|)$$

for $\zeta \in \mathbb{C}$ sufficiently small. Then, $f_{\overline{z}}(z)$ is called the *complex conjugate derivative* of f at z. f is called *anti-holomorphic* at z iff f is complex conjugate differentiable in an open neighborhood of z.

1. Show that the total derivative of f as a real 2×2-matrix takes the form

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad \text{for} \quad a, b \in \mathbb{R},$$

   where f is complex conjugate differentiable.

2. Deduce the corresponding modified Cauchy-Riemann equations.

3. Show that a function is anti-holomorphic iff it is the complex conjugate of a holomorphic function.

1.2 Elementary properties of holomorphic functions

Proposition 1.4. *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Then, f is constant iff $f'(z) = 0$ for all $z \in D$.*

Proof. If f is constant it follows immediately that $f' = 0$. Conversely, suppose that $f' = 0$. Then, viewing f as a function from an open set D in $\mathbb{R}^2$ to $\mathbb{R}^2$ we know that its total derivative is zero. By results of real analysis it follows that f is constant along any path in D. But since D is connected it is also path connected and f must be constant on D. □

Proposition 1.5. *Let $D \subseteq \mathbb{C}$ be a region.*

1. *If $f \in \mathcal{O}(D)$ and $\lambda \in \mathbb{C}$, then $\lambda f \in \mathcal{O}(D)$ and $(\lambda f)'(z) = \lambda f'(z)$.*
2. *If $f, g \in \mathcal{O}(D)$, then $f + g \in \mathcal{O}(D)$ and $(f + g)'(z) = f'(z) + g'(z)$.*
3. *If $f, g \in \mathcal{O}(D)$, then $fg \in \mathcal{O}(D)$ with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.*
4. *If $f, g \in \mathcal{O}(D)$ and $g(z) \neq 0$ for all $z \in D$, then $f/g \in \mathcal{O}(D)$ and*
$$(f/g)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.$$

Proof. The proofs are completely analogous to those for real functions on open subsets of the real line with the ordinary real differential. Alternatively, 1.-3. follow from statements in real analysis by viewing $\mathbb{C}$ as $\mathbb{R}^2$. □

Note that items 1.-3. imply that $\mathcal{O}(D)$ is an *algebra* over the complex numbers.

Proposition 1.6. *Let $D_1, D_2 \subseteq \mathbb{C}$ be regions. Let $f \in \mathcal{O}(D_1)$ such that $f(D_1) \subseteq D_2$ and let $g \in \mathcal{O}(D_2)$. Then $g \circ f \in \mathcal{O}(D_1)$ and moreover the* chain rule *applies,*
$$(g \circ f)'(z) = g'(f(z))f'(z) \quad \forall z \in D_1.$$

Proof. This is again a result of real analysis, obtained by viewing $\mathbb{C}$ as $\mathbb{R}^2$. (Note that g' and f' are then 2×2-matrices whose multiplication translates to multiplication of complex numbers here.) □

Proposition 1.7. *Let $D_1, D_2 \subseteq \mathbb{C}$ be regions. Let $f : D_1 \to \mathbb{C}$ be continuous and such that $f(D_1) \subseteq D_2$. Let $g \in \mathcal{O}(D_2)$ be such that $g \circ f(z) = z$ for all $z \in D_1$. Let $z \in D_1$. Suppose that $g'(f(z)) \neq 0$ and that g' is continuous at $f(z)$. Then, f is complex differentiable at z and*
$$f'(z) = \frac{1}{g'(f(z))}.$$

Proof. Again, this is a statement imported from real analysis on $\mathbb{R}^2$. (There, the condition $g'(f(z)) \neq 0$ is the condition that the determinant of the 2×2-matrix $g'(f(z))$ does not vanish.) □

A few elementary examples together with the properties of holomorphic functions we have identified so far already allow us to generate considerable families of holomorphic functions.

Example 1.8. The following are elementary entire functions.

- The *constant functions*: They have vanishing complex derivative.
- The *identity function*: $f(z) = z$ has complex derivative $f'(z) = 1$.

Example 1.9. The following are (classes of) holomorphic functions produced from the elementary entire functions of Example 1.8 by addition, multiplication and division.

- *Polynomials*: Any polynomial $p(z) = \sum_n \lambda_n z^n$, where $\lambda_n \in \mathbb{C}$, is entire with $p'(z) = \sum_{n\neq 0} \lambda_n n z^{n-1}$.
- *Rational functions*: Let $p(z)$ and $q(z)$ be polynomials with $q \neq 0$ and suppose that p and q have no common zeros. Let $D = \mathbb{C} \setminus N$, where N is the set of zeros of q. Then, $f(z) = p(z)/q(z) \in \mathcal{O}(D)$.

1.3 The exponential function

The most important example of a transcendental entire function is the complex exponential function.

Definition 1.10. We define the *complex exponential function* $\exp : \mathbb{C} \to \mathbb{C}$ as follows. For all $a, b \in \mathbb{R}$ define

$$\exp(a + \mathrm{i}b) := \exp(a) \left(\cos(b) + \mathrm{i}\sin(b)\right),$$

where exp, cos and sin are the functions known from real analysis.

Proposition 1.11. *The complex exponential function has the following properties:*

1. $\exp$ *is entire.*
2. $\exp'(z) = \exp(z)$ *for all* $z \in \mathbb{C}$.
3. $\exp(-z) = 1/\exp(z)$ *for all* $z \in \mathbb{C}$.
4. $\exp(z + 2\pi \mathrm{i} n) = \exp(z)$ *for all* $z \in \mathbb{C}$ *and all* $n \in \mathbb{Z}$.
5. $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$.

6. *For each $z \in \mathbb{C} \setminus \{0\}$ there is a unique angle $\theta \in [0, 2\pi)$, called the* argument *or* phase*, so that $z = |z| \exp(\mathrm{i}\theta)$.*

Proof. **Exercise.**[Suggestion: Use properties of the functions exp, cos and sin defined on the real numbers.] $\square$

Proposition 1.12. *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Then, the following statements are equivalent:*

1. *$f(z) = a \exp(bz)$ for all $z \in D$, where $a, b \in \mathbb{C}$ are constants.*
2. *$f'(z) = bf(z)$ for all $z \in D$, where $b \in \mathbb{C}$ is a constant.*

Proof. The implication 1. $\Longrightarrow$ 2. is straightforward using elementary properties of the derivative (Propositions 1.5 and 1.6) together with Proposition 1.11.2. For the implication 2. $\Longrightarrow$ 1. consider the holomorphic function $g : D \to \mathbb{C}$ given by $g(z) := f(z) \exp(-bz)$ for all $z \in D$. Then, $g' = 0$, so by Proposition 1.4 there exists a constant $a \in \mathbb{C}$ such that $g(z) = a$ for all $z \in D$. But since $\exp(-bz) = 1/\exp(bz)$ due to Proposition 1.11.3, we obtain 1. as desired. $\square$

Remark 1.13. This Proposition shows in particular that the complex exponential function is uniquely determined by the properties $\exp' = \exp$ and $\exp(0) = 1$.

Proposition 1.14 (Addition Theorem)**.**

$$\exp(z + \zeta) = \exp(z) \exp(\zeta) \qquad \forall z, \zeta \in \mathbb{C}.$$

Proof. Fix $\zeta \in \mathbb{C}$. Then, $f(z) := \exp(z + \zeta)$ is holomorphic for $z \in \mathbb{C}$ and $f'(z) = f(z)$. So $f(z) = a \exp(z)$ for some $a \in \mathbb{C}$ by Proposition 1.12. Since $f(0) = \exp(\zeta) = a$ we obtain the stated result. $\square$

Proposition 1.14 motivates us to use the notation $e^z := \exp(z)$ as in the real case.

Remark 1.15. It might seem somewhat unsatisfactory to define the complex exponential function by recurrence to transcendental functions from real analysis. Indeed, one could instead start from a definition in terms of a power series. One can then derive properties 1.,2.,3. of Proposition 1.11 and consequently Propositions 1.12 and 1.14 from properties of this power series. We come back to the power series of the exponential function in Proposition 1.21.

Example 1.16. The following are transcendental entire functions produced using the exponential function.

- *Hyperbolic functions*:

$$\cosh(z) := \frac{\exp(z) + \exp(-z)}{2} \quad \text{and} \quad \sinh(z) := \frac{\exp(z) - \exp(-z)}{2}.$$

- *Trigonometric functions*:

$$\cos(z) := \frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2} \quad \text{and} \quad \sin(z) := \frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}.$$

Example 1.17 (The logarithm)**.** Since $\exp(z + 2\pi\mathrm{i}) = \exp(z)$ we have to restrict the domain of exp in order to find a unique inverse. It is customary to make the following choice: Consider the region $D_2 := \mathbb{R} + \mathrm{i}(-\pi, \pi)$. Then exp is a bijective function $D_2 \to D_1$, where $D_1 = \mathbb{C} \setminus \mathbb{R}_0^-$. We define log as the unique function such that $\exp(\log(z)) = z$ for all $z \in D_1$ and such that the image of log lies in $D_2 \subseteq \mathbb{C}$. Then, $\log \in \mathcal{O}(D_1)$ and $\log'(z) = 1/z$ for all $z \in D_1$. This version of the logarithm is also called the *principal branch*.

Exercise 2. Suppose f is a holomorphic function on a region $D \subseteq \mathbb{C}$. Suppose that the real or the imaginary part of f is constant. Show that f must be constant on D.

Exercise 3. At which points in the complex plane are the following functions complex differentiable and at which points are they holomorphic?

1. $f(x + \mathrm{i}y) = x^4 y^5 + \mathrm{i}xy^3$
2. $f(x + \mathrm{i}y) = \sin^2(x + y) + \mathrm{i}\cos^2(x + y)$

Exercise 4. Define another version ("branch") of the logarithm function that is holomorphic in the region $D = \mathbb{C} \setminus \mathbb{R}_0^+$.

Exercise 5. Define $\tan z := \frac{sinz}{\cos z}$. Where is this function defined and where is it holomorphic?

Exercise 6. Define a function $z \mapsto \sqrt{z}$ on $\mathbb{C}$ or on a subset of $\mathbb{C}$. Is this function holomorphic and if yes, where? Comment on possible choices in the construction.

1.4 Power series and analytic functions

With each sequence $\{c_n\}_{n\in\mathbb{N}}$ of complex numbers and each point $z_0 \in \mathbb{C}$ we can associate a power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

around z_0. Recall the following result from real analysis.

Lemma 1.18. *The* radius of convergence r *of the power series is given by*

$$\frac{1}{r} = \limsup_{n\to\infty} |c_n|^{1/n}.$$

That is, the power series converges absolutely in the open disk $B_r(z_0)$ to a complex function $f : B_r(z_0) \to \mathbb{C}$. For any $0 < \rho < r$ the convergence is uniform in the open disk $B_\rho(z_0)$. It diverges for z outside of the closed disk $\overline{B_r(z_0)}$.

Proof. **<u>Exercise</u>.** □

Definition 1.19. Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$. We say that f is *analytic* in D iff for every point $z \in D$ and any $r > 0$ such that $B_r(z) \subseteq D$ the function f can be expressed as a power series around z with radius of convergence greater or equal to r.

Theorem 1.20. *Let $D \subseteq \mathbb{C}$ be a region. Suppose that f is analytic in D. Then $f \in \mathcal{O}(D)$ and $f^{(k)}$ is also analytic in D. Moreover, if*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \tag{1}$$

converges in $B_r(z_0)$, then

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} c_{n+k} (z - z_0)^n, \tag{2}$$

converges in $B_r(z_0)$. Moreover, the coefficients c_n of the power series around a given point are unique.

Proof. (Adapted from Rudin.) Fix $z_0 \in D$ and $r > 0$ such that $B_r(z_0) \subseteq D$. Suppose f is given by the power series (1) and converges in $B_r(z_0)$. Consider the power series

$$g(z) := \sum_{n=0}^{\infty} (n+1) c_{n+1} (z - z_0)^n.$$

It is then enough to show that $g(z)$ converges in $B_r(z_0)$ and that $g(z)$ is the complex derivative of f for all $z \in B_r(z_0)$. The statement (2) about the k-th derivative follows then by iteration.

Firstly, it is clear by Lemma 1.18 that $g(z)$ has the same radius of convergence as $f(z)$. In particular, $g(z)$ converges in $B_r(z_0)$. Fix $z \in B_r(z_0)$ and define $\xi := z - z_0$. Then, set ρ arbitrarily such that $|\xi| < \rho < r$. Let $\zeta \in B_s(0) \setminus \{0\}$ where $s := \rho - |\xi|$ and set

$$h(\zeta) := \frac{f(z+\zeta) - f(z)}{\zeta} - g(z).$$

We have to show that $h(\zeta) \to 0$ when $|\zeta| \to 0$. $h(\zeta)$ can be written as

$$h(\zeta) = \sum_{n=0}^{\infty} c_n a_n(\zeta),$$

where

$$a_n(\zeta) := \frac{(\xi + \zeta)^n - \xi^n}{\zeta} - n\xi^{n-1}.$$

Note that $a_0(\zeta) = 0$ and $a_1(\zeta) = 0$. By explicit computation we find for $n \geq 2$,

$$a_n(\zeta) = \zeta \sum_{k=1}^{n-1} k\xi^{k-1}(\xi + \zeta)^{n-k-1}.$$

Now, $|\xi| < \rho$ and $|\xi + \zeta| < \rho$ so that we get the estimate,

$$|a_n(\zeta)| < |\zeta|\frac{1}{2}n(n-1)\rho^{n-2}.$$

This implies

$$|h(\zeta)| < |\zeta|\frac{1}{2}\sum_{n=2}^{\infty} |c_n| n(n-1)\rho^{n-2}.$$

However, since $\rho < r$, the sum converges by Lemma 1.18 showing that there is a constant M such that

$$|h(\zeta)| < |\zeta| M.$$

This completes the proof of (2). Finally , the uniqueness of the coefficients c_n follows from the special case of (2) given by

$$f^{(k)}(z_0) = k! c_k.$$

□

The first remarkable result of complex analysis is that the converse of this theorem is also valid: Every holomorphic function is analytic. However, in order to show this we will have to introduce the integral calculus in the complex plane. We will do this in the next chapter.

Proposition 1.21. *The exponential function is analytic in $\mathbb{C}$ and has a power series representation given as follows:*

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

Proof. **Exercise.** □

Lemma 1.22. *Let $n \in \mathbb{Z}$. Then,*

$$\int_0^{2\pi} e^{in\theta}\, d\theta = \begin{cases} 2\pi & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \neq 0 \end{cases}$$

Proof. **Exercise.** □

Lemma 1.23. *Let $z_0 \in \mathbb{C}$ and $r > 0$, and suppose the power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

has radius of convergence greater than r. Then,

$$c_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta} \, \mathrm{d}\theta.$$

Proof. By Lemma 1.18 the power series converges uniformly in $\overline{B_r(z_0)}$. We can thus in the following interchange integration and summation,

$$\begin{aligned}
&\int_0^{2\pi} f(z_0 + re^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta} \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \sum_{k=0}^{\infty} c_k r^k e^{\mathrm{i}(k-n)\theta} \, \mathrm{d}\theta \\
&= \sum_{k=0}^{\infty} c_k r^k \int_0^{2\pi} e^{\mathrm{i}(k-n)\theta} \, \mathrm{d}\theta \\
&= 2\pi c_n r^n.
\end{aligned}$$

Here we have used Lemma 1.22. □

Lemma 1.24 (Gutzmer Formula)**.** *Let $z_0 \in \mathbb{C}$ and $r > 0$, and suppose the power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

has radius of convergence greater than r. Set $M := \sup_{z \in \partial B_r(z_0)} |f(z)|$. Then,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{\mathrm{i}\theta}) \right|^2 \mathrm{d}\theta \le M^2.$$

Proof. Since

$$\overline{f(z_0 + re^{\mathrm{i}\theta})} = \sum_{n=0}^{\infty} \overline{c_n} r^n e^{-\mathrm{i}n\theta}$$

we have,

$$\left| f(z_0 + re^{\mathrm{i}\theta}) \right|^2 = \sum_{n=0}^{\infty} \overline{c_n} r^n f(z_0 + re^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta},$$

where the series converges uniformly as a series of functions on the interval $\theta \in [0, 2\pi]$. Thus, we can interchange integration and summation in the following and use Lemma 1.23 to obtain,

$$\int_0^{2\pi} \left| f(z_0 + re^{\mathrm{i}\theta}) \right|^2 \,\mathrm{d}\theta = \sum_{n=0}^{\infty} \overline{c_n} r^n \int_0^{2\pi} f(z_0 + re^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta} \,\mathrm{d}\theta = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

This shows the claimed equality. The stated inequality is obtained by estimating the integral through the maximum of its integrand. □

Proposition 1.25 (Cauchy's Estimates)**.** *Let $D \subseteq \mathbb{C}$ be a region, $f : D \to \mathbb{C}$ analytic, $z_0 \in D$ and $r > 0$ such that $\overline{B_r(z_0)} \subset D$. Set $M := \sup_{z \in \partial B_r(z_0)} |f(z)|$. Then,*

$$|f^{(n)}(z_0)| \leq \frac{n! M}{r^n}.$$

Proof. **Exercise.** □

Theorem 1.26 (Liouville Theorem)**.** *Every bounded function analytic in $\mathbb{C}$ is constant.*

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic in C and be bounded by N, i.e., $|f(z)| \leq N$ for all $z \in \mathbb{C}$. Since f is analytic in $\mathbb{C}$ and its power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ around 0 has infinite radius of convergence. Thus, for a radius $r > 0$ we have from Lemma 1.24 the estimate,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq M^2 \leq N^2.$$

Since r can be arbitrarily large, this implies $c_k = 0$ for all $k \in \mathbb{N}$. □

Exercise 7. Let $a, b, c, d \in \mathbb{C}$ such that $c \neq 0$ and $ad - bc \neq 0$. Show that $f(z) := \frac{az+b}{cz+d}$ is analytic in $D := \mathbb{C} \setminus \{-\frac{d}{c}\}$.

Exercise 8. Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic in $\mathbb{C}$ and satisfies

$$|f(z)| \leq a + b|z|^c \,\forall z \in \mathbb{C},$$

where a, b, c are positive constants. Show that f is a polynomial of degree less than or equal to c.

Exercise 9. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic in $\mathbb{C}$. Show that the power series of f at 0 converges uniformly in all of $\mathbb{C}$ if an only if f is a polynomial.

2 Complex integration

2.1 Integration along paths

Definition 2.1. Let $I = [a, b] \subset \mathbb{R}$ be a compact interval. A continuous map $\gamma : I \to \mathbb{C}$ is called a *curve* in $\mathbb{C}$. We denote the image of the curve γ by $|\gamma|$. If $\gamma(a) = \gamma(b)$, the curve is called *closed*. A curve $\gamma : I \to \mathbb{C}$ is called a *path* iff it is *piecewise continuously differentiable*. That is, there exist points $a = x_0 < x_1 < \cdots < x_n = b$ such that γ restricted to $[x_{k-1}, x_k]$ is continuously differentiable for all $k \in \{1, \ldots, n\}$.

Recall that *continuous differentiability* in a closed interval $[a, b]$ means differentiability in (a, b) such that the differential is continuous and has a continuous extension to $[a, b]$.

For the theory of integration along paths what is important in a path is its image in and in which direction this is retraced. In contrast, the concrete parametrization of a path via an interval $I \subset \mathbb{R}$ is not important. To make this more precise we define the concept of reparametrization of a path.

Definition 2.2. Let $\gamma : [a, b] \to \mathbb{C}$ and $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ be paths. We say that $\tilde{\gamma}$ is a *reparametrization* of γ iff there exists a monotonous, continuous and piecewise continuously differentiable map $\phi : [\tilde{a}, \tilde{b}] \to [a, b]$ with $\phi(\tilde{a}) = a$ and $\phi(\tilde{b}) = b$ and such that $\tilde{\gamma} = \gamma \circ \phi$.

We will be interested only in properties and usages of paths that are invariant under reparametrization. The first such property we consider is the length of a path. Intuitively it is quite clear what we mean by this. If a path $\gamma : [a, b] \to \mathbb{C}$ is a straight line

$$\gamma(t) := \frac{(b - t)x_1 + (t - a)x_2}{b - a}$$

with end points x_1 and x_2, then its length should be $|x_2 - x_1|$ where we use the standard Euclidean inner product on $\mathbb{C}$. In general, we can approximate a path by subdividing the interval on which it is defined and replacing the pieces of paths in subdivisions by straight lines. The length of the path should then be the limit of the sum of the lengths of these straight lines when we make the subdivisions arbitrarily fine. That this limit exists is due to the piecewise continuous differentiability property we have imposed. (The limit does not necessarily exist for arbitrary curves, even if their image is bounded.) The result is the following, which we state as a definition.

Definition 2.3. Let $\gamma : [a, b] \to \mathbb{C}$ be a path. The *length* of γ, denoted $l(\gamma)$ is defined by,

$$l(\gamma) := \int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

Exercise 10. (a) Show that the definition indeed agrees with the result of the procedure described above. (b) Give an example of a curve that has bounded image, but no well defined length.

Exercise 11. Show that the length of a path is invariant under reparametrization. That is, show that if γ is a path and $\tilde{\gamma}$ is a reparametrization of γ, then $l(\gamma) = l(\tilde{\gamma})$.

Definition 2.4. Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be a continuous map. Let $\gamma : I \to \mathbb{C}$ be a path such that $|\gamma| \subset U$. We define the *complex integral* of f *along the path* γ as follows,

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t. \tag{3}$$

To make sense of this definition we note that $t \mapsto f(\gamma(t))\gamma'(t)$ is a piecewise continuous function $I \to \mathbb{C}$ and is therefore bounded and integrable.

Proposition 2.5. *The complex integral is invariant under reparametrizations: Given an open set $U \subseteq \mathbb{C}$, a continuous function $f : U \to \mathbb{C}$, a path γ with $|\gamma| \subset U$ and a reparametrization $\tilde{\gamma}$ of γ. Then,*

$$\int_{\tilde{\gamma}} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

Proof. **Exercise.** □

Similarly to what we have seen in the context of the concept of derivative, the concept of integration introduced is quite similar to what we are familiar with in the case of $\mathbb{R}$ or $\mathbb{R}^n$. Nevertheless, again, there is an important difference that makes crucial use of the fact that the complex numbers form a field. If we were to discuss integration along paths in $\mathbb{R}^2$ weighted by path length, the formula to use would be almost identical to (3), with one important difference: γ' would be a 2×1-matrix and we would insert $|\gamma'(t)|$ instead of $\gamma'(t)$ on the right hand side. Decomposing $\gamma' = re^{\mathrm{i}\theta}$ the difference is that in the real case we would only put the absolute value r. We might think of the complex case as letting the direction of the curve (encoded in θ) enter the integrand. As we shall see, this leads to a remarkable interplay between complex integral and derivative.

Suppose $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [b, c] \to \mathbb{C}$ are paths such that $\gamma_1(b) = \gamma_2(b)$. Then, we can form the *composite path* $\gamma_1 \cdot \gamma_2 : [a, c] \to \mathbb{C}$ in the obvious way. We have then,

$$\int_{\gamma_1 \cdot \gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Because of Proposition 2.5 we are usually interested in paths only up to reparametrization. That is, we consider two paths as equivalent if one is a reparametrization of the other. We may then talk about the composition of paths whenever the endpoint of the first coincides with the initial point of the second.

Given a path $\gamma : [0, 1] \to \mathbb{C}$ we may form the *opposite path* $\gamma^{-1} : [0, 1] \to \mathbb{C}$ given by $\gamma^{-1}(t) = \gamma(1 - t)$. Then clearly, $(\gamma^{-1})^{-1} = \gamma$. As is easy to see,

$$\int_{\gamma^{-1}} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$$

We also find that the integral of any function along $\gamma \cdot \gamma^{-1}$ vanishes. $\gamma \cdot \gamma^{-1}$ is called a *retracing*. Because the integral along a retracing vanishes, we consider a retracing as equivalent to the trivial path.

Exercise 12. The concept of reparametrization can be generalized to include some form of retracing. To this end remove the monotonicity condition from Definition 2.2. (a) Is the length of a path invariant under generalized reparametrization? (b) Is the complex integral along a path invariant under generalized reparametrization?

Exercise 13 (Transformation rule)**.** Prove the following Proposition: Let $D \subseteq \mathbb{C}$ be a region, $g \in \mathcal{O}(D)$ such that $g' : D \to \mathbb{C}$ is continuous and γ a path with $|\gamma| \subset D$. Then, $g \circ \gamma$ is a path and for any continuous function $f : U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is open and $|g \circ \gamma| \subset U$ we have,

$$\int_{g\circ\gamma} f(z)\,\mathrm{d}z = \int_\gamma f(g(z))g'(z)\,\mathrm{d}z.$$

Proposition 2.6. *Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ continuous, γ be a path with $|\gamma| \subset U$. Set $\|f\|_\gamma := \sup_{z\in|\gamma|} |f(z)|$. Then,*

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \|f\|_\gamma l(\gamma).$$

Proof. **Exercise.** □

Proposition 2.7. *Let $U \subseteq \mathbb{C}$ be open and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of continuous functions $f_n : U \to \mathbb{C}$ converging uniformly. Let γ be a path in U. Then,*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,\mathrm{d}z = \int_\gamma \lim_{n\to\infty} f_n(z)\,\mathrm{d}z.$$

Proof. **Exercise.**[Hint: Use Proposition 2.6.] □

2.2 Closed paths and winding

Definition 2.8. Let γ be a closed path. Let $z \in \mathbb{C} \setminus |\gamma|$ and define the *index* of z with respect to γ as,

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\zeta - z}\,\mathrm{d}\zeta.$$

Theorem 2.9. *Let γ be a closed path and $U := \mathbb{C} \setminus |\gamma|$. Then, $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$ for all $z \in U$. Moreover, $\mathrm{Ind}_\gamma(z) = \mathrm{Ind}_\gamma(z')$ if z and z' are in the same connected component of U. Also, $\mathrm{Ind}_\gamma(z) = 0$ if $|z|$ is sufficiently large.*

Proof. Parametrizing $\gamma : [a,b] \to \mathbb{C}$ we have,

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi\mathrm{i}} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z}\,\mathrm{d}t.$$

In order to show that $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$ we define $\phi : [a,b] \to \mathbb{C}$ via,

$$\phi(t) := \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\,\mathrm{d}s\right).$$

It is then sufficient to show that $\phi(b) = 1$, which we proceed to do. Observe that ϕ is continuous and piecewise continuously differentiable with piecewise differential

$$\phi'(t) = \frac{\phi(t)\gamma'(t)}{\gamma(t) - z}.$$

The quotient function $t \mapsto \phi(t)/(\gamma(t) - z)$ is also continuous and piecewise continuously differentiable with piecewise differential given by,

$$\left(\frac{\phi(t)}{\gamma(t) - z}\right)' = 0.$$

Thus, this function is piecewise constant and continuous. So it must be constant on the connected set $[a,b]$. Equating its value at a with its value at b yields,

$$\phi(b) = \phi(a)\frac{\gamma(b) - z}{\gamma(a) - z} = 1,$$

since $\phi(a) = \exp(0) = 1$ and γ is closed.

Exercise. Show that $\mathrm{Ind}_\gamma(z) = \mathrm{Ind}_\gamma(z')$ if z and z' are in the same connected component of U. [Hint: Show first that $\mathrm{Ind}_\gamma : U \to \mathbb{C}$ is continuous.]

It remains to show that $\mathrm{Ind}_\gamma(z) = 0$ if z is sufficiently large. Let $M := \sup_{t\in[a,b]} |\gamma(t)|$. Then, if $|z| > M + l(\gamma)$ we have, using Proposition 2.6,

$$|\mathrm{Ind}_\gamma(z)| \le \frac{1}{2\pi}\frac{l(\gamma)}{|z| - M} < 1.$$

On the other hand $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$, so we must have in this case $\mathrm{Ind}_\gamma(z) = 0$. This completes the proof. □

Exercise 14. Let $\gamma : [0,1] \to \mathbb{C}$ be the path $\gamma(t) := z_0 + re^{2\pi\mathrm{i}kt}$ with $z_0 \in \mathbb{C}$ and $r > 0$ and $k \in \mathbb{Z}$. Show that $\mathrm{Ind}_\gamma(z) = k$ if $z \in B_r(z_0)$ and $\mathrm{Ind}_\gamma(z) = 0$ if $z \in \mathbb{C} \setminus \overline{B_r(z_0)}$.

Let B be an open disk in $\mathbb{C}$. We denote by ∂B its boundary, i.e., $\partial B = \overline{B} \setminus B$. We also denote by ∂B a closed path that traces the boundary ∂B once with positive (anti-clockwise) direction. If B has center z_0 and radius r, the path ∂B can be represented by the corresponding path γ of Exercise 14 with $k = 1$.

Lemma 2.10. *Let $z \in \mathbb{C}$, $r > 0$ and $k \in \mathbb{Z}$. Then,*

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(z)} (\zeta - z)^k \, \mathrm{d}\zeta = \begin{cases} 1 & \textit{if} \quad k = -1 \\ 0 & \textit{if} \quad k \in \mathbb{Z} \setminus \{-1\} \end{cases}.$$

Proof. **Exercise.** □

2.3 Integrable functions

Definition 2.11. Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$. If $F \in \mathcal{O}(D)$ such that $F' = f$, then F is called a *primitive* of f. f is called *integrable* in D if there exists such a primitive.

Theorem 2.12. *Let $D \subseteq \mathbb{C}$ be a region, $f : D \to \mathbb{C}$ be continuous and $F : D \to \mathbb{C}$. Then, F is a primitive of f iff for every path $\gamma : [a, b] \to D$*

$$\int_\gamma f(z) \, \mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Suppose F is a primitive of f. Assume without loss of generality that γ is continuously differentiable everywhere. Then, using the chain rule,

$$\int_\gamma f(z) \, \mathrm{d}z = \int_a^b F'(\gamma(t))\gamma'(t) \, \mathrm{d}t = \int_a^b (F \circ \gamma)'(t) \, \mathrm{d}t = F(\gamma(b)) - F(\gamma(a)).$$

Conversely, suppose that F satisfies the stated formula for every path γ in D. Let $z \in D$ and choose $r > 0$ such that $B_r(z) \subseteq D$. For $\xi \in B_r(0)$ let $\gamma_\xi : [0, 1] \to \mathbb{C}$ be the path $\gamma_\xi(t) := z + t\xi$. By assumption,

$$F(z + \xi) - F(z) = \int_{\gamma_\xi} f(\zeta) \, \mathrm{d}\zeta = \int_0^1 f(z + t\xi)\xi \, \mathrm{d}t$$

For $\xi \neq 0$ we get,

$$\frac{F(z + \xi) - F(z)}{\xi} = \int_0^1 f(z + t\xi) \, \mathrm{d}t.$$

The right hand side of this expression converges to $f(z)$ when $|\xi| \to 0$ since,

$$\begin{aligned} \left| \left(\int_0^1 f(z + t\xi) \, \mathrm{d}t \right) - f(z) \right| \leq \int_0^1 |f(z + t\xi) - f(z)| \, \mathrm{d}t & \\ \leq \sup_{\zeta \in B_{|\xi|}(0)} |f(z + \zeta) - f(z)|, \end{aligned}$$

where the right hand side expression converges to zero for $|\xi| \to 0$ by continuity of f. Thus, F is complex differentiable at z with the differential being $F'(z) = f(z)$. This completes the proof. □

Proposition 2.13. *Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ be continuous. Then, f is integrable in D iff for every closed path γ in D we have:*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. If f is integrable, then by Theorem 2.12 the integral along any close path must be zero. Conversely, suppose the integral of f along any closed path is zero. Choose $z_0 \in D$ arbitrarily. Define

$$F(z) := \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta,$$

where $\gamma_z : [a, b] \to D$ is a path such that $\gamma_z(a) = z_0$ and $\gamma_z(b) = z$. Such a path always exists by the path-connectedness of D. Also, the definition of $F(z)$ is well, since any other path with the same end points must yield the same value by assumption. $F : D \to \mathbb{C}$ defined in this way satisfies the assumption of Theorem 2.12 and is thus a primitive of f. $\square$

Definition 2.14. Let $D \subset \mathbb{C}$ be a region. We call D *star-shaped* with *center* $z_0 \in D$ iff for every element $z \in D$ the path $\gamma : [0, 1] \to \mathbb{C}$ given by $\gamma(t) := z_0 + t(z - z_0)$ lies entirely in D.

A triangle Δ is a closed subset of $\mathbb{C}$ with the shape of a triangle. Its boundary $\partial\Delta$ is the union of three straight line segments. We also denote by $\partial\Delta$ a closed path that traces the boundary of the triangle once in positive (i.e., counter-clockwise) direction.

Lemma 2.15. *Let $D \subseteq \mathbb{C}$ be a star-shaped region with center z_0. Let $f : D \to \mathbb{C}$ be continuous. Then, f is integrable in D iff for every triangle Δ in D with z_0 a corner,*

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0.$$

Proof. If f is integrable, we obtain the required implication as a special case of Proposition 2.13. Conversely, we show that f is integrable if the integral along all triangles in D with one vertex in z_0 vanishes. We define a function $F : D \to \mathbb{C}$ as follows. Let $z \in D$ and define the path $\gamma_z : [0, 1] \to \mathbb{C}$ by $\gamma_z(t) := z_0 + t(z - z_0)$. Since D is star-shaped with center z_0, the path γ_z lies entirely in D. Then set,

$$F(z) := \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta.$$

Fix $z \in D$. By star-shapedness of D there exist $r > 0$ such that $B_r(z) \subseteq D$ and for all $\zeta \in B_r(z)$ the path γ_ζ lies entirely in D. For all $\xi \in B_r(0)$ set $\tilde{\gamma}_\xi : [0, 1] \to \mathbb{C}$ to be the path $\tilde{\gamma}_\xi(t) = z + t\xi$. Then, by assumption,

$$F(z + \xi) - F(z) = \int_{\gamma_{z+\xi}} f(\zeta)\,\mathrm{d}\zeta - \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta = \int_{\tilde{\gamma}_\xi} f(\zeta)\,\mathrm{d}\zeta,$$

and we may proceed as in the proof of Theorem 2.12 to show that F is a primitive of f at z. This completes the proof. □

Proposition 2.16 (Integral Lemma of Goursat)**.** *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ and $\Delta \subset D$ a triangle. Then,*

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0.$$

Proof. We produce a sequence of triangles $\{\Delta_n\}_{n\in\mathbb{N}}$ with $\Delta_n \subset D$ by iteration. Set $\Delta_1 := \Delta$. To produce Δ_{n+1} from Δ_n proceed as follows. Subdivide Δ_n into four triangles $\Delta_{n,1}, \ldots, \Delta_{n,4}$ by subdividing each of its sides into two pieces of equal length. Now choose $k \in \{1,2,3,4\}$ such that the absolute value

$$\left|\int_{\partial\Delta_{n,k}} f(z)\,\mathrm{d}z\right|$$

is maximized and set $\Delta_{n+1} := \Delta_{n,k}$. This defines a sequence of triangles. Note that the intersection $\bigcap_{n\in\mathbb{N}} \Delta_n$ is a single point $z_0 \in D$.

By the addition property of the integral along paths we have for every $n \in \mathbb{N}$ the identity

$$\int_{\partial\Delta_n} f = \int_{\partial\Delta_{n,1}} f + \int_{\partial\Delta_{n,2}} f + \int_{\partial\Delta_{n,3}} f + \int_{\partial\Delta_{n,4}} f.$$

By the maximality condition of our construction, this implies, for all $n \in \mathbb{N}$,

$$\left|\int_{\partial\Delta_n} f\right| \le 4\left|\int_{\partial\Delta_{n+1}} f\right|, \tag{4}$$

and thus,

$$\left|\int_{\partial\Delta} f\right| \le 4^{n-1}\left|\int_{\partial\Delta_n} f\right|.$$

For the circumference of the triangles we obtain the relation,

$$l(\partial\Delta_n) = \frac{1}{2^{n-1}} l(\partial\Delta). \tag{5}$$

Now set $\epsilon > 0$ arbitrarily and choose $r > 0$ such that $B_r(z_0) \subseteq D$ and

$$|g(z)| \le \epsilon|z - z_0|, \quad \text{where} \quad g(z) := f(z) - f(z_0) - f'(z_0)(z - z_0)$$

for all $z \in B_r(z_0)$. (This is possible since f is complex differentiable at z_0.) Now fix $n \in \mathbb{N}$ such that $\Delta_n \subset B_r(z_0)$. Note that the constant function and the identity function are integrable so that with Proposition 2.13 we have,

$$\int_{\partial\Delta_n} f(z)\,\mathrm{d}z = \int_{\partial\Delta_n} \left(f(z_0) + f'(z_0)(z - z_0) + g(z)\right)\,\mathrm{d}z = \int_{\partial\Delta_n} g(z)\,\mathrm{d}z.$$

Using the estimate of Proposition 2.6, and (5),

$$\left|\int_{\partial\Delta_n} f\right| \leq \|g\|_{\partial\Delta_n} l(\partial\Delta_n) \leq \frac{\epsilon}{2}\, l(\partial\Delta_n)^2 = \frac{\epsilon}{2^{2n-1}} l(\partial\Delta)^2.$$

On the other hand, combining this with (4) we get,

$$\left|\int_{\partial\Delta} f\right| \leq \frac{\epsilon}{2} l(\partial\Delta)^2.$$

Since ϵ was arbitrary, we conclude that the integral of f along $\partial\Delta$ vanishes. □

Proposition 2.17 (Cauchy Integral Theorem for star-shaped regions)**.** *Let $D \subseteq \mathbb{C}$ be a star-shaped region and $f \in \mathcal{O}(D)$. Then, f is integrable in D.*

Proof. This is obtained by combining Lemma 2.15 with Proposition 2.16. □

We arrive at the important conclusion that a holomorphic function is integrable (in star-shaped regions). Soon we will see that the converse is also true: An integrable function is holomorphic.

Exercise 15. Let $D := \mathbb{C} \setminus [0, 1]$. Show that $f(z) := \frac{1}{z(z-1)}$ is integrable in D. [Hint: Observe that $f(z) = \frac{1}{z-1} - \frac{1}{z}$ and use primitives for the summands. Be careful about the domain of definition.]

Exercise 16. Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of continuous integrable functions converging uniformly to a function $f : D \to \mathbb{C}$. Show that f is integrable in D.

Exercise 17. Let $D_1, D_2 \subseteq \mathbb{C}$ be regions such that $D_1 \cap D_2$ is connected. Let $f : D_1 \cup D_2 \to \mathbb{C}$ be continuous. (a) Show that if f is integrable in D_1 and also integrable in D_2, then f is integrable in $D_1 \cup D_2$. (b) Give a counter example in the case when the connectedness condition is removed.

2.4 The Cauchy Integral Formula

In order to obtain Cauchy's integral formula, a key result of complex analysis, we need a sharpened version of Proposition 2.16.

Proposition 2.18. *Let $D \subseteq \mathbb{C}$ be a region and $p \in D$. Let $f : D \to \mathbb{C}$ be continuous function which is moreover holomorphic in $D \setminus \{p\}$. Let Δ be a triangle in D such that p is one of the corners of Δ. Then,*

$$\int_{\partial\Delta} f = 0.$$

Proof. Fix $\epsilon > 0$. Denote the corner points of Δ by (p, x, y). Define the triangle Δ_t for $t \in [0, 1]$ as the triangle with corner points (p, x_t, y_t), where $x_t := p + t(x - p)$ and $y_t := p + t(y - p)$. Then, $l(\partial\Delta_t) \to 0$ as $t \to 0$. By continuity of f on the compact set Δ, Proposition 2.6 implies that there exists $t > 0$ such that

$$\left| \int_{\partial\Delta_t} f \right| < \epsilon.$$

Now, subdivide the triangle Δ into the triangle Δ_t and the triangles with corners given by (x_t, x, y) and (x_t, y, y_t). The integral over boundary paths of the latter two triangles vanishes by Proposition 2.16. On the other hand, the sum of the integrals over the boundary paths of the three triangles equals the integral over the boundary path of Δ. Thus,

$$\left| \int_{\partial\Delta} f \right| = \left| \int_{\partial\Delta_t} f \right| < \epsilon.$$

Since ϵ was arbitrary the statement follows. □

Exercise 18. The above Proposition can be strengthened considerably. Show the following: Let $\Delta \subset \mathbb{C}$ be a triangle and let $f : \Delta \to \mathbb{C}$ be continuous. Furthermore, assume that f is holomorphic in the interior of Δ. Then,

$$\int_{\partial\Delta} f = 0.$$

The above proposition implies a corresponding stronger version of Proposition 2.17.

Proposition 2.19. *Let $D \subseteq \mathbb{C}$ be a star-shaped region with center $z_0 \in D$ and $f : D \to \mathbb{C}$ continuous. Furthermore assume that f is holomorphic in $D \setminus \{z_0\}$. Then, f is integrable in D.*

Proof. Combine Lemma 2.15 with Proposition 2.18. □

Theorem 2.20 (Cauchy Integral Formula)**.** *Let $D \subseteq \mathbb{C}$ be a star-shaped region with center z, $f \in \mathcal{O}(D)$, γ a closed path in $D \setminus \{z\}$. Then,*

$$f(z)\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

Proof. Define the function $g : D \to \mathbb{C}$ as follows,

$$g(\zeta) := \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{if}\, \zeta \neq z \\ f'(z) & \text{if}\, \zeta = z \end{cases}.$$

By the property of the complex derivative of f at z, g is continuous in all of D. Moreover, g is holomorphic in $D\setminus\{z\}$. So, by Proposition 2.19, g is integrable in D. By Proposition 2.13 this implies,

$$0 = \int_\gamma g = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - f(z)\,2\pi\mathrm{i}\,\mathrm{Ind}_\gamma(z).$$

□

The Cauchy Integral Formula is often used in the special case where the path is the boundary of a disk: Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$, $z \in D$ and $r > 0$ such that $\overline{B_r(z)} \subset D$. Then,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Lemma 2.21. *Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ continuous and γ a closed path in U. Define the function $F : \mathbb{C} \setminus |\gamma| \to \mathbb{C}$ via*

$$F(z) := \int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Then, F is analytic in $\mathbb{C} \setminus |\gamma|$. Moreover, for all $n \in \mathbb{N}_0$,

$$F^{(n)}(z) = n! \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta.$$

Proof. Fix $z_0 \in \mathbb{C} \setminus |\gamma|$ and define for all $n \in \mathbb{N}_0$,

$$c_n := \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta.$$

Set $r := \inf_{t\in[a,b]} |\gamma(t) - z_0|$. We proceed to show that the power series

$$G(z) := \sum_{n=0}^\infty c_n (z - z_0)^n$$

converges in $B_r(z_0)$ and agrees there with $F(z)$. Fix $z \in B_r(z_0)$. Define the partial sums $g_n : |\gamma| \to \mathbb{C}$ for $n \in \mathbb{N}_0$ via,

$$g_n(\zeta) := \sum_{k=0}^n \frac{f(\zeta)(z - z_0)^k}{(\zeta - z_0)^{k+1}}.$$

Since $|\zeta - z_0| \geq r > |z - z_0|$ and f is bounded on $|\gamma|$, the sequence of functions $\{g_n\}_{n\in\mathbb{N}_0}$ converges uniformly on $|\gamma|$. Thus, by Proposition 2.7,

$$G(z) = \lim_{n\to\infty} \int_\gamma g_n(\zeta)\,\mathrm{d}\zeta = \int_\gamma \lim_{n\to\infty} g_n(\zeta)\,\mathrm{d}\zeta.$$

In particular, $G(z)$ is well defined and its radius of convergence is at least r. Consider now the identity

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,$$

for $x \in B_1(0) \subset \mathbb{C}$. Inserting $x = (z - z_0)/(\zeta - z_0)$ and dividing by $(\zeta - z_0)$ we get,

$$\frac{1}{\zeta - z} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}.$$

This implies,

$$\lim_{n\to\infty} g_n(\zeta) = \frac{f(\zeta)}{\zeta - z},$$

and hence $G(z) = F(z)$.

Finally, Theorem 1.20 tells us that F is holomorphic and its complex derivatives are again analytic in the same region. In particular, formula (2) of that Theorem yields,

$$F^{(n)}(z) = n!\, c_n,$$

and thus the stated formula. □

Theorem 2.22 (Cauchy-Taylor Representation Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$. Then, f is analytic in D. Moreover, for any $z_0 \in D$ and $r > 0$ such that $\overline{B_r(z_0)} \subseteq D$ we have,*

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \,\mathrm{d}\zeta.$$

for all $z \in B_r(z_0)$.

Proof. Fix $z_0 \in D$ and $\rho > 0$ such that $B_\rho(z_0) \subseteq D$. Then choose r such that $0 < r < \rho$. This implies, $\overline{B_r(z_0)} \subset D$ and by Theorem 2.20 and Exercise 14 we have,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta.$$

for $z \in B_r(z_0)$. Lemma 2.21 then tells us that f is analytic in $B_r(z_0)$ and that

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \,\mathrm{d}\zeta.$$

for $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$. But since r can be chosen arbitrarily close to ρ, the radius of convergence of the power series for f around z_0 is actually at least ρ. Thus, f is analytic in D. This completes the proof. □

This Theorem finally yields the remarkable result that holomorphic functions are analytic. Together with Theorem 1.20 this means that the properties of holomorphicity and analiticity are really equivalent. Furthermore, it implies that the derivative of a holomorphic function is again a holomorphic function.

Definition 2.23. Let $D \subseteq \mathbb{C}$ be a region. We call $f : D \to \mathbb{C}$ *locally analytic* iff for every point $z \in D$ there is $r > 0$ so that f can be represented by a power series around z with radius of convergence r.

Definition 2.24. Let $D \subseteq \mathbb{C}$ be a region. We call $f : D \to \mathbb{C}$ *locally integrable* iff for every point $z \in D$ there is a neighborhood $U \subseteq D$ of z such that f is integrable in U.

Theorem 2.25. *Let $D \subseteq \mathbb{C}$ be a region. For a function $f : D \to \mathbb{C}$ the following statements are equivalent:*

1. *f is holomorphic in D.*
2. *f is analytic in D.*
3. *f is locally analytic in D.*
4. *f is locally integrable in D.*

Proof. **Exercise.** □

Exercise 19. Calculate the following integrals. [Hint: Use the Cauchy Integral formula]

1.
$$\int_{\partial B_2(0)} \frac{e^z}{(z+1)(z-3)^2}\,\mathrm{d}z$$

2.
$$\int_{\partial B_2(-2\mathrm{i})} \frac{1}{z^2+1}\,\mathrm{d}z$$

Exercise 20. Determine all entire functions $f \in \mathcal{O}(\mathbb{C})$ which satisfy the differential equation $f'' + f = 0$.

2.5 General Cauchy Theory

We have already seen that the index $\mathrm{Ind}_\gamma(z)$ of a point z with respect to a path γ is zero, if z lies in the connected component of $\mathbb{C} \setminus |\gamma|$ which is unbounded. This motivates the following definition.

Definition 2.26. Let γ be a closed path in $\mathbb{C}$. We define the *interior* of γ as the subset $\mathrm{Int}_\gamma := \{z \in \mathbb{C} \setminus |\gamma| : \mathrm{Ind}_\gamma(z) \neq 0\}$. Similarly, we define the *exterior* of γ as the subset $\mathrm{Ext}_\gamma := \{z \in \mathbb{C} \setminus |\gamma| : \mathrm{Ind}_\gamma(z) = 0\}$.

Obviously, we have the disjoint union $\mathbb{C} = \mathrm{Int}_\gamma \cup |\gamma| \cup \mathrm{Ext}_\gamma$.

Lemma 2.27. *Let $D \subseteq \mathbb{C}$ be a region and γ a path in D. Suppose $g : |\gamma| \times D \to \mathbb{C}$ is a continuous function such that $z \mapsto g(\zeta, z)$ is holomorphic for all $\zeta \in |\gamma|$. Then, the function $h : D \to \mathbb{C}$ given by*

$$h(z) := \int_\gamma g(\zeta, z)\,\mathrm{d}\zeta$$

is holomorphic.

Proof. Fix $z_0 \in D$. Let $U \subseteq D$ be a star-shaped neighborhood of z_0 with center z_0 (e.g. a disc centered at z_0). Then, for all $\zeta \in |\gamma|$ and all closed paths $\tilde{\gamma}$ in U we have,

$$\int_{\tilde{\gamma}} g(\zeta, z)\,\mathrm{d}z = 0,$$

by Proposition 2.13 since $z \mapsto g(\zeta, z)$ is holomorphic and thus integrable in U by Proposition 2.17. But we can interchange the order of integration by Fubini's Theorem to get

$$\int_{\tilde{\gamma}} h(z)\,\mathrm{d}z = \int_{\tilde{\gamma}} \left(\int_\gamma g(\zeta, z)\,\mathrm{d}\zeta \right) \mathrm{d}z = \int_\gamma \left(\int_{\tilde{\gamma}} g(\zeta, z)\,\mathrm{d}z \right) \mathrm{d}\zeta = 0.$$

Thus, h is integrable in U by Proposition 2.13 and therefore holomorphic in U by Theorem 2.25. Since z_0 was arbitrary, h is holomorphic in D. □

Theorem 2.28. *Let $D \subseteq \mathbb{C}$ be a region and γ a closed path in D. Then, the following conditions are equivalent:*

1. *All $f \in \mathcal{O}(D)$ satisfy*
$$\int_\gamma f = 0.$$
2. *For every $f \in \mathcal{O}(D)$ and every $z \in D \setminus |\gamma|$ we have,*
$$f(z)\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$
3. $\mathrm{Int}_\gamma \subset D$.

Proof. To show 2.⇒1. for a given $f \in \mathcal{O}(D)$, choose $z \in D \setminus |\gamma|$ arbitrarily and define $h \in \mathcal{O}(D)$ via $h(\zeta) := (\zeta - z)f(\zeta)$. Applying the formula of 2. to h yields 1. since $h(z) = 0$ by construction.

We proceed to show 1.⇒3. If $D = \mathbb{C}$ there is nothing to demonstrate. So assume the contrary and let $z_0 \in \mathbb{C} \setminus D$. We have to demonstrate that $\mathrm{Ind}_\gamma(z_0) = 0$. Define $f \in \mathcal{O}(D)$ via $f(z) := (z - z_0)^{-1}$. By 1.,

$$0 = \int_\gamma f = 2\pi\mathrm{i}\,\mathrm{Ind}_\gamma(z_0).$$

It remains to demonstrate 3.$\Rightarrow$2. Fix $f \in \mathcal{O}(D)$. Define the function $g : D \times D \to \mathbb{C}$ as follows,

$$g(\zeta, z) := \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{if } \zeta \neq z \\ f'(z) & \text{if } \zeta = z \end{cases}.$$

We proceed to show that g is continuous. For $(\zeta, z) \in D \times D$ such that $\zeta \neq z$ this is immediate. Thus, to consider the case $\zeta = z$ and fix $z \in D$. Let $r > 0$ such that $B_r(z) \subseteq D$. Consider the power series expansion of f around z,

$$f(\zeta) = \sum_{n=0}^{\infty} c_n (\zeta - z)^n,$$

for all $\zeta \in B_r(z)$. Then, for $\zeta, \xi \in B_r(z)$,

$$g(\zeta, \xi) = f'(z) + \sum_{n=2}^{\infty} c_n \sum_{k=1}^{n} (\zeta - z)^{n-k} (\xi - z)^{k-1}.$$

For $\zeta, \xi \in B_\rho(z)$ with $0 < \rho < r$ we have thus the estimate,

$$|g(\zeta, \xi) - g(z, z)| \le \left| \sum_{n=2}^{\infty} c_n \sum_{k=1}^{n} (\zeta - z)^{n-k} (\xi - z)^{k-1} \right| \le \sum_{n=2}^{\infty} n |c_n| \rho^{n-1}.$$

However, the series on the right hand side converges for all $\rho < r$ to a continuous function which goes to 0 when $\rho \to 0$. Thus, g is continuous at (z, z). Since z was arbitrary in D this completes the proof that g is continuous in $D \times D$. **Exercise.**Show that g is holomorphic in the second argument. Now we apply Lemma 2.27 to conclude that the function $h : D \to \mathbb{C}$ defined by

$$h(z) := \int_\gamma g(\zeta, z) \, \mathrm{d}\zeta$$

is holomorphic in D.

Now observe that for $z \in D \setminus |\gamma|$ we have

$$h(z) = \int_\gamma g(\zeta, z) \, \mathrm{d}\zeta = -2\pi \mathrm{i} \, f(z) \mathrm{Ind}_\gamma(z) + \int_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

Thus, to show 2. we need to show that $h = 0$. But if $z \in D \cap \mathrm{Ext}\gamma$, then $\mathrm{Ind}_\gamma(z) = 0$ and we get

$$h(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

However, this formula actually defines a holomorphic function in all of $\mathrm{Ext}\gamma$ by Lemma 2.21. Thus, we use it to extend h to a holomorphic function on the open set $D \cup \mathrm{Ext}\gamma$. Now recall that the assumption is that $\mathrm{Int}\gamma \subset D$. But this means $D \cup \mathrm{Ext}\gamma = \mathbb{C}$, i.e. h is entire. **Exercise.**Complete the proof. [Hint: Use Liouville's Theorem 1.26]. $\square$

3 Basic properties

3.1 Holomorphic convergence

Proposition 3.1. *Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of holomorphic functions $f_n \in \mathcal{O}(D)$ that converges uniformly on any compact subset of D to f. Then, $f \in \mathcal{O}(D)$ and the sequence $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ converges uniformly on any compact subset of D to $f^{(k)}$ for all $k \in \mathbb{N}$.*

Proof. Let $z_0 \in D$ and set $r > 0$ such that $\overline{B_r(z)} \subset D$. By Proposition 2.17 f_n is integrable in $B_r(z_0)$. For any closed path γ in $B_r(z_0)$ we thus have

$$\int_\gamma f = \int_\gamma \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\gamma f_n = 0,$$

where we have used Proposition 2.7 to interchange the integral with the limit. Thus, f is integrable in $B_r(z_0)$ and hence holomorphic there by Theorem 2.25. Since the choice of z_0 was arbitrary we find that f is holomorphic in all of D.

Fix $k \in \mathbb{N}$ and consider $z_0 \in D$. Choose $r > 0$ such that $\overline{B_{2r}(z_0)} \subseteq D$. Now for each $z \in B_r(z_0)$ we have the Cauchy estimate (Proposition 1.25),

$$|f_n^{(k)}(z) - f^{(k)}(z)| \le \frac{k!}{r^k}\|f_n - f\|_{\partial B_r(z)} \le \frac{k!}{r^k}\|f_n - f\|_{\overline{B_{2r}(z_0)}}.$$

For $\epsilon > 0$ there is by uniform convergence of $\{f_n\}_{n\in\mathbb{N}}$ an $n_0 \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon\, r^k/k!$ for all $n \ge n_0$ and all $z \in \overline{B_{2r}(z_0)}$. Hence, $|f_n^{(k)}(z) - f^{(k)}(z)| < \epsilon$ for all $n \ge n_0$ and all $z \in B_r(z_0)$. That is, $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ converges to $f^{(k)}$ uniformly on some neighborhood of every point of D. To obtain uniform convergence on a compact subset $K \subset D$ it is merely necessary to cover K with finitely many such neighborhoods. □

3.2 From local to global structure

Definition 3.2. Let T be a topological space and A a subset. We say that $p \in A$ is an *isolated point* of A in T iff there exists a neighborhood $U \subseteq T$ of p such that $U \cap A = \{p\}$. We say that A is *discrete* in T iff all its points are isolated.

Theorem 3.3 (Riemann Continuation Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region and $A \subset D$ a discrete and relatively closed subset. Suppose that $f \in \mathcal{O}(D \setminus A)$. Then, the following assertions are equivalent.*

1. *f extends to a holomorphic function on D.*
2. *f extends to a continuous function on D.*
3. *f is bounded in some neighborhood of any point of A.*

4. $\lim_{z \to z_0}(z - z_0)f(z) = 0$ *for each point* $z_0 \in A$.

Proof. The implications 1.⇒2.⇒3.⇒4. are clear. It remains to show 4.⇒1. It is sufficient to consider a single point $z_0 \in A$. Moreover, without loss of generality we may assume $z_0 = 0$. Since 0 is isolated, there exists an open neighborhood $U \subseteq D$ of 0 such that $U \cap A = \{0\}$. Define $g : U \to \mathbb{C}$ as follows,

$$g(z) := \begin{cases} zf(z) & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}.$$

By assumption, g is continuous in U. Define $h : U \to \mathbb{C}$ by $h(z) := zg(z)$. Since g is holomorphic in $U \setminus \{0\}$ so is h. Moreover, $h(z) = h(0) + zg(z) = h(0) + \mathrm{o}(|z|)$, so h is complex differentiable at 0 with differential $h'(0) = 0$. Thus, h is actually holomorphic in U. By Theorem 2.22 it can be represented for some radius of convergence $r > 0$ as a power series $h(z) = \sum_{n=0}^{\infty} c_n z^n$ around 0. But since $h(0) = 0$ and $h'(0) = 0$ we actually have $c_0 = 0$ and $c_1 = 0$ and thus $h(z) = z^2 \sum_{n=0}^{\infty} c_{n+2} z^n$, where the series still converges pointwise in $B_r(0)$. But since $h(z) = z^2 f(z)$ in $U \setminus \{0\}$, this implies that the power series $\sum_{n=0}^{\infty} c_{n+2} z^n$ coincides with f in $U \cap B_r(0) \setminus \{0\}$. This yields an analytic (and therefore holomorphic) extension of f to $(D \setminus A) \cup \{z_0\}$. □

Theorem 3.4 (Identity Theorem)**.** *Let D be a region and $f, g \in \mathcal{O}(D)$. The following statements are equivalent:*

1. $f = g$
2. *The coincidence set* $\{z \in D | f(z) = g(z)\}$ *is not empty and not discrete.*
3. *There exists a point* $z_0 \in D$ *such that* $f^{(n)}(z_0) = g^{(n)}(z_0)$ *for all* $n \in \mathbb{N}$.

Proof. The implication 1.⇒2. is trivial. We show 2.⇒3. Let $h := f - g$. Suppose $z_0 \in \{z \in D | h(z) = 0\}$ is not an isolated point. Suppose there exists $m \in \mathbb{N}_0$ such that $h^{(m)}(z_0) \neq 0$ and choose the smallest such m. Since h is holomorphic in D it is also analytic by Theorem 2.22 and has a power series expansion around z_0 for some radius $r > 0$, given by

$$h(z) = \sum_{n=m}^{\infty} \frac{h^{(n)}(z_0)}{n!}(z - z_0)^n = (z - z_0)^m k(z),$$

where $k : B_r(0) \to \mathbb{C}$ is the analytic function given by the power series,

$$k(z) = \sum_{n=0}^{\infty} \frac{h^{(n+m)}(z_0)}{(n+m)!}(z - z_0)^n.$$

In particular, $k(z_0) = h^{(m)}(z_0)/m! \neq 0$. But continuity of k at z_0 implies that there must be a neighborhood $U \subseteq D$ of z_0 such that $k(z) \neq 0$ for $z \in U$. But this implies $h(z) \neq 0$

for $z \in U \setminus \{z_0\}$, a contradiction to the assumption that z_0 is not an isolated point of the coincidence set.

We proceed to show the implication 3.⇒1. Set $S_n := \{z \in D | h^{(n)}(z) = 0\}$ for all $n \in \mathbb{N}_0$. Then, each S_n is closed in D and so is the intersection $S := \bigcap_{n=0}^{\infty} S_n$. On the other hand, S is open since given $z_1 \in S$ the power series expansion of h around z_1 has non-zero radius r of convergence by Theorem 2.22, but is identical to zero. So every point $z \in B_r(z_1)$ is element of S. Thus S is both open and closed in D. Connectedness of D implies that S is either empty or $S = D$. The first possibility is excluded by the assumption that $z_0 \in S$. So the power series of h is zero around any point of D, hence $h = 0$, implying $f = g$ in D. □

Corollary 3.5. *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{C}$ some function. For any region $D \subseteq \mathbb{C}$ such that $I \subset D$ there is at most one holomorphic function $g : D \to \mathbb{C}$ such that $f(z) = g(z)$ for all $z \in I$.*

This is relevant when we are interested in extending functions on $\mathbb{R}$ or some interval $I \subset \mathbb{R}$ to holomorphic functions on the complex plane.

Theorem 3.6 (Maximum Modulus Principle)**.** *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Suppose that $|f|$ has a local maximum at some point $z \in D$, i.e., that $|f(z)| = \|f\|_U := \sup_{\zeta \in U} |f(\zeta)|$ for some neighborhood $U \subseteq D$ of z, then f is constant.*

Proof. Given a point $z \in D$ and a neighborhood U of z as described, consider the power series expansion $f(\zeta) = \sum_{n=0}^{\infty} c_n (\zeta - z)^n$ of f around z. Let $\rho > 0$ such that $\overline{B_\rho(z)} \subseteq U$. Then, the power series converges with radius at least ρ and for $0 < r < \rho$ we have, by Lemma 1.24,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \le \|f\|^2_{\partial B_r(z)} \le \|f\|^2_U = |f(z)|^2 = |c_0|^2.$$

This implies $c_k = 0$ for all $k \in \mathbb{N}$, i.e., f is constant in $B_\rho(z)$. But then the Identity Theorem (Theorem 3.4) ensures that f is constant in all of D. □

Proposition 3.7. *Let $D \subseteq \mathbb{C}$ be a bounded region and K its closure. Suppose $f : K \to \mathbb{C}$ is continuous and its restriction to D is holomorphic. Then,*

$$|f(z)| \le \|f\|_{\partial D} \quad \forall z \in D.$$

In case of equality for some $z \in D$, f is constant.

Proof. If f is constant the inequality is an equality and is valid trivially. Thus, suppose that f is not constant. Since K is compact and f is continuous on K there exists a point $z \in K$ such that $|f(z)| = \|f\|_K$. We have to show that necessarily $z \in \partial D = K \setminus D$. Assume to the contrary that $z \in D$. Since $|f(z)| = \|f\|_K = \|f\|_D$ we can apply Theorem 3.6 with $U = D$, concluding that f is constant, a contradiction. □

Proposition 3.8 (Minimum Principle)**.** *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Suppose that $|f|$ has a local minimum at some point $z \in D$, i.e., that $|f(z)| = \inf_{\zeta \in U} |f(\zeta)|$ for some neighborhood $U \subseteq D$ of z. Then, $f(z) = 0$ or f is constant in D.*

Proof. Let $z \in D$ be a local minimum and U a neighborhood of z as described. Without loss of generality we may assume that U is connected, i.e. a region. If $f(z) = 0$ we are done. Thus, suppose $f(z) \neq 0$. Since z is local minimum of $|f|$ in U, $f(\zeta) \neq 0$ for all $\zeta \in U$. So, $1/f \in \mathcal{O}(U)$. But $|1/f|$ has a local maximum at z and we may apply Theorem 3.6 to conclude that $1/f$ is constant in U. But then f is constant in U and by Theorem 3.4 constant in D. $\square$

Proposition 3.9. *Let $D \subseteq \mathbb{C}$ be a bounded region and K its closure. Suppose $f : K \to \mathbb{C}$ is continuous and its restriction to D is holomorphic. Then, either f has zeros in D or*

$$|f(z)| \geq \inf_{\zeta \in \partial D} |f(\zeta)| \quad \forall z \in D.$$

Proof. **Exercise.** $\square$

Exercise 21. Let $D \subseteq \mathbb{C}$ a region, $a \in D$. Suppose that $f \in \mathcal{O}(D \setminus \{a\})$. Show that f has a holomorphic extension to D if f' has.

Exercise 22. Let f, g be entire functions satisfying $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there is $a \in \mathbb{C}$ such that $f = ag$.

Exercise 23. Let $D \subseteq \mathbb{C}$ be a region and $L \subset \mathbb{C}$ be a straight line. Let $f : D \to \mathbb{C}$ be continuous and f holomorphic in $D \setminus L$. Show that f is actually holomorphic in all of D.

Exercise 24. Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Suppose that there exists $z \in D$ such that $f^{(n)}(z) = 0$ for almost all $n \in \mathbb{N}$. Show that f is a polynomial.

Exercise 25. Let $D \subseteq \mathbb{C}$ be a region such that if $z \in D$ then $\overline{z} \in D$. Show that for $f \in \mathcal{O}(D)$ the following statements are equivalent:

1. $f(D \cap \mathbb{R}) \subseteq \mathbb{R}$.
2. $f(\overline{z}) = \overline{f(z)}$ for all $z \in D$.

Exercise 26. For each of the following properties give an example for a holomorphic function defined in some disk around 0 with that property or show that there can be no such function.

1. $f(1/n) = (-1)^n/n$ for almost all $n \in \mathbb{N}$.
2. $f(1/n) = 1/(n^2 - 1)$ for almost all $n \in \mathbb{N} \setminus \{1\}$.
3. $|f^{(n)}(0)| \geq (n!)^2$ for almost all $n \in \mathbb{N}_0$.
4. $|f(1/n)| \leq e^{-n}$ for almost all $n \in \mathbb{N}$ and $f \neq 0$.

3.3 The Open Mapping Theorem

Definition 3.10. Let X, Y be topological spaces. A map $f : X \to Y$ is called *open* iff for every open set $U \subseteq X$ the image $f(U)$ is open in Y.

Lemma 3.11. *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Let $z \in D$ and $r > 0$ such that $\overline{B_r(z)} \subset D$ and $2\delta := \inf_{\zeta \in \partial B_r(z)} |f(\zeta) - f(z)| > 0$. Then, $B_\delta(f(z)) \subseteq f(B_r(z))$.*

Proof. Let $a \in B_\delta(f(z))$. Then,

$$|f(\zeta) - a| \geq |f(\zeta) - f(z)| - |a - f(z)| > \delta \quad \forall \zeta \in \partial B_r(z).$$

In particular, $\inf_{\zeta \in \partial B_r(z)} |f(\zeta) - a| > |f(z) - a|$. Thus, by Proposition 3.9 $f - a$ must have zeros in the region $B_r(z)$. That is, there exists $\xi \in B_r(z)$ such that $f(\xi) = a$. □

Theorem 3.12 (Open Mapping Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$ such that f is not constant. Then f is an open map $D \to \mathbb{C}$.*

Proof. Let $U \subseteq D$ be open. Let $z \in U$. It is enough to show that $f(U)$ contains a disc centered around $f(z)$. Since f is not constant, by the Identity Theorem (Theorem 3.4) there is a radius $r > 0$ such that $f(z) \notin f(\partial B_r(z))$ while $B_r(z) \subseteq U$. Then $2\delta := \inf_{\zeta \in \partial B_r(z)} |f(\zeta) - f(z)| > 0$ and Lemma 3.11 can be applied, showing that $B_\delta(f(z)) \subseteq f(B_r(z)) \subseteq f(U)$. □

3.4 Zeros

Definition 3.13. Let $D \subseteq \mathbb{C}$ be a region, $z_0 \in D$ and $f \in \mathcal{O}(D)$ such that $f(z_0) = 0$. We say that f has a *zero of order* n at z_0 iff there exists $g \in \mathcal{O}(D)$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^n g(z)$ for all $z \in D$.

Proposition 3.14. *Let $D \subseteq \mathbb{C}$ be a region, $z_0 \in D$ and $f \in \mathcal{O}(D)$ such that $f(z_0) = 0$. If f is not constant, then there exists a unique $n \in \mathbb{N}$ such that f has a zero of order n at z_0. Moreover, $n = \inf\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\}$.*

Proof. **Exercise.** □

Proposition 3.15 (Fundamental Theorem of Algebra)**.** *Let $n \in \mathbb{N}$ and $p(z) = \sum_{k=0}^{n} c_k z^k$ be a polynomial of degree n (i.e., $c_n \neq 0$). Then, there are constants $a_1, \ldots, a_n \in \mathbb{C}$ such that p factorizes as*

$$p(z) = c_n (z - a_1) \cdots (z - a_n).$$

Proof. **Exercise.**[Hint: First show the existence of one zero and factorize it, then proceed recursively.] □

Theorem 3.16. *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ such that it has distinct zeros $a_1, \ldots, a_m \in D$ with orders $n_1, \ldots, n_m$. Suppose γ is a closed path in $D \setminus \{a_1, \ldots, a_n\}$ such that $\mathrm{Int}_\gamma \subset D$. Then,*

$$\sum_{k=1}^{m} n_k \mathrm{Ind}_\gamma(a_k) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z.$$

Proof. Knowing the zeros, we can factorize f as

$$f(z) = (z - a_1)^{n_1} \cdots (z - a_m)^{n_m} g(z),$$

where $g \in \mathcal{O}(D)$ has no zeros in D. Using the product rule for the derivative we find for $z \in D \setminus \{a_1, \ldots, a_n\}$,

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{k=1}^{m} \frac{n_k}{z - a_k}.$$

The term g'/g on the right hand side is a holomorphic function in D. So, by Theorem 2.28 its integral along γ vanishes. The second term yields the desired sum over the indices of the a_k. □

Exercise 27. Let $D \subseteq \mathbb{C}$ be a region and $a \in D$. For a function $f \in \mathcal{O}(D)$ we denote by $n_a(f)$ the order of its zero at a. (If $f(a) \neq 0$ then $n_a(f) = 0$.) For all $f, g \in \mathcal{O}(D)$ show the following:

1. $n_a(fg) = n_a(f) + n_a(g)$.
2. $n_a(f + g) \geq \min\{n_a(f), n_a(g)\}$ and equality if $n_a(f) \neq n_a(g)$.

3.5 Holomorphic logarithms and roots

Definition 3.17. A region $D \subseteq \mathbb{C}$ is called *homologically simply connected* iff all holomorphic functions in D are integrable.

Remark 3.18. Theorem 2.28 together with Proposition 2.13 imply that all holomorphic functions are integrable in a region $D \subseteq \mathbb{C}$ iff every closed path γ in D satisfies $\mathrm{Int}_\gamma \subset D$. So this provides an alternative definition of homologically simple connectedness. In fact it turns out that the adjective "homologically" is superfluous as the notion is equivalent to simple connectedness. However, we will not prove this here.

Definition 3.19. Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Then, $g \in \mathcal{O}(D)$ is called a *holomorphic logarithm* of f iff $f = \exp g$.

Theorem 3.20. *Let $D \subseteq \mathbb{C}$ be a homologically simply connected region and $f \in \mathcal{O}(D)$ zero-free. Then, there exists a holomorphic logarithm of f in D.*

Proof. By the assumptions $f'/f \in \mathcal{O}(D)$ and integrable. Let $h \in \mathcal{O}(D)$ be a primitive. Define $k := f \exp(-h) \in \mathcal{O}(D)$. As is easy to check, $k' = 0$ so $k = c$ for all $z \in D$ for some constant $c \in \mathbb{C}$. This implies $f = c \exp h$ and $c \neq 0$ since f is zero-free. Since exp takes all complex values except zero, there is $b \in \mathbb{C}$ with $c = \exp(b)$. Then, $g := h + b \in \mathcal{O}(D)$ is the looked for holomorphic logarithm with $f = \exp g$. □

Definition 3.21. Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ and $n \in \mathbb{N}$. Then, a *(holomorphic) nth root* of f is a function $g \in \mathcal{O}(D)$ such that $f = g^n$.

Theorem 3.22. *Let $D \subseteq \mathbb{C}$ be a homologically simply connected region and $f \in \mathcal{O}(D)$ zero-free. Then, there exists an nth root of f for every $n \in \mathbb{N}$.*

Proof. According to Theorem 3.20 there is a holomorphic logarithm $g \in \mathcal{O}(D)$ of f. An nth root of f is given by

$$z \mapsto \exp\left(\frac{1}{n} g(z)\right) \quad \forall z \in D.$$

□

__Exercise__ **28.** Let $D, D' \subseteq \mathbb{C}$ be homologically simply connected regions. Suppose that $D'' := D \cap D'$ is connected and non-empty. Show that D'' is homologically simply connected.

__Exercise__ **29.** Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ such that f is not constant. Let $a \in D$. Show the equivalence of the following statements:

1. There exists a neighborhood $U \subseteq D$ of a such that f has a holomorphic square-root in U.

2. $f(a) \neq 0$ or $f(a) = 0$ and the order of the zero is even.

4 Singularities

4.1 Types of singularities

Definition 4.1. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Then, we say that f has an *isolated singularity* at a. Moreover, a is called a *removable singularity* iff f can be extended to a holomorphic function on all of D.

We have already seen criteria for identifying removable singularities in the Riemann Continuation Theorem 3.3.

Definition 4.2. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. We say that a is a *pole* of f iff f diverges at a, i.e. if for any $M > 0$ there exists $r > 0$ such that $|f(z)| > M$ for all $z \in B_r(a) \setminus \{a\}$. We say that a is an *essential singularity* of f iff a is not removable and is not a pole.

We now consider poles.

Proposition 4.3. *Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Suppose that a is a pole of f. Then, there exists a unique $m \in \mathbb{N}$ such that there is a $g \in \mathcal{O}(D)$ with $g(a) \neq 0$ and*

$$f(z) = \frac{g(z)}{(z-a)^m} \quad \forall z \in D \setminus \{a\}.$$

Proof. Since f has a pole at a there exist $r > 0$ such that $B_r(a) \subseteq D$ and $f(z) \neq 0$ for all $z \in B_r(a) \setminus \{a\}$. Thus we can define $h \in \mathcal{O}(B_r(a) \setminus \{a\})$ by $h(z) := 1/f(z)$. But $\lim_{z \to a} h(z) = 0$, so by Theorem 3.3, a is a removable singularity of h and h can be extended to a holomorphic function on all of $B_r(a)$. By Proposition 3.14 there exists a unique $m \in \mathbb{N}$ such that $h(z) = (z-a)^m k(z)$, where $k \in \mathcal{O}(B_r(a))$ and $k(a) \neq 0$. Moreover, $k(z) \neq 0$ for all $z \in B_r(a)$ so we can invert it, defining $g \in \mathcal{O}(B_r(a))$ by $g(z) = 1/k(z)$. But notice that $g(z) = (z-a)^m f(z)$ for all $z \in B_r(a) \setminus \{a\}$, which obviously extends to a holomorphic function on $D \setminus \{a\}$. So g really extends to a holomorphic function on all of D. Observe also that $g(a) \neq 0$. This completes the proof. □

Definition 4.4. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$, $f \in \mathcal{O}(D \setminus \{a\})$ such that a is a pole of f. Then, the integer $m \in \mathbb{N}$ such that $g(z) := (z-a)^m f(z)$ extends to a holomorphic function in D with $g(a) \neq 0$ is called the *order* of the pole. If $m = 1$ we also say that the pole is *simple*.

Proposition 4.5. *Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$ with a pole at a of order m. Then, there is a function $g \in \mathcal{O}(D)$ and there are constants $b_1, \ldots, b_m \in \mathbb{C}$ with $b_m \neq 0$ such that*

$$f(z) = g(z) + \sum_{n=1}^{m} \frac{b_n}{(z-a)^n} \quad \forall z \in D \setminus \{a\}.$$

Proof. **Exercise.** □

The second term on the right hand side of the equation above is also called the *singular part* of f at a.

We now turn to essential singularities. In some sense they are more "wild" than poles, as shows the following Theorem.

Theorem 4.6 (Casorati, Weiserstrass)**.** *Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. The following statements are equivalent:*

1. *The point a is an essential singularity of f.*
2. *For every neighborhood $U \subseteq D$ of a the set $f(U \setminus \{a\})$ is dense in $\mathbb{C}$.*
3. *There exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $D \setminus \{a\}$ such that $\lim_{n\to\infty} z_n = a$, but $\{f(z_n)\}_{n\in\mathbb{N}}$ has no limit in $\mathbb{C} \cup \{\infty\}$.*

Proof. We start with the implication 1.⇒2. Assume the contrary of 2. Let $U \subseteq D$ be a neighborhood of a such that $f(U \setminus \{a\})$ is not dense in $\mathbb{C}$. Thus, there exists $p \in \mathbb{C}$ and $r > 0$ such that $f(U \setminus \{a\}) \cap B_r(p) = \emptyset$. This implies $|f(z) - p| \geq r$ for all $z \in U \setminus \{a\}$. Define $g \in \mathcal{O}(U \setminus \{a\})$ by $g(z) := 1/(f(z) - p)$. Then, $|g(z)| \leq 1/r$ for all $z \in U \setminus \{a\}$ so by Theorem 3.3, g has a removable singularity at a. Thus, $c := \lim_{z\to a} g(z)$ exists. If $c \neq 0$, $f(z) = p + 1/g(z)$ is bounded near a and thus has a removable singularity at a. If $c = 0$, then $\lim_{z\to a} |f(z)| = \infty$ and f has a pole at a. In both cases, a is not an essential singularity, contradicting 1. **Exercise.**Complete the proof. □

Exercise 30. Find and classify the isolated singularities of the following functions and specify the order in case of a pole:

1.	$\dfrac{z^4}{(z^4+16)^2}$	2.	$\dfrac{1-\cos(z)}{\sin z}$
3.	$\exp(1/z)$	4.	$\dfrac{1}{\cos(1/z)}$

Exercise 31. Let f be a function that is holomorphic in $\mathbb{C}$ except for poles. Show that the set of poles is discrete and closed.

Exercise 32. Investigate how the different types of singularities behave with respect to addition, multiplication, quotienting and composition (whenever the corresponding operations make sense)!

Exercise 33. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Show that if a is a non-removable singularity of f, then $\exp \circ f \in \mathcal{O}(D \setminus \{a\})$ has an essential singularity at a.

Exercise 34. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Let $P \in \mathcal{O}(\mathbb{C})$ be a non-constant polynomial. Show that f and $P \circ f$ have the same type of singularity at a.

4.2 Meromorphic functions

Definition 4.7. Let $D \subseteq \mathbb{C}$ be a region and $A \subset D$ a discrete and relatively closed subset. Then, $f \in \mathcal{O}(D \setminus A)$ is called *meromorphic* in D if all points $a \in A$ are either removable singularities or poles of f. The set of meromorphic functions in D is denoted by $\mathcal{M}(D)$.

Proposition 4.8. *Let $D \subseteq \mathbb{C}$ be a region. Then, the set $\mathcal{M}(D)$ forms a vector space over $\mathbb{C}$ and moreover forms a field. That is, sums, scalar multiples, products and quotients of meromorphic functions are meromorphic. (Except the quotient by the zero function.)*

Proof. **Exercise.** □

Exercise 35. Show that the set of rational functions forms a proper subfield of $\mathcal{M}(\mathbb{C})$.

Theorem 4.9 (Argument Principle)**.** *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{M}(D)$. Suppose $Z \subset D$ is the set of zeros of f and $P \subset D$ is the set of poles of f. Suppose γ is a closed path in $D \setminus (Z \cup P)$ such that* $\mathrm{Int}_\gamma \subset D$. *Then,*

$$\sum_{z\in Z} N(z)\mathrm{Ind}_\gamma(z) - \sum_{z\in P} N(z)\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z,$$

where $N(z)$ is the order of the zero or pole z.

Proof. **Exercise.**[Hint: Generalize the proof of Theorem 3.16.] □

Theorem 4.10 (Rouché's Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region and $f, g \in \mathcal{M}(D)$. Let $Z_f, Z_g \subset D$ be the sets of zeros of f and g and $P_f, P_g \subset D$ the sets of poles of f and g. Let γ be a closed path such that $|\gamma| \subset D \setminus (P_f \cup P_g)$ and* $\mathrm{Int}\gamma \subset D$. *Suppose that*

$$|f(\zeta) + g(\zeta)| < |f(\zeta)| + |g(\zeta)| \quad \forall \zeta \in |\gamma|.$$

Then,

$$\sum_{z\in Z_f} N(z)\mathrm{Ind}_\gamma(z) - \sum_{z\in P_f} N(z)\mathrm{Ind}_\gamma(z) = \sum_{z\in Z_g} N(z)\mathrm{Ind}_\gamma(z) - \sum_{z\in P_g} N(z)\mathrm{Ind}_\gamma(z),$$

where $N(z)$ denotes the order of the zero or pole z.

Proof. First, note that the inequality also implies $|\gamma| \cap Z_f = \emptyset$ and $|\gamma| \cap Z_g = \emptyset$. Set $U := D \setminus (Z_f \cup Z_g \cup P_f \cup P_g)$ and $h(z) := f(z)/g(z)$ for all $z \in U$. Then, $h \in \mathcal{O}(U)$. Note that the hypothesis is equivalent to the inequality

$$|h(z) + 1| < |h(z)| + 1 \quad \forall z \in |\gamma|.$$

This inequality implies that $h(z)$ cannot be a non-negative real number (since in that case there would be equality). That is, $h(z) \in \mathbb{C} \setminus \mathbb{R}_0^+$ for all $z \in |\gamma|$. But since $\mathbb{C} \setminus \mathbb{R}_0^+$ is open, there must a neighborhood $V \subseteq U$ of $|\gamma|$ such that $h(z) \in \mathbb{C} \setminus \mathbb{R}_0^+$ for all $z \in V$. Now, $\mathbb{C} \setminus \mathbb{R}_0^+$ is star-shaped so that $z \mapsto 1/z$ is integrable there (Proposition 2.17), i.e., has a primitive $l \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R}_0^+)$. ($l$ is in fact a branch of the logarithm.) But $l \circ h \in \mathcal{O}(V)$ is a primitive of $h'/h \in \mathcal{O}(V)$, so the integral of h'/h along γ vanishes (by Proposition 2.13). This means,

$$0 = \int_\gamma \frac{h'(z)}{h(z)} \,\mathrm{d}z = \int_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z - \int_\gamma \frac{g'(z)}{g(z)} \,\mathrm{d}z.$$

The result follows then from Theorem 4.9. □

Theorem 4.11 (Hurwitz)**.** *Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $f_n \in \mathcal{O}(D)$ converging uniformly in every compact subset of D to f. Let $a \in D$ and $r > 0$ such that $\overline{B_r(a)} \subset D$. Suppose that $f(z) \neq 0$ for all $z \in \partial B_r(a)$. Then, there exists $n_0 \in \mathbb{N}$ such that f and f_n have the same number of zeros in $B_r(a)$ for all $n \geq n_0$.*

Proof. Set $\delta := \inf\{|f(z)| : z \in \partial B_r(a)\}$. By the assumptions $\delta > 0$ and $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $\partial B_r(a)$. Thus, there exists $n_0 \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \delta/2$ for all $n \geq n_0$ and all $z \in \partial B_r(a)$. But this implies,

$$|f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)| \leq |f(z)| + |f_n(z)| \quad \forall n \geq n_0, \forall z \in \partial B_r(a).$$

Applying Rouché's Theorem 4.10 yields the desired result. □

Proposition 4.12. *Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $f_n \in \mathcal{O}(D)$ converging uniformly in every compact subset of D to f. Suppose that for all $n \in \mathbb{N}$, f_n has no zeros. Then, either $f = 0$ or f has no zeros.*

Proof. **Exercise.** □

Exercise 36. Let $D, D' \subseteq \mathbb{C}$ be regions such that $D' \subset D$. Consider the linear map $\mathcal{O}(D) \to \mathcal{O}(D')$ induced by the restriction of functions on D to D'. (a) Show either that this map must be injective or that it cannot be injective. (b) Show either that this map must be surjective or that it cannot be surjective.

Exercise 37. Let $D \subseteq \mathbb{C}$ be a bounded region. Define $\tilde{\mathcal{O}}(D) \subseteq \mathcal{O}(D)$ to be the set of holomorphic functions f on D such that f extends to a holomorphic function on some open neighborhood of $\overline{D}$. Likewise, define $\tilde{\mathcal{M}}(D) \subseteq \mathcal{M}(D)$ to be the set of meromorphic functions f on D such that f extends to a meromorphic function on some neighborhood of $\overline{D}$. (a) Show that $\tilde{\mathcal{O}}(D)$ is a proper subalgebra of $\mathcal{O}(D)$. Likewise, show that $\tilde{\mathcal{M}}(D)$ is a proper subfield of $\mathcal{M}(D)$. (b) Show that $\tilde{\mathcal{M}}(D)$ is the quotient field of $\tilde{\mathcal{O}}(D)$. In other words, show that for every element $f \in \tilde{\mathcal{M}}(D)$ there exist elements $g, h \in \tilde{\mathcal{O}}(D)$ such that $f = g/h$. (c) Comment on the possible problems that would appear if one replaces in this exercise $\tilde{\mathcal{O}}(D)$ with $\mathcal{O}(D)$ and $\tilde{\mathcal{M}}(D)$ with $\mathcal{M}(D)$.

Exercise 38. Let $D \subseteq \mathbb{C}$ be a region such that $\overline{B_1(0)} \subset D$ and $f \in \mathcal{O}(D)$. Suppose $|f(z)| < 1$ for all $z \in \partial B_1(0)$. Show that f has precisely one fixed point in $B_1(0)$.

Exercise 39. Determine the number of zeros (counted with order) of the following functions in the specified domain:

1. $z^5 + \frac{1}{3}z^3 + \frac{1}{4}z^2 + \frac{1}{3}$ in $B_1(0)$ and in $B_{1/2}(0)$.
2. $z^5 + 3z^4 + 9z^3 + 10$ in $B_1(0)$ and $B_2(0)$.
3. $9z^5 + 5z - 3$ in $B_5(0) \setminus \overline{B_{1/2}(0)}$.
4. $z^8 + z^7 + 4z^2 - 1$ in $B_1(0)$ and $B_2(0)$.

4.3 Laurent Series

The representation of a holomorphic function with a pole as in Proposition 4.5 can be written as an "extended" power series that starts not with the power 0, but with the power $-n$. Indeed, we will see that even essential singularities can be captured by such an "extended" power series, if we start at $-\infty$. Such series are called *Laurent series.*

Let $z \in \mathbb{C}$ and $0 < r_1 < r_2$. In the following we use the notation

$$A_{r_1,r_2}(z) := B_{r_2}(z) \setminus \overline{B_{r_1}(z)}.$$

This type of region is called an (open) annulus. Note the special case of the punctured disk $A_{0,r}(z) = B_r(z) \setminus \{z\}$.

Definition 4.13. Let $\{a_n\}_{n\in\mathbb{Z}}$ be an indexed set of complex numbers. We say that $\sum_{n\in\mathbb{Z}} a_n$ *converges (absolutely)* iff $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ both converge (absolutely). Let S be a set and $\{f_n\}_{n\in\mathbb{Z}}$ be an indexed set of functions $f_n : S \to \mathbb{C}$. We say that $\sum_{n\in\mathbb{Z}} f_n$ converges uniformly iff $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=1}^{\infty} f_{-n}$ both converge uniformly.

Proposition 4.14. *Let $\{c_n\}_{n\in\mathbb{Z}}$ be an indexed set of complex numbers. Define $r_1, r_2 \in [0,\infty]$ via*

$$r_1 := \limsup_{n\to\infty} |c_{-n}|^{1/n} \quad \textit{and} \quad 1/r_2 := \limsup_{n\to\infty} |c_n|^{1/n}.$$

Iff $r_1 < r_2$ then the Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} c_n z^n$$

converges absolutely for all $z \in A_{r_1,r_2}(0)$ and uniformly on $A_{\rho_1,\rho_2}(0)$ where $r_1 < \rho_1 < \rho_2 < r_2$. Moreover, it diverges for $z \in \mathbb{C} \setminus \overline{A_{r_1,r_2}(0)}$.

Proof. **Exercise.**[Hint: Split the series into the parts with positive and negative indices and apply Lemma 1.18.] □

Proposition 4.15. *Let $D \subseteq \mathbb{C}$ be a region, $z_0 \in \mathbb{C}$ and $0 \le r_1 < r_2$ such that $\overline{A_{r_1,r_2}(z_0)} \subset D$. Then, for all $f \in \mathcal{O}(D)$ we have,*

$$\int_{\partial B_{r_1}(z_0)} f = \int_{\partial B_{r_2}(z_0)} f.$$

Moreover, for all $z \in A_{r_1,r_2}(z_0)$ we have,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

Proof. **Exercise.** □

Theorem 4.16 (Laurent Decomposition)**.** *Let $z_0 \in \mathbb{C}$ and $0 \le r_1 < r_2 \le \infty$ and $f \in \mathcal{O}(A_{r_1,r_2}(z_0))$. Then, there exists a unique pair of holomorphic functions $f^+ \in \mathcal{O}(B_{r_2}(z_0))$ and $f^- \in \mathcal{O}(\mathbb{C} \setminus \overline{B_{r_1}(z_0)})$ such that*

$$f(z) = f^+(z) + f^-(z), \quad \forall z \in A_{r_1,r_2}(z_0) \quad \textit{and} \quad \lim_{|z|\to\infty} f^-(z) = 0$$

Proof. For any $r_1 < s < r_2$ define $f_s : \mathbb{C} \setminus \partial B_s(z_0) \to \mathbb{C}$ via

$$f_s(z) := \frac{1}{2\pi \mathrm{i}} \int_{\partial B_s(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta,$$

By Lemma 2.27, f_s is holomorphic. Now define $f^+ : B_{r_2}(z_0) \to \mathbb{C}$ as follows. For a given z choose $r_1 < s < r_2$ such that $|z - z_0| < s$ and set $f^+(z) := f_s(z)$. Proposition 4.15 ensures that this definition does not depend on the choice of s. Moreover, it is clear that this defines a holomorphic function. Similarly, we define $f^- : \mathbb{C} \setminus \overline{B_{r_1}(z_0)} \to \mathbb{C}$ as follows. For a given z choose $r_1 < s < r_2$ such that $s < |z - z_0|$ and set $f^-(z) := -f_s(z)$. Again, this definition does not depend on the choice of s and f^- is holomorphic.

Now let $z \in A_{r_1,r_2}(z_0)$ and choose s_1, s_2 such that $r_1 < s_1 < |z - z_0| < s_2 < r_2$. Then, by Proposition 4.15 we have,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{s_2}(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{s_1}(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = f^+(z) + f^-(z).$$

Fix $r_1 < s < r_2$ and choose $\epsilon > 0$. Now if

$$|z| > \frac{s\|f\|_{\partial B_s(z_0)}}{\epsilon} + s + |z_0|,$$

then we have $|f^-(z)| < \epsilon$ by an application of the integral estimate of Proposition 2.6. Thus $\lim_{|z|\to\infty} f^-(z) = 0$.

To see uniqueness suppose there is another pair of holomorphic functions $g^+ \in \mathcal{O}(B_{r_2}(z_0))$ and $g^- \in \mathcal{O}(\mathbb{C} \setminus \overline{B_{r_1}(z_0)})$ with the same properties. Then, $h(z) := f^+(z) - g^+(z)$ defines a holomorphic function on $B_{r_2}(z_0)$. Moreover, for $z \in A_{r_1,r_2}(z_0)$ we also have $h(z) = g^-(z) - f^-(z)$. But the latter are even defined on $\mathbb{C} \setminus \overline{B_{r_1}(z_0)}$. So h extends to an entire function. But, $\lim_{|z|\to\infty} h(z) = \lim_{|z|\to\infty} g^-(z) - \lim_{|z|\to\infty} f^-(z) = 0$. So by Liouville's Theorem (Theorem 1.26) h must be constant and therefore can only be equal to zero. □

Definition 4.17. In the above Theorem, f^+ is called the *regular* part of f while f^- is called the *principal* or *singular* part of f.

Theorem 4.18 (Laurent Series)**.** *Let $z_0 \in \mathbb{C}$ and $0 \le r_1 < r_2$ and $f \in \mathcal{O}(A_{r_1,r_2}(z_0))$. Then, there exist a unique set of coefficients $\{c_n\}_{n\in\mathbb{Z}}$ such that*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z - z_0)^n,$$

where the series converges absolutely for all $z \in A_{r_1,r_2}(z_0)$ and uniformly on $A_{s_1,s_2}(z_0)$, when $r_1 < s_1 < s_2 < r_2$. Also, the coefficients are given by

$$c_n = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \,\mathrm{d}\zeta,$$

where $r_1 < r < r_2$.

Proof. We use the decomposition $f = f^+ + f^-$ of Theorem 4.16. Define $g \in \mathcal{O}(B_{1/r_1}(0) \setminus \{0\})$ via

$$g(z) := f^-\left(\frac{1}{z} + z_0\right).$$

Since $\lim_{|z|\to\infty} f^-(z) = 0$ it follows that $\lim_{z\to 0} g(z) = 0$. In particular, g has a continuous extension to $B_{1/r_1}(0)$ and thus a holomorphic one by the Riemann Continuation Theorem (Theorem 3.3). Consider its power series expansion

$$g(z) = \sum_{n=1}^{\infty} b_n z^n,$$

which converges absolutely in $B_{1/r_1}(0)$ and uniformly in $B_{1/s_1}(0)$ for any $s_1 > r_1$. Thus

$$f^-(z) = g\left(\frac{1}{z - z_0}\right) = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

converges absolutely in $\mathbb{C} \setminus \overline{B_{r_1}(z_0)}$ and uniformly on $\mathbb{C} \setminus \overline{B_{s_1}(z_0)}$ for any $s_1 > r_1$. On the other hand, the power series expansion

$$f^+(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

converges absolutely in $B_{r_2}(z_0)$ and uniformly on $B_{s_2}(z_0)$ for any $0 < s_2 < r_2$. Summing both expansions and setting $c_{-n} := b_n$ for all $n \in \mathbb{N}$ yields the Laurent series with the desired properties.

Set $r_1 < r < r_2$. Using Lemma 2.10 together with convergence of the Laurent series and interchangeability of limit and integral (Proposition 2.7) yields the desired formula for the coefficients c_n. □

Proposition 4.19. *Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Let $r > 0$ such that $A_{0,r}(a) \subset D$. Let*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-a)^n$$

be the Laurent series for f in $A_{0,r}(a)$. Then,

1. *a is a removable singularity of f iff $c_n = 0$ for all $n < 0$.*
2. *a is a pole of order m of f iff $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$.*
3. *a is an essential singularity of f iff there exist infinitely many $n < 0$ such that $c_n \neq 0$.*

Proof. **Exercise.** □

Exercise 40. Let $f \in \mathcal{O}(\mathbb{C} \setminus \{0, 1, 2\})$ be given by

$$f(z) := \frac{1}{z(z-1)(z-2)}.$$

Give the Laurent series expansion of f in the following regions: $A_{0,1}(0)$, $A_{1,2}(0)$, $A_{2,\infty}(0)$.

Exercise 41. Give the Laurent series expansion of $z \mapsto \exp(1/z)$.

4.4 Residues

Definition 4.20. Let $a \in \mathbb{C}$ and $0 < r$, $f \in \mathcal{O}(B_r(a) \setminus \{a\})$ and

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-a)^n$$

the Laurent series of f at a. Then, $\operatorname{Res}(f, a) := c_{-1}$ is called the *residue* of f at a.

Theorem 4.21 (Residue Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region, $A \subset D$ a discrete and relatively closed subset, and $f \in \mathcal{O}(D \setminus A)$. Let γ be a closed path with $|\gamma| \subset D \setminus A$ and $\operatorname{Int}_\gamma \subset D$. Then,*

$$\sum_{a\in A} \operatorname{Res}(f, a)\operatorname{Ind}_\gamma(a) = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(z)\, \mathrm{d}z.$$

Proof. Define $\tilde{A} := \mathrm{Int}_\gamma \cap A$. This is finite since $\mathrm{Int}_\gamma \cup |\gamma|$ is compact. Thus, suppose $\tilde{A} = \{a_1, \ldots, a_n\}$. Observe that the sum in the statement really only runs over $\tilde{A}$, since the index of the other elements of A vanishes. Now, decompose f into a sum

$$f(z) = f_1(z) + \cdots + f_n(z) + g(z) \quad \forall z \in D \setminus A,$$

where $f_k \in \mathcal{O}(\mathbb{C}\setminus\{a_k\})$ and $g \in \mathcal{O}((D\setminus A)\cup\tilde{A})$ as follows. Let f_1 be the singular part f^- of f at a_1 (according to Theorem 4.16). In particular $\mathrm{Res}(f, a_1) = \mathrm{Res}(f_1, a_1)$. Note that $f - f_1$ has one singularity less than f (the one at a_1) and moreover $\mathrm{Res}(f, a_k) = \mathrm{Res}(f - f_1, a_k)$ for all $k > 1$. Now, take f_2 to be the singular part of $f - f_1$ at a_2 etc. Finally, let $g := f - f_1 - \cdots - f_n$ and notice that g has no singularities in Int_γ left. Note that the integral over g along γ vanishes by Theorem 2.28. Thus, the Theorem reduces to proving the identity,

$$\mathrm{Res}(h, a)\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi\mathrm{i}} \int_\gamma h(z)\,\mathrm{d}z$$

for functions $h \in \mathcal{O}(\mathbb{C} \setminus \{a\})$ such that $\lim_{|z|\to\infty} h(z) = 0$. Consider the Laurent series of h around a,

$$h(z) = \sum_{n=-\infty}^{-1} c_n (z - a)^n.$$

Since this converges uniformly on the compact set $|\gamma|$, we can interchange integration and summation,

$$\int_\gamma h(z)\,\mathrm{d}z = \sum_{n=-\infty}^{-1} c_n \int_\gamma (z - a)^n\,\mathrm{d}z.$$

Now note that $(z - a)^n$ has a primitive if $n \leq -2$, i.e., is then integrable in $\mathbb{C} \setminus \{a\}$. Thus, by Proposition 2.13 its integral vanishes. Hence,

$$\int_\gamma h(z)\,\mathrm{d}z = c_{-1} \int_\gamma (z - a)^{-1}\,\mathrm{d}z = \mathrm{Res}(h, a)2\pi\mathrm{i}\mathrm{Ind}_\gamma(a).$$

This completes the proof. □

Exercise 42. Let $D \subseteq \mathbb{C}$ be a region and $a \in D$. Let $g, h \in \mathcal{O}(D)$ such that $g(a) \neq 0$ and $h(a) = 0$, but $h'(a) \neq 0$. Show that $f := g/h \in \mathcal{M}(D)$ has a simple pole at a and,

$$\mathrm{Res}(f, a) = \frac{g(a)}{h'(a)}.$$

Exercise 43. Calculate the following integrals:

1. $\displaystyle\int_0^\infty \frac{x^2}{x^4 + x^2 + 1}\,\mathrm{d}x$
2. $\displaystyle\int_0^\infty \frac{\cos(x) - 1}{x^2}\,\mathrm{d}x$
3. $\displaystyle\int_0^\pi \frac{\cos(2\theta)}{1 - 2a\cos(\theta) + a^2}\,\mathrm{d}\theta, \quad a^2 < 1$
4. $\displaystyle\int_0^\pi \frac{1}{(a + \cos(\theta))^2}, \quad a > 1$

Exercise 44. Show that the following identities hold:

1. $$\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}$$

2. $$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{\pi}{4a^3}, \quad a > 0$$

5 Conformal mappings

5.1 Conformal mappings as holomorphic functions

Recall that we have the standard Euclidean *scalar product* on the complex plane, by viewing $\mathbb{C}$ as a two-dimensional real vector space. That is, we have

$$\langle z, z'\rangle := aa' + bb' = \Re(\overline{z}z'),$$

where $z = a + \mathrm{i}b$ and $z' = a' + \mathrm{i}b'$. Recall also that $|z| = \sqrt{\langle z, z\rangle}$. In geometric terms we have,

$$\langle z, z'\rangle = |z||z'|\cos\theta,$$

where θ is the angle between z and z', viewed as vectors in the complex plane.

We shall now be interested in mappings $A : \mathbb{C} \to \mathbb{C}$ that preserve angles between intersecting curves. First, we consider $\mathbb{R}$-linear mappings. Then, for A to be angle-preserving clearly is equivalent to the identity,

$$|z||z'|\langle A(z), A(z')\rangle = |A(z)||A(z')|\langle z, z'\rangle \quad \forall z, z' \in \mathbb{C}.$$

(We also require of course that A not be zero.)

We write A as a real 2×2 matrix

$$A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}, \quad \text{acting as} \quad a + \mathrm{i}b \mapsto ra + sb + \mathrm{i}(ta + ub).$$

Lemma 5.1. *Let $A : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$-linear mapping. Then, A preserves angles iff*

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \textit{or} \quad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and a and b are not both equal to zero.

Proof. **Exercise.** □

More generally, to make sense of the concept of angle-preservation for a map $f : D \to \mathbb{C}$, where D is a region, it is necessary that f possesses a continuous total differential. Then, f preserves angles iff its total differential f' preserves angles at every point of D.

Proposition 5.2. *Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ a function possessing a continuous total differential in D. Then, f is angle-preserving iff f is holomorphic in D or anti-holomorphic in D and its derivative never vanishes.*

Proof. **Exercise.** □

A *conformal* mapping is a mapping that preserves both angles and orientation. Recall that a linear map is orientation preserving iff its determinant is positive. More generally, a mapping is orientation preserving iff its total derivative has positive determinant everywhere.

Proposition 5.3. *Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ a function possessing a continuous total differential in D. Then, f is conformal iff f is holomorphic in D and its derivative never vanishes.*

Proof. **Exercise.** □

5.2 Biholomorphic mappings

Definition 5.4. Let X be a topological space and S a set. A function $f : X \to S$ is called *locally injective at $x \in X$* iff there is a neighborhood $U \subseteq X$ of x such that f restricted to U is injective. f is called *locally injective* iff it is locally injective at each $x \in X$.

Theorem 5.5. *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$, $a \in D$ and $p := f(a)$. Suppose that $f - p$ has a zero of order m at a. Then there exist $\epsilon > 0$ and $\delta > 0$ with $\overline{B_\delta(a)} \subset D$ such that for $q \in B_\epsilon(p) \setminus \{p\}$ the function $f - q$ has exactly m distinct simple zeros for $z \in B_\delta(a)$ and $f - p$ has no further zeros in $z \in B_\delta(a)$.*

Proof. Since f is not constant (otherwise $f - p$ could not have a zero of finite order according to Proposition 3.14), neither $f - p$ nor f' are constant zero. So the zeros of both $f - p$ and f' are isolated. This implies that we can find $\delta > 0$ with $\overline{B_\delta(a)} \subset D$ such that $f(z) - p \neq 0$ and $f'(z) \neq 0$ for all $z \in \overline{B_\delta(a)} \setminus \{a\}$. Now set $\epsilon := \min_{\zeta \in \partial B_\delta(a)}\{|f(\zeta) - p|\}$. Then, if $q \in B_\epsilon(p)$,

$$|(f(\zeta) - p) - (f(\zeta) - q)| < \epsilon \leq |f(\zeta) - p| \quad \forall \zeta \in \partial B_\delta(a).$$

So, by Rouché's Theorem (Theorem 4.10), $f - p$ and $f - q$ must have the same numbers of zeros, counted with multiplicity, in $B_\delta(a)$, namely m. If $q \neq p$ these are all simple by Proposition 3.14 because $f'(z) \neq 0$ for $z \in B_\delta(a) \setminus \{a\}$. □

Proposition 5.6. *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Then, f is locally injective at $a \in D$ iff $f'(a) \neq 0$. Moreover, f is locally injective in D iff f' is nowhere zero in D.*

Proof. Let $a \in D$ and $p := f(a)$. Suppose first that $f'(a) = 0$. Then, either f is constant or $f - p$ has a zero of order $m \geq 2$ at a. In the first case the lack of local injectivity is trivial. In the second case consider an open neighborhood $U \subseteq D$ of a. Applying Theorem 5.5, there exists $\epsilon > 0$ such that for $q \in B_\epsilon(p) \setminus \{p\}$ the equation $f(z) = q$ has at least two distinct solutions for $z \in U$. In particular, f is not injective in U. Since U was arbitrary, f is not locally injective at a.

Now suppose $f'(a) \neq 0$. Then, $f - p$ has a simple zero at a. Applying Theorem 5.5, there exist $\epsilon > 0$ and $\delta > 0$ with $B_\delta(a) \subset D$ such that for all $q \in B_\epsilon(p)$ the equation

$f(z) = q$ has exactly one solution in $B_\delta(a)$. By continuity of f, $U := f^{-1}(B_\epsilon(p)) \cap B_\delta(a)$ is an open neighborhood of a. Clearly, f is injective in U, showing that f is locally injective at a. $\square$

Recalling Section 5.1 we see that the concept of conformality is equivalent to holomorphicity combined with local injectivity.

Definition 5.7. Let $D, D' \subseteq \mathbb{C}$ be regions. A map $f : D \to \mathbb{C}$ with $f(D) = D'$ is called a *biholomorphic map* from D to D' iff f is holomorphic and has a holomorphic inverse $f^{-1} : D' \to \mathbb{C}$. If such a map exists, D and D' are said to be *conformally equivalent*.

Theorem 5.8. *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Then, f is a biholomorphic mapping from D to $f(D)$ iff f is injective.*

Proof. Clearly, biholomorphicity implies injectivity. For the converse assume that f is injective. By continuity, the image $D' := f(D)$ is connected. Moreover, by the Open Mapping Theorem 3.12, D' is open. So D' is a region as it cannot be empty. Since f is injective, the inverse map $f^{-1} : D' \to D$ exists. Again using the Open Mapping Theorem, f^{-1} is continuous. Moreover, by Proposition 5.6 f' is nowhere zero. Applying Proposition 1.7 we conclude that f^{-1} is everywhere complex differentiable, i.e., it is holomorphic. $\square$

Proposition 5.9. *Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of injective functions $f_n \in \mathcal{O}(D)$ converging uniformly in every compact subset of D to f. Then, either f is constant or f is injective.*

Proof. Suppose that f is not constant. Let a in D and set $p := f(a)$ and $p_n := f_n(a)$ for all $n \in \mathbb{N}$. By injectivity $f_n - p_n$ never vanishes on $D \setminus \{a\}$. On the other hand, the sequence $\{f_n - p_n\}_{n\in\mathbb{N}}$ converges uniformly in any compact subset of D to $f - p$. Since $f - p \neq 0$, Proposition 4.12 implies that $f - p$ has no zeros in $D \setminus \{a\}$. In other words, f does not take the value p at any point of $D \setminus \{a\}$. Since we chose a arbitrarily it follows that f is injective. $\square$

In the following $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ denotes the *upper half-plane* in $\mathbb{C}$.

Exercise 45. Show that $z \mapsto -z^2$ restricted to $\mathbb{H}$ is a biholomorphic mapping. Onto which region?

Exercise 46. Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ such that f is not constant. Show that for any $a \in D$ there exists a neighborhood $U \subseteq D$ of a such that there is $m \in \mathbb{N}$ and $g \in \mathcal{O}(U)$ biholomorphic with the property $f(z) = f(a) + (g(z))^m$ for all $z \in U$.

5.3 Conformal automorphisms of $\mathbb{C}$ and $\mathbb{C}^\times$

Definition 5.10. Let $D \subseteq \mathbb{C}$ be a region. A biholomorphic mapping from D to D is called a *conformal automorphism* of D. The group of conformal automorphisms of D is denoted $\mathrm{Aut}(D)$.

As a first example we consider conformal automorphisms of $\mathbb{C}$. The following ones are obvious:

1. $T_a : z \mapsto z + a$ where $a \in \mathbb{C}$ is the *translation* by a.
2. $R_\theta : z \mapsto e^{\mathrm{i}\theta} z$ where $\theta \in [0, 2\pi)$ is the *rotation* by the angle θ around the origin in positive direction.
3. $S_r : z \mapsto rz$ where $r \in \mathbb{R}^+$ is the *scaling* by the factor r around the origin.

Exercise 47. Show that the group generated by translations, rotations and scalings of $\mathbb{C}$ consists precisely of the biholomorphic transformations $\mathbb{C} \to \mathbb{C}$ of the form

$$z \mapsto az + b \quad \text{with} \quad a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}.$$

As we shall see soon there are in fact no further automorphisms of $\mathbb{C}$. Another interesting example is the *punctured plane* $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. In addition to the rotations and scalings already seen above, there is another elementary automorphism of $\mathbb{C}^\times$ given by

$$I : z \mapsto \frac{1}{z}, \quad \text{called } \textit{inversion}.$$

We shall see that there are no further automorphisms of $\mathbb{C}^\times$ than those generated by rotations, scalings and inversions.

Lemma 5.11. *Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$ be injective. Then, either a is a pole of order one or it is a removable singularity and the continuation of f to D is injective.*

Proof. Suppose that a is a removable singularity and denote the continuation of f by $\tilde{f} \in \mathcal{O}(D)$. Assume that $\tilde{f}$ is not injective. Since f is injective this means there exists $z \in D \setminus \{a\}$ such that $\tilde{f}(a) = \tilde{f}(z)$. Choose $r > 0$ such that $r < |z - a|/2$ and $B_r(a) \subseteq D$ and $B_r(z) \subseteq D$. By the Open Mapping Theorem (Theorem 3.12) $\tilde{f}(B_r(z))$ and $\tilde{f}(B_r(a))$ are open and so is their intersection $U := \tilde{f}(B_r(z)) \cap \tilde{f}(B_r(a))$. But by assumption U is not empty as it contains $\tilde{f}(a)$. Since U is open there exists $p \in U$ with $p \neq \tilde{f}(a)$. Then there must exist $z_1 \in B_r(a) \setminus \{a\}$ and $z_2 \in B_r(z)$ such that $f(z_1) = p = f(z_2)$ contradicting the injectivity of f. Thus, $\tilde{f}$ must be injective.

Suppose now that a is not a removable singularity. Let $r > 0$ such that $\overline{B_r(a)} \subset D$ and define $D' := D \setminus \overline{B_r(a)}$. By the Open Mapping Theorem (Theorem 3.12) the sets

$f(D')$ and $f(B_r(a) \setminus \{a\})$ are both open and non-empty, but their intersection is empty by injectivity. Thus, $f(B_r(a) \setminus \{a\})$ cannot be dense in $\mathbb{C}$. By the Casorati-Weierstrass Theorem (Theorem 4.6) this implies that a is not an essential singularity. Hence, it must be a pole. This implies that there is $s > 0$ such that $B_s(a) \subseteq D$ and $f(z) \neq 0$ for all $z \in B_s(a) \setminus \{a\}$. Define $g \in \mathcal{O}(B_s(a) \setminus \{a\})$ by $g(z) := 1/f(z)$. Note that g is injective since f is. Also, a is a pole of f, so a is a removable singularity of g. This implies by the above part of the proof that the continuation $g \in \mathcal{O}(B_s(a))$ is still injective. In particular, g is locally injective at a, so Proposition 5.6 implies that $g'(a) \neq 0$. On the other hand $g(a) = 0$, so a is a zero of order one of g, implying that it is a pole of order one of f. $\square$

Theorem 5.12. *Every injective holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is an automorphism of $\mathbb{C}$ and can be written in the form*

$$z \mapsto az + b \quad \textit{for some} \quad a \in \mathbb{C}^\times, b \in \mathbb{C}.$$

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

be the power series expansion of f. Define the function $g \in \mathcal{O}(\mathbb{C}^\times)$ by $g(z) := f(1/z)$. Then, g is injective and has the Laurent series expansion

$$g(z) = \sum_{n=0}^{\infty} c_n z^{-n}$$

in $A_{0,\infty}(0)$. By Lemma 5.11, 0 is either a removable singularity of g or a pole of order one. This implies $c_n = 0$ for all $n \geq 2$ by Proposition 4.19. By injectivity $c_1 \neq 0$, so f has the stated form and is an automorphism of $\mathbb{C}$. $\square$

Corollary 5.13. $\mathbb{C}$ *is not conformally equivalent to any proper subset.*

Theorem 5.14. *Every injective holomorphic mapping $f : \mathbb{C}^\times \to \mathbb{C}^\times$ is an automorphism of $\mathbb{C}^\times$ and takes either the form*

$$z \mapsto az \quad \textit{or} \quad z \mapsto \frac{a}{z} \quad \textit{for some} \quad a \in \mathbb{C}^\times.$$

Proof. According to Lemma 5.11, 0 can either be a removable singularity of f or a pole of order one. In the first case, the continuation $\tilde{f} \in \mathcal{O}(\mathbb{C})$ is injective by the same Lemma. Thus, $\tilde{f}$ is automorphism of $\mathbb{C}$ and $\tilde{f}(z) = az + b$ for some $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$ by Theorem 5.12. But must have $\tilde{f}^{-1}(\{0\}) \neq \emptyset$ while $f^{-1}(\{0\}) = \emptyset$, implying $\tilde{f}(0) = 0$. Thus, $b = 0$. In the second case define the injective holomorphic function $g : \mathbb{C}^\times \to \mathbb{C}^\times$ by $g(z) := 1/f(z)$. Since f has a pole at 0, g has a removable singularity at 0. So we can apply the first part of the proof to g showing that $g(z) = \tilde{a}z$ for some $\tilde{a} \in \mathbb{C}^\times$. Setting $a := 1/\tilde{a}$ we find $f(z) = a/z$, completing the proof. $\square$

Exercise 48. Show that $\mathbb{C}^\times$ is conformally equivalent to $\mathbb{C} \setminus \{p\}$ for any $p \in \mathbb{C}$, but not to any other subset of $\mathbb{C}$.

5.4 Conformal automorphisms of $\mathbb{D}$

We now consider the conformal automorphisms of the open unit disk $\mathbb{D} := B_1(0)$. Among the transformations we have seen so far, the rotation by an angle θ around the origin is obviously an automorphism of $\mathbb{D}$. A less obvious automorphism is given by

$$D_w : z \mapsto \frac{z - w}{\overline{w}z - 1}, \quad \text{where} \quad w \in \mathbb{D}.$$

Exercise 49. Verify the following properties of the transformation D_w: (a) it is an automorphism of $\mathbb{D}$, (b) it is self-inverse, i.e., composing the transformation with itself yields the identity on $\mathbb{D}$, (c) it interchanges the points 0 and w.

We shall see that the group generated by rotations R_θ and by transformations D_w is already the full automorphism group of $\mathbb{D}$.

Lemma 5.15 (Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that $f(0) = 0$. Then,*

$$|f(z)| \le |z| \quad \forall z \in \mathbb{D} \quad \textit{and} \quad |f'(0)| \le 1.$$

Moreover, if $|f(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$ or if $|f'(0)| = 1$, then there is $a \in \mathbb{C}$ with $|a| = 1$ such that $f(z) = az$ for all $z \in \mathbb{D}$.

Proof. Since f has a zero at 0, there is $g \in \mathcal{O}(\mathbb{D})$ such that $f(z) = zg(z)$ and moreover, $f'(0) = g(0)$. Since $|f(z)| < 1$ for all $z \in \mathbb{D}$, we have for any $0 < r < 1$,

$$\|g\|_{\partial B_r(0)} < \frac{1}{r}.$$

On the other hand, applying Proposition 3.7 to $B_r(0)$ we have

$$|g(z)| \le \|g\|_{\partial B_r(0)} < \frac{1}{r} \quad \forall z \in B_r(0).$$

Since r can be chosen arbitrarily close to 1, we get, for all $z \in \mathbb{D}$, $|g(z)| \le 1$. This translates to the first stated inequality if $z \neq 0$ and to the second stated inequality if $z = 0$. If either $|f(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$ or if $|f'(0)| = 1$, then $|g(z)| = 1$ for some $z \in \mathbb{D}$. Then, by Theorem 3.6, g is constant, i.e, there is $a \in \mathbb{C}$ such that $g(z) = a$ for all $z \in \mathbb{D}$. Consequently, $f(z) = az$. Observe also that $|a| = 1$. □

Proposition 5.16. *Let $f : \mathbb{D} \to \mathbb{D}$ be biholomorphic and $f(0) = 0$. Then, f is a rotation, i.e., there exists $\theta \in [0, 2\pi)$ such that $f = R_\theta$.*

Proof. Applying Lemma 5.15 to both f and f^{-1} yields,

$$|f(z)| \le |z| \quad \text{and} \quad |f^{-1}(z)| \le |z| \quad \forall z \in \mathbb{D}.$$

Replacing z by $f(z)$ in the second inequality yields, $|z| \le |f(z)|$ for all $z \in \mathbb{D}$. Thus, we actually find $|f(z)| = |z|$ for all $z \in \mathbb{D}$. By Lemma 5.15 this implies that there exists $a \in \mathbb{C}$ with $|a| = 1$ and $f(z) = az$, i.e., f is a rotation. □

Theorem 5.17. *The group of automorphisms of $\mathbb{D}$ is generated by rotations R_θ and transformations D_w. In particular, any automorphism of $\mathbb{D}$ can be written uniquely as a composition $R_\theta \circ D_w$ for some $\theta \in [0, 2\pi)$ and some $w \in \mathbb{D}$.*

Proof. Let $f \in \mathrm{Aut}(\mathbb{D})$. Set $w := f^{-1}(0)$ and define $g := f \circ D_w$. Then $g \in \mathrm{Aut}(\mathbb{D})$ with the property that $g(0) = 0$. Applying Proposition 5.16 to g yields that g is a rotation. That is, there exists $\theta \in [0, 2\pi)$ such that $g = R_\theta$. Then, $f = R_\theta \circ D_w$, since $D_w \circ D_w = \mathrm{id}$. To see uniqueness suppose that also $f = R_{\theta'} \circ D_{w'}$. Then $f^{-1}(0) = (R_{\theta'} \circ D_{w'})^{-1}(0) = D_{w'}^{-1}(0) = w'$, so $w' = w$. But composing with D_w yields then $R_{\theta'} = R_\theta$ which implies $\theta' = \theta$. □

Exercise 50. Show that the set of automorphisms of $\mathbb{D}$ is identical to the set of transformations $\mathbb{D} \to \mathbb{D}$ of the form

$$z \mapsto \frac{xz + y}{\overline{y}z + \overline{x}} \quad \text{with} \quad x, y \in \mathbb{C} \quad \text{and} \quad |x| > |y|.$$

Exercise 51. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and $a \in \mathbb{D}$ such that $f(a) = 0$. Show that

$$|f(z)| \le \frac{|z - a|}{|\overline{a}z - 1|} \quad \forall z \in \mathbb{D}.$$

Moreover, in case of equality for some $z \in \mathbb{D} \setminus \{a\}$, f is automorphism of $\mathbb{D}$.

5.5 Möbius Transformations

It turns out that all the biholomorphic transformations we have considered so far can be written as rational maps that arise as quotients of polynomials of degree one. It turns out that maps of this type are always biholomorphic and permit the understanding of a variety of conformal equivalences and automorphism groups.

To each complex matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \neq 0$ or $d \neq 0$ we associate the rational function $M_A \in \mathcal{M}(\mathbb{C})$ given by

$$M_A(z) := \frac{az + b}{cz + d}.$$

Since

$$M_A'(z) = \frac{\det A}{(cz+d)^2}$$

we see that M_A is constant if $\det A = 0$. In the following we shall restrict to the case $\det A \neq 0$. M_A is then called a *Möbius transformation* or *fractional linear transformation.* We denote the set of these meromorphic functions by **Möb**. Recall that $\mathrm{GL}_2(\mathbb{C})$, the group of *general linear transformations* in $\mathbb{C}^2$, is the group of complex 2×2-matrices with non-zero determinant.

Proposition 5.18. *The set of Möbius transformations* ***Möb*** *forms a group by composition. Moreover, the map* $\mathrm{GL}_2(\mathbb{C}) \to$ ***Möb*** *given by* $A \mapsto M_A$ *is a group homomorphism, i.e., we have*

$$M_{AB} = M_A \circ M_B \quad \forall A, B \in \mathrm{GL}_2(\mathbb{C}).$$

Proof. **Exercise.** □

Exercise 52. Verify that the upper triangular matrices (with non-vanishing determinant) form a subgroup of $\mathrm{GL}_2(\mathbb{C})$. Show that the image of this subgroup under the map $\mathrm{GL}_2(\mathbb{C}) \to$ **Möb** is the group $\mathrm{Aut}(\mathbb{C})$. Identify the upper triangular matrices corresponding to translations, rotations and dilations.

Exercise 53. Verify that the other Möbius transformations also define biholomorphic mappings. Between which regions?

Recall that $\mathrm{GL}_2^+(\mathbb{R})$ is the group of *orientation-preserving general linear transformations* of $\mathbb{R}^2$, i.e., these are 2×2-matrices with real entries and positive determinant.

Proposition 5.19. *The restriction of the map* $\mathrm{GL}_2(\mathbb{C}) \to$ ***Möb*** *to the subgroup* $\mathrm{GL}_2^+(\mathbb{R})$ *yields Möbius transformations that are conformal automorphisms of* $\mathbb{H}$*. That is, we obtain a group homomorphism* $\mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{Aut}(\mathbb{H})$.

Proof. **Exercise.** □

Proposition 5.20. *Let* $D, D' \subseteq \mathbb{C}$ *be regions such that* D *and* D' *are conformally equivalent. Then* $\mathrm{Aut}(D)$ *and* $\mathrm{Aut}(D')$ *are isomorphic. In particular, every biholomorphic mapping* $D \to D'$ *yields such an isomorphism.*

Proof. Let $f : D \to D'$ be a biholomorphic mapping. Then, an isomorphism $\mathrm{Aut}(D) \to \mathrm{Aut}(D')$ is given by $g \mapsto f \circ g \circ f^{-1}$. □

Exercise 54. Show that the *Cayley map* $M_C \in \mathcal{M}(\mathbb{C})$ given by

$$C := \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}$$

is a biholomorphic map from $\mathbb{H}$ to $\mathbb{D}$.

Proposition 5.21. *Consider the group homomorphism* $\mathrm{GL}^+(\mathbb{R}) \to \mathrm{Aut}(\mathbb{D})$ *given by* $A \mapsto M_C \circ M_A \circ M_C^{-1}$ *induced by the Cayley map* $M_C : \mathbb{H} \to \mathbb{D}$*. This group homomorphism is surjective, i.e., every automorphism of* $\mathbb{D}$ *can be obtained in this way.*

Proof. If C is the matrix of Exercise 54, then $C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}$ and $M_C^{-1} = M_{C^{-1}}$. It is easy to verify by matrix multiplication that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the indicated group homomorphism yields the automorphism $\mathbb{D} \to \mathbb{D}$ given by

$$z \mapsto \frac{xz + y}{\overline{y}z + \overline{x}},$$

where $x := a + d + \mathrm{i}b - \mathrm{i}c$ and $y := a - d - \mathrm{i}b - \mathrm{i}c$. If a, b, c, d were arbitrary real numbers, x, y would be arbitrary complex numbers. It is easy to verify that $|x|^2 - |y|^2 = 4 \det A$. Thus, the condition $\det A > 0$ on (a, b, c, d) corresponds precisely to the condition $|x| > |y|$ on (x, y). Recalling Exercise 50, we recognize that we obtain all automorphisms of $\mathbb{D}$. □

Exercise 55. Let $A, B \in \mathrm{GL}_2(\mathbb{C})$. Show that $M_A = M_B$ iff there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $B = \lambda A$.

$\mathrm{PGL}_2(\mathbb{C})$ is the group of *projective general linear transformations* of $\mathbb{C}^2$. It is the quotient $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^*$, where $\mathbb{C}^*$ is the subgroup of $\mathrm{GL}_2(\mathbb{C})$ given by non-zero complex multiples of the unit matrix.

Exercise 56. Show that $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to $\mathrm{SL}_2(\mathbb{C})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the subgroup of $\mathrm{SL}_2(\mathbb{C})$ consisting of $\{\mathbf{1}, -\mathbf{1}\}$.

Proposition 5.22. $\mathrm{PGL}_2(\mathbb{C}) \approx$ ***Möb****.*

$\mathrm{PGL}_2^+(\mathbb{R})$ is the group of *projective orientation-preserving general linear transformations* of $\mathbb{R}^2$. It is the quotient $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^*$, where $\mathbb{R}^*$ is the subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ given by non-zero real multiples of the unit matrix.

Exercise 57. Show that $\mathrm{PGL}_2^+(\mathbb{R})$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the subgroup of $\mathrm{SL}_2(\mathbb{R})$ consisting of $\{\mathbf{1}, -\mathbf{1}\}$.

Proposition 5.23. $\mathrm{PGL}_2^+(\mathbb{R}) \approx \mathrm{Aut}(\mathbb{H}) \approx \mathrm{Aut}(\mathbb{D})$.

5.6 Montel's Theorem

Let X be a topological space. We denote by $\mathcal{C}(X)$ the set of complex valued continuous functions on X.

Definition 5.24. A topological space is called *separable* iff it contains a countable dense subset.

Definition 5.25. Let X be a topological space, $F \subseteq \mathcal{C}(X)$. F is called *pointwise bounded* iff for each $a \in X$ there is a constant $M > 0$ such that $|f(a)| < M$ for all $f \in F$. F is called *locally bounded* iff for each $a \in X$ there is a constant $M > 0$ and a neighborhood $U \subseteq X$ of a such that $|f(x)| < M$ for all $x \in U$ and for all $f \in F$.

Definition 5.26. Let X be a topological space. A subset $F \subseteq \mathcal{C}(X)$ is called *equicontinuous at $a \in X$* iff for every $\epsilon > 0$ there exists a neighborhood $U \subseteq X$ of a such that

$$|f(x) - f(y)| < \epsilon \quad \forall\, x, y \in U, \forall\, f \in F.$$

A subset $F \subseteq \mathcal{C}(X)$ is called *locally equicontinuous* iff F is equicontinuous at a for all $a \in X$.

Definition 5.27. Let X be a topological space. A subset $F \subseteq \mathcal{C}(X)$ is called *normal* iff every sequence of elements of F has a subsequence that converges uniformly on every compact subset of X.

Theorem 5.28 (Arzela-Ascoli)**.** *Let X be a separable topological space and $F \subseteq \mathcal{C}(X)$. Suppose that F is pointwise bounded and locally equicontinuous. Then, F is normal.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of F. We have to show that there exists a subsequence that converges uniformly on any compact subset of X. We encode subsequences of a sequence through infinite subsets of $\mathbb{N}$ in the obvious way. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of points which is dense in X. Set $N_0 := \mathbb{N}$ and construct iteratively $N_k \subseteq N_{k-1}$ as follows. The sequence $\{f_n(x_k)\}_{n\in N_{k-1}}$ is bounded by the assumption of pointwise boundedness of F. Thus there exists a convergent subsequence given by an infinite subset $N_k \subseteq N_{k-1}$. Proceeding in this way we obtain a sequence of decreasing infinite subsets $N_0 \supseteq N_1 \supseteq N_2 \supseteq \dots$. Now consider the sequence $\{n_l\}_{l\in\mathbb{N}}$ of strictly increasing natural numbers n_l obtained as follows: n_l is the lth element of the set N_l. It is then clear that the sequence $\{f_{n_l}(x_k)\}_{l\in\mathbb{N}}$ converges for every $k \in \mathbb{N}$.

Now let $K \subseteq X$ be compact and choose $\epsilon > 0$. Since F is locally equicontinuous, we find for each $a \in K$ an open neighborhood $U_a \subseteq X$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in F$ if $x, y \in U_a$. Since K is compact there are finitely many points $a_1, \dots, a_m \in K$ such that $U_{a_1}, \dots, U_{a_m}$ cover K. Since $\{x_k\}_{k\in\mathbb{N}}$ is dense in X there exists for each $j \in 1, \dots, m$ an index k_j such that $x_{k_j} \in U_{a_j}$. Now, $\{f_{n_l}(x_{k_j})\}_{l\in\mathbb{N}}$ converges and is Cauchy for all $j \in \{1, \dots, m\}$. In particular, by taking a maximum if necessary we can find $l_0 \in \mathbb{N}$ such that $|f_{n_i}(x_{k_j}) - f_{n_l}(x_{k_j})| < \epsilon$ for all $i, l \geq l_0$ and for all $j \in \{1, \dots, m\}$.

Now fix $p \in K$. Then, there is $j \in \{1, \dots, m\}$ such that $p \in U_{a_j}$. For $i, l \geq l_0$ we thus obtain the estimate

$$|f_{n_i}(p) - f_{n_l}(p)| \leq |f_{n_i}(p) - f_{n_i}(x_{k_j})| \\ + |f_{n_i}(x_{k_j}) - f_{n_l}(x_{k_j})| + |f_{n_l}(x_{k_j}) - f_{n_l}(p)| < 3\epsilon.$$

In particular, this implies that $\{f_{n_l}\}_{l\in\mathbb{N}}$ converges uniformly on K. $\square$

Theorem 5.29 (Montel)**.** *Let $D \subseteq \mathbb{C}$ be a region and $F \subseteq \mathcal{O}(D)$. Suppose that F is locally bounded. Then, F is normal.*

Proof. We show that F is locally equicontinuous. The result follows then from the Arzela-Ascoli Theorem 5.28. Let $z_0 \in D$ and choose $\epsilon > 0$. Since F is locally bounded, there exists a constant $M > 0$ and $r > 0$ with $\overline{B_{2r}(z_0)} \subset D$ and such that $|f(z)| < M$ for all $z \in \overline{B_{2r}(z_0)}$ and all $f \in F$. The Cauchy Integral Formula (Theorem 2.20) yields for all $f \in F$ and $z, w \in B_{2r}(z_0)$

$$\begin{aligned} f(z) - f(w) &= \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{2r}(z_0)} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) \mathrm{d}\zeta \\ &= \frac{z - w}{2\pi \mathrm{i}} \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)} \mathrm{d}\zeta. \end{aligned}$$

If we restrict to $z, w \in B_r(z_0)$ we have the estimate $|(\zeta - z)(\zeta - w)| > r^2$ for all $\zeta \in \partial B_{2r}(z_0)$. Combining this with the standard integral estimate (Proposition 2.6) we obtain,

$$|f(z) - f(w)| \le |z - w| \frac{2\|f\|_{\partial B_{2r}(z_0)}}{r} < |z - w| \frac{2M}{r}.$$

Choosing $\delta := \min\left\{r, \frac{r\epsilon}{4M}\right\}$ yields the estimate

$$|f(z) - f(w)| < \epsilon \quad \forall z, w \in B_\delta(z_0),$$

showing local equicontinuity. This completes the proof. $\square$

Exercise 58. Let X be a metric space and $F \subseteq \mathcal{C}(X)$. Suppose that F is normal. Show that F is (a) locally bounded and (b) locally equicontinuous.

Exercise 59 (Vitali's Theorem)**.** Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n \in \mathbb{N}}$ a locally bounded sequence of holomorphic functions on D. Set $A := \{z \in D : \lim_{n \to \infty} f_n(z) \text{ exists}\}$. Suppose that A has a limit point in D. Show that $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly on compact subsets of D.

5.7 The Riemann Mapping Theorem

Theorem 5.30 (Riemann Mapping Theorem)**.** *Every homologically simply connected region which is different from $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$.*

Proof. Let D be the region in question. Fix $z_0 \in D$ arbitrarily. Let $F \subseteq \mathcal{O}(D)$ be the set of holomorphic functions $f \in \mathcal{O}(D)$ which are injective, whose image is contained in $\mathbb{D}$ and such that $f(z_0) = 0$. Our strategy is to find an element of F which is a biholomorphism $D \to \mathbb{D}$.

First we show that F is not empty. By assumption $D \neq \mathbb{C}$, so we can choose $a \in \mathbb{C} \setminus D$. The function $f(z) := z - a$ is holomorphic and zero-free in D, so according to

Theorem 3.22 there is a holomorphic square root $g \in \mathcal{O}(D)$ with $g^2 = f$. If $g(z_1) = g(z_2)$ then $(g(z_1))^2 = (g(z_2))^2$ and so $z_1 = z_2$ since f is injective. Therefore also g is injective. Moreover, if $g(z_1) = -g(z_2)$ we can draw the same conclusion $z_1 = z_2$, but this time we get a contradiction, since g is zero-free. Thus, if $z \in \mathbb{C}$ is in the image of g, then $-z$ cannot be in the image of g. Now since g is not constant the Open Mapping Theorem 3.12 ensures that $g(D)$ is open. In particular there exists $w \in \mathbb{C}$ and $r > 0$ such that $\overline{B_r(w)} \subset g(D)$. But applying the previous statement to all elements of $B_r(w)$ we obtain $\overline{B_r(-w)} \cap g(D) = \emptyset$. It is now easy to see that the function $h \in \mathcal{O}(D)$ defined by $h(z) := r/(g(z) + w)$ is also injective and satisfies $h(D) \subseteq \mathbb{D}$. Setting $v := h(z_0)$, we have $D_v \circ h \in F$ since $D_v \in \mathrm{Aut}(\mathbb{D})$ and $D_v(v) = 0$.

Since D is open, there exists $r > 0$ such that $\overline{B_r(z_0)} \subset D$. Using the Cauchy estimate (Proposition 1.25) we find the bound $|f'(z_0)| < 1/r$ for all $f \in F$. This implies that

$$M := \sup\{|f'(z_0)| : f \in F\}$$

is well defined. On the other hand we will show that if $f(D) \neq \mathbb{D}$ for some $f \in F$, then there exists $g \in F$ such that $|g'(z_0)| > |f'(z_0)|$. This implies that $h \in F$ is a biholomorphism $D \to \mathbb{D}$ if $|h'(z_0)| = M$. We will then show that such an h exists.

Consider some $f \in F$ such that $f(D) \neq \mathbb{D}$. Choose $p \in \mathbb{D} \setminus f(D)$. Since $D_p \in \mathrm{Aut}(\mathbb{D})$, the composition $D_p \circ f$ is injective and $D_p \circ f(D) \subset \mathbb{D}$. Furthermore, $D_p \circ f$ is zero-free since $D_p^{-1}(0) = \{p\}$. Since D is homologically simply connected we can find a holomorphic square root $g \in \mathcal{O}(D)$ with $g^2 = D_p \circ f$ according to Theorem 3.22. In fact, it is clear that g is injective and $g(D) \subseteq \mathbb{D}$. Set $w := g(z_0)$. Then $h := D_w \circ g \in F$. Consider now the holomorphic map $k : \mathbb{D} \to \mathbb{D}$ given by $k(z) = D_p((D_w(z))^2)$. Then, $f = k \circ h$ and applying the chain rule for derivatives we obtain

$$f'(z_0) = k'(h(z_0))h'(z_0) = k'(0)h'(z_0).$$

Noting that $k(0) = 0$ we can apply the Schwarz Lemma 5.15. Since k is not a rotation, this implies $|k'(0)| < 1$. Hence, $|f'(z_0)| < |h'(z_0)|$ since $h'(z_0) \neq 0$ by injectivity of h.

The image of all functions in F is contained in the bounded set $\mathbb{D}$, so in particular F is locally bounded. According to Montel's Theorem 5.29 this implies that F is normal. Consider now a sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of F such that $|f_n'(z_0)| \to M$ as $n \to \infty$. Since F is normal, there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ which converges uniformly on any compact subset of D to a function $f \in \mathcal{O}(D)$ by Proposition 3.1. By the same Proposition we have convergence of the derivative and thus $|f'(z_0)| = M$ as desired. It remains to show that $f \in F$. From the limit process it is clear that $f(z_0) = 0$ and $f(D) \subseteq \overline{\mathbb{D}}$. Since f is not constant (in particular, $f'(z_0) \neq 0$) the Open Mapping Theorem 3.12 implies that $f(D)$ must be open and so we must have $f(D) \subseteq \mathbb{D}$. The injectivity of f follows from Proposition 5.9. Hence $f \in F$. This completes the proof. □

Proposition 5.31. *Let $D \subset \mathbb{C}$ be a homologically simply connected region, $a \in D$. Then, there exists exactly one biholomorphism $f : D \to \mathbb{D}$ such that $f(a) = 0$ and $f'(a) > 0$.*

Proof. **Exercise.** □

Exercise 60. Show that a homologically simply connected region cannot be conformally equivalent to a region that is not homologically simply connected.

6 Harmonic functions

6.1 Mean value and maximum

We coordinatize the complex plane by coordinates $(x, y) \in \mathbb{R}^2$ with $z = x + \mathrm{i}y \in \mathbb{C}$. The Laplace operator on the complex plane is then given by

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Definition 6.1. Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{R}$ be twice continuously partially differentiable. Then, f is called *harmonic* iff it satisfies the *Laplace equation*

$$\Delta f = 0.$$

Proposition 6.2. *The real and the imaginary part of a holomorphic function are harmonic.*

Proof. **Exercise.** □

Proposition 6.3. *Let $U, V \subseteq \mathbb{C}$ be open and $f \in \mathcal{O}(U)$ such that $f(U) \subseteq V$. If $g : V \to \mathbb{R}$ is harmonic, then $g \circ f : U \to \mathbb{R}$ is also harmonic.*

Proof. **Exercise.** □

Lemma 6.4. *Let $D = \mathbb{C}$ or $D = \mathbb{D}$ and $u : D \to \mathbb{R}$ a harmonic function. Then, there exists a harmonic function $v : D \to \mathbb{R}$ such that $u + \mathrm{i}v \in \mathcal{O}(D)$.*

Proof. Define the continuously partially differentiable function $v : D \to \mathbb{R}$ given by

$$v(x, y) := \int_0^y u_x(x, t)\, \mathrm{d}t - \int_0^x u_y(s, 0)\, \mathrm{d}s \quad \forall (x, y) \in D.$$

Differentiating by $\partial/\partial x$ and using that u is harmonic we get

$$\begin{aligned} v_x(x, y) &= \int_0^y u_{xx}(x, t)\, \mathrm{d}t - u_y(x, 0) \\ &= -\int_0^y u_{yy}(x, t)\, \mathrm{d}t - u_y(x, 0) \\ &= -u_y(x, y) + u_y(x, 0) - u_y(x, 0) \\ &= -u_y(x, y). \end{aligned}$$

Note that the interchange of differentiation and integration in the first step is permitted since the integrand is continuously differentiable and the integration range compact. On the other hand, differentiating by $\partial/\partial y$ we obtain

$$v_y(x, y) = u_x(x, y).$$

Thus, the pair (u, v) satisfies the Cauchy-Riemann equations so that $u + \mathrm{i}v$ is holomorphic according to Proposition 1.3. □

Theorem 6.5. *Let $D \subseteq \mathbb{C}$ be a homologically simply connected region and $u : D \to \mathbb{R}$ be harmonic. Then, there exists a harmonic function $v : D \to \mathbb{R}$ such that $u + \mathrm{i}v : D \to \mathbb{C}$ is holomorphic.*

Proof. If $D = \mathbb{C}$ then Lemma 6.4 directly applies and we are done. Suppose therefore that $D \neq \mathbb{C}$. By the Riemann Mapping Theorem 5.30 there exists a biholomorphic map $f : D \to \mathbb{D}$. By Proposition 6.3, $u \circ f^{-1} : \mathbb{D} \to \mathbb{R}$ is harmonic. Applying Lemma 6.4, there exists a harmonic function $w : \mathbb{D} \to \mathbb{R}$ such that $u \circ f^{-1} + \mathrm{i}w : \mathbb{D} \to \mathbb{C}$ is holomorphic. Define $v : D \to \mathbb{R}$ by $v := w \circ f$. Then, v is harmonic by Proposition 6.3 and $u + \mathrm{i}v : D \to \mathbb{C}$ is holomorphic. □

Proposition 6.6. *Harmonic functions are infinitely differentiable.*

Proof. **Exercise.** □

Theorem 6.7 (Mean Value Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region and $u : D \to \mathbb{R}$ harmonic. Suppose $a \in D$ and $r > 0$ such that $\overline{B_r(a)} \subset D$. Then,*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u\left(a + re^{\mathrm{i}\theta}\right) \mathrm{d}\theta.$$

Proof. Choose $s > r$ such that $B_s(a) \subseteq D$. By Theorem 6.5 there exist a harmonic function $v : B_s(a) \to \mathbb{R}$ such that $f := u + \mathrm{i}v : B_s(a) \to \mathbb{C}$ is holomorphic. Applying the Cauchy Integral Formula (Theorem 2.20) to f at the point a with path $\partial B_r(a)$ we obtain,

$$f(a) = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(a)} \frac{f(\zeta)}{\zeta - a} \mathrm{d}\zeta = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + re^{\mathrm{i}\theta}\right) \mathrm{d}\theta.$$

Taking the real part on both sides yields the desired result. □

Definition 6.8. Let $D \subseteq \mathbb{C}$ be a region and $u : D \to \mathbb{R}$ continuous. We say that u has the *mean value property* iff for all $a \in D$ and all $r > 0$ such that $\overline{B_r(a)} \subset D$ we have

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u\left(a + re^{\mathrm{i}\theta}\right) \mathrm{d}\theta.$$

It turns out that the mean value property implies harmonicity.

Theorem 6.9 (Maximum Principle)**.** *Let $D \subseteq \mathbb{C}$ be a region and $u : D \to \mathbb{R}$ a continuous function with the mean value property. Suppose that u has a maximum at some point $a \in D$, i.e., that $u(z) \leq u(a)$ for all $z \in D$. Then u is constant.*

Proof. Define

$$A := \{z \in D : u(z) = u(a)\}.$$

Since u is continuous, A must be closed in D. We proceed to show that A is also open. Let $z_0 \in A$ and $r > 0$ such that $B_r(z_0) \subseteq D$. Choose $b \in B_r(z_0)$ and set $s := |b - z_0|$. By the mean value property

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + se^{i\theta}\right) d\theta.$$

The integrand is continuous and everywhere smaller or equal to $u(z_0)$. Hence, for the equality to hold, we must have $u\left(z_0 + se^{i\theta}\right) = u(z_0)$ for all $\theta \in [0, 2\pi)$. In particular, $u(b) = u(z_0) = u(a)$ and hence $b \in A$. Since b was chosen arbitrarily we have $B_r(z_0) \subseteq A$, showing that A is open. Since A is non-empty, closed in D and open, we must have $A = D$. Thus, $u(z) = u(a)$ for all $z \in D$ and u is constant. □

Proposition 6.10. *Let $D \subset \mathbb{C}$ be a bounded region and $u : \overline{D} \to \mathbb{R}$ a continuous function with the mean value property in D, satisfying $u|_{\partial D} = 0$. Then, $u = 0$.*

Proof. **Exercise.** □

Exercise 61. Show the following version of the maximum principle, which is more similar to Theorem 3.6: Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ a continuous function satisfying the mean value property. Suppose that $|f|$ has a maximum at some point $a \in D$, i.e., that $|f(z)| \leq |f(a)|$ for all $z \in D$. Then f is constant. [Hint: Consider the function $g(z) := \Re(f(z)/f(a))$.]

6.2 The Dirichlet Problem

Definition 6.11. The function $P : \mathbb{D} \to \mathbb{R}$ given by

$$P(z) := \Re\left(\frac{1+z}{1-z}\right) \quad \forall z \in \mathbb{D}$$

is called the *Poisson kernel*. For $0 \leq r < 1$ and $\theta \in \mathbb{R}$ it is also common to use the notation

$$P_r(\theta) := P\left(re^{i\theta}\right).$$

Proposition 6.12. *The Poisson kernel P has the following properties:*

1. *P is harmonic.*
2. *For all $z = re^{i\theta} \in \mathbb{D}$,*
$$P_r(\theta) = P(z) = \frac{1 - |z|^2}{|1 - z|^2} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$
3. *$P(z) > 0$ for all $z \in \mathbb{D}$.*

4. *$P_r(-\theta) = P(\overline{z}) = P(z) = P_r(\theta)$ for all $z = re^{\mathrm{i}\theta} \in \mathbb{D}$.*

5. *for all $z = re^{\mathrm{i}\theta} \in \mathbb{D}$,*

$$P_r(\theta) = P(z) = 1 + \sum_{n=1}^{\infty} (z^n + \overline{z}^n) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

6. *For all $0 \le r < 1$ we have*

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(0)} \frac{P(\zeta)}{\zeta} \,\mathrm{d}\zeta = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) \,\mathrm{d}\theta = 1.$$

7. *For all $0 < r < 1$ and $0 < |\delta| < |\theta| \le \pi$ we have $P_r(\theta) < P_r(\delta)$.*

8. *For each $0 < \delta < \pi$ and $\epsilon > 0$ there exists $0 < \rho < 1$ such that for all $\rho < r < 1$ and $\delta < |\theta| \le \pi$ we have $|P_r(\theta)| < \epsilon$.*

Proof. 1. By definition, the Poisson kernel is the real part of a holomorphic function. Thus, it is harmonic by 6.2. 2. Elementary calculation. 3. This follows immediately from 2. 4. This follows immediately from 2. 5. **Exercise.** 6. Note that for $0 \le r < 1$ the series representation given in 5. converges uniformly. So, we can exchange summation and integration to get,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) \,\mathrm{d}\theta = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\mathrm{i}n\theta} \,\mathrm{d}\theta = 1.$$

7. This follows easily from 2. 8. Fix $0 < \delta < \pi$ and $\epsilon > 0$. Then, $P_r(\delta) \to 0$ for $r \to 1-$ using 2. Thus, there is $0 < \rho < 1$ so that $|P_r(\delta)| < \epsilon$ if $\rho < r < 1$. Using 7. completes the proof of 8. □

Theorem 6.13. *Let $b : \partial\mathbb{D} \to \mathbb{R}$ be continuous. Then, there exists a unique continuous function $u : \overline{\mathbb{D}} \to \mathbb{R}$ such that $u|_{\partial\mathbb{D}} = b$ and u is harmonic in $\mathbb{D}$. Moreover, for all $0 \le r < 1$ and $\theta \in \mathbb{R}$,*

$$u\left(re^{\mathrm{i}\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \phi) b\left(e^{\mathrm{i}\phi}\right) \,\mathrm{d}\phi.$$

Proof. Define $u(z)$ for $z \in \mathbb{D}$ by the stated formula and $u(z) := b(z)$ for $z \in \partial\mathbb{D}$. We first show that u is harmonic in $\mathbb{D}$. We note that for $z \in \mathbb{D}$,

$$\begin{aligned} u(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re\left(\frac{1 + ze^{-\mathrm{i}\phi}}{1 - ze^{-\mathrm{i}\phi}}\right) b\left(e^{\mathrm{i}\phi}\right) \,\mathrm{d}\phi \\ &= \Re\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{\mathrm{i}\phi} + z}{e^{\mathrm{i}\phi} - z} b\left(e^{\mathrm{i}\phi}\right) \,\mathrm{d}\phi\right). \end{aligned}$$

Note that the integrand in the last expression is continuous as a function of $(\phi, z) \in \mathbb{R} \times \mathbb{D}$ and holomorphic as a function of $z \in \mathbb{D}$ for each value of $\phi \in \mathbb{R}$. That is, we can apply Lemma 2.27 to conclude that the integral defines a holomorphic function in $\mathbb{D}$. But by Proposition 6.2, the real part of this function is harmonic.

We proceed to show that u is continuous in $\overline{\mathbb{D}}$. Since continuity in $\mathbb{D}$ follows from harmonicity it suffices to consider points in the boundary of $\overline{\mathbb{D}}$. In particular, it is enough to show the following: Given $\psi \in [-\pi, \pi)$ and $\epsilon > 0$, there exist $\delta > 0$ and $0 < \rho < 1$ such that

$$\left| u\left(r e^{\mathrm{i}\theta}\right) - b\left(e^{\mathrm{i}\psi}\right) \right| < \epsilon \quad \forall \rho < r \leq 1, \forall \theta \in (\psi - \delta, \psi + \delta).$$

We proceed to find such δ and ρ given ψ and ϵ. By continuity of b, there exists $\delta > 0$ such that

$$\left| b\left(e^{\mathrm{i}\theta}\right) - b\left(e^{\mathrm{i}\psi}\right) \right| < \frac{\epsilon}{2} \quad \forall \theta \in (\psi - 2\delta, \psi + 2\delta).$$

By Proposition 6.12.8, there exists $0 < \rho < 1$ such that

$$P_r(\theta) < \frac{\epsilon}{4(\|b\|_{\partial\mathbb{D}} + 1)} \quad \forall \rho < r < 1, \forall \delta < |\theta| \leq \pi.$$

Now let $\theta \in (\psi - \delta, \psi + \delta)$ and $\rho < r \leq 1$. Then,

$$\begin{aligned}
\left| u\left(r e^{\mathrm{i}\theta}\right) - b\left(e^{\mathrm{i}\psi}\right) \right| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \phi) b\left(e^{\mathrm{i}\phi}\right) \,\mathrm{d}\phi - b\left(e^{\mathrm{i}\psi}\right) \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \phi) \left(b\left(e^{\mathrm{i}\phi}\right) - b\left(e^{\mathrm{i}\psi}\right)\right) \,\mathrm{d}\phi \right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \phi) \left| b\left(e^{\mathrm{i}\phi}\right) - b\left(e^{\mathrm{i}\psi}\right) \right| \,\mathrm{d}\phi \\
&= \frac{1}{2\pi} \int_{|\phi - \psi| < 2\delta} P_r(\theta - \phi) \left| b\left(e^{\mathrm{i}\phi}\right) - b\left(e^{\mathrm{i}\psi}\right) \right| \,\mathrm{d}\phi \\
&\quad + \frac{1}{2\pi} \int_{|\phi - \psi| \geq 2\delta} P_r(\theta - \phi) \left| b\left(e^{\mathrm{i}\phi}\right) - b\left(e^{\mathrm{i}\psi}\right) \right| \,\mathrm{d}\phi \\
&\leq \frac{1}{2\pi} \int_{|\phi - \psi| < 2\delta} P_r(\theta - \phi) \frac{\epsilon}{2} \,\mathrm{d}\phi \\
&\quad + \frac{1}{2\pi} \int_{|\phi - \psi| \geq 2\delta} \frac{\epsilon}{4(\|b\|_{\partial\mathbb{D}} + 1)} 2\|b\|_{\partial\mathbb{D}} \,\mathrm{d}\phi \\
&< \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \phi) \frac{\epsilon}{2} \,\mathrm{d}\phi + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\epsilon}{2} \,\mathrm{d}\phi \\
&= \epsilon.
\end{aligned}$$

Here, we have used the properties of the Poisson kernel given in Proposition 6.12 parts 3. and 6.

It remains to show uniqueness of the function u. Suppose there was another function $v : \overline{\mathbb{D}} \to \mathbb{R}$ with the required properties. Then, the difference $u - v$ would be continuous on $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$. Furthermore, $(u-v)|_{\partial\mathbb{D}} = 0$, so by Proposition 6.10, $u - v = 0$, i.e., $u = v$. □

Definition 6.14. We call a region $D \subseteq \mathbb{C}$ *disk-like* iff there exists a conformal equivalence $D \to \mathbb{D}$ which extends to a homeomorphism $\overline{D} \to \overline{\mathbb{D}}$.

Remark 6.15. A disk-like region is in particular homologically simply connected and bounded.

Theorem 6.16. *Let $D \subset \mathbb{C}$ be a disk-like region. Let $b : \partial D \to \mathbb{R}$ be continuous. Then, there exists a unique continuous function $u : \overline{D} \to \mathbb{R}$ such that $u|_{\partial\mathbb{D}} = b$ and u is harmonic in D.*

Proof. **Exercise.** □

Theorem 6.17. *Let $U \subseteq \mathbb{C}$ be open and $u : U \to \mathbb{R}$ be continuous with the mean value property. Then, u is harmonic.*

Proof. Let $a \in U$ and $r > 0$ such that $\overline{B_r(a)} \subset U$. It is sufficient to show that u is harmonic in $B_r(a)$. Since $B_r(a)$ is disk-like there exists by Theorem 6.16 a continuous function $v : \overline{B_r(a)} \to \mathbb{R}$ which is harmonic in $B_r(a)$ and coincides with u in $\partial B_r(a)$. But the difference $u - v : \overline{B_r(a)} \to \mathbb{R}$ is continuous, has the mean value property in $B_r(a)$ and vanishes on the boundary $\partial B_r(a)$. Thus $u = v$ also in $B_r(a)$ by Proposition 6.10. In particular, u is harmonic in $B_r(a)$. □

7 The Riemann Sphere

7.1 Definition

Definition 7.1. A topological space is called *locally compact* iff every point has a compact neighborhood.

Proposition 7.2 (One-Point Compactification)**.** *Let X be a Hausdorff topological space that is locally compact. Consider the set $\hat{X} := X \cup \{\infty\}$ equipped with the following topology: A set $U \subseteq \hat{X}$ is open iff $U \subseteq X$ and U is open in X or if $U = V \cup \{\infty\}$ where $V \subseteq X$ such that $X \setminus V$ is compact in X. Then, $\hat{X}$ is a compact Hausdorff space.*

Proof. **Exercise.** □

Proposition 7.3. *Consider the topological space $\hat{\mathbb{C}}$ with the subsets $U_0 := \hat{\mathbb{C}} \setminus \{\infty\}$ and $U_\infty := \hat{\mathbb{C}} \setminus \{0\}$. Consider the maps $\phi_0 : U_0 \to \mathbb{C}$ given by $\phi_0(z) := z$ for all $z \in U_0$ and $\phi_\infty : U_\infty \to \mathbb{C}$ given by $\phi_\infty(z) := 1/z$ for all $z \in U_\infty \setminus \{\infty\}$ and $\phi_\infty(\infty) := 0$. Then, ϕ_0 and ϕ_∞ are homeomorphisms. Moreover, $\phi_0 \circ \phi_\infty^{-1}|_{\mathbb{C}\setminus\{0\}}$ is the biholomorphism $\mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ given by $z \mapsto 1/z$.*

Proof. **Exercise.** □

Remark 7.4. The topological space $\hat{\mathbb{C}}$ together with the structures introduced in the preceding Proposition is called the *Riemann sphere*. It is an example of a *complex manifold*. The maps ϕ_0, ϕ_∞ are called *charts*.

Exercise 62. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. Show that $\lim_{n\to\infty} z_n = \infty$ in $\hat{\mathbb{C}}$ if and only if for each $M > 0$ there exists $n_0 \in \mathbb{N}$ such that $|z_n| > M$ for all $n \geq n_0$.

Exercise 63. Consider the symmetric function $d : \hat{\mathbb{C}} \times \hat{\mathbb{C}} \to \mathbb{R}_0^+$ given by

$$\begin{aligned} d(z, z') &:= \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}} \quad \forall z, z' \in \mathbb{C} \\ d(\infty, z) &:= \frac{2}{\sqrt{1 + |z|^2}} \quad \forall z \in \mathbb{C} \\ d(\infty, \infty) &:= 0. \end{aligned}$$

Show that d defines a metric on the Riemann sphere that is compatible with its topology.

Remark 7.5. The metric introduced above can be obtained from the *stereographic projection* of $\hat{\mathbb{C}}$ identified with the unit disk to the complex plane.

7.2 Functions on $\hat{\mathbb{C}}$

Exercise 64. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $f : D \to \mathbb{C}$ be continuous. Let $a \in D \setminus \{0, \infty\}$. Show that $f \circ \phi_0^{-1}$ is holomorphic/conformal at $\phi_0(a)$ iff $f \circ \phi_\infty^{-1}$ is holomorphic/conformal at $\phi_\infty(a)$.

Definition 7.6. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $f : D \to \mathbb{C}$ be continuous. Let $a \in D$. If $a \neq \infty$, we say that f is holomorphic/conformal at a iff $f \circ \phi_0^{-1}$ is holomorphic/conformal at $\phi_0(a)$. If $a \neq 0$, we say that f is holomorphic/conformal at a iff $f \circ \phi_\infty^{-1}$ is holomorphic/conformal at $\phi_\infty(a)$. We say that f is holomorphic/conformal in D iff f is holomorphic/conformal at each point $a \in D$.

Exercise 65. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $a \in D \setminus \{0, \infty\}$. Let $f \in \mathcal{O}(D \setminus \{a\})$. Show that the type and order of the singularity of $f \circ \phi_0^{-1}$ at $\phi_0(a)$ is the same as the type and order of the singularity of $f \circ \phi_\infty^{-1}$ at $\phi_\infty(a)$.

Definition 7.7. Let $D \subseteq \hat{\mathbb{C}}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. If $a \neq \infty$, we say that f has a removable singularity/a pole of order n/an essential singularity at a iff $f \circ \phi_0^{-1}$ has a removable singularity/a pole of order n/an essential singularity at $\phi_0(a)$. If $a \neq 0$, we say that f has a removable singularity/a pole of order n/an essential singularity at a iff $f \circ \phi_\infty^{-1}$ has a removable singularity/a pole of order n/an essential singularity at $\phi_\infty(a)$.

Proposition 7.8. *Let $f \in \mathcal{O}(\hat{\mathbb{C}})$. Then, f is constant.*

Proof. **Exercise.** □

Definition 7.9. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $A \subset D$ be a discrete and relatively closed subset. A function $f \in \mathcal{O}(D \setminus A)$ is called *meromorphic* iff each point $a \in A$ is either a removable singularity or a pole of f.

Proposition 7.10. *Let $f \in \mathcal{M}(\hat{\mathbb{C}})$. Then, f is a rational function.*

Proof. **Exercise.**[Hint: First assume that f has a pole only at ∞ and show that $|f(z)| < M|z|^n$ for some constants $M > 0$ and $n \in \mathbb{N}$. Conclude that f must be a polynomial. In the general case show and use the fact that f can only have finitely many poles.] □

7.3 Functions onto $\hat{\mathbb{C}}$ and $\mathrm{Aut}(\hat{\mathbb{C}})$

Exercise 66. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $f \in \mathcal{M}(D)$. Let $P \subset D$ be the set of poles of f and $Z \subseteq D$ the set of zeros of f. Define $\hat{f} : D \to \hat{\mathbb{C}}$ by $\hat{f}(z) := \phi_0^{-1}(f(z))$ if $z \in D \setminus P$ and $\hat{f}(z) := \infty$ if $z \in P$. Show that $\hat{f}$ is continuous and that $\phi_0 \circ \hat{f}|_{D\setminus P}$ as well as $\phi_\infty \circ \hat{f}|_{D\setminus Z}$ are holomorphic.

Exercise 67. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be continuous. Let $Z := \{z \in \hat{\mathbb{C}} : \hat{f}(z) = 0\}$ and $P := \{z \in \hat{\mathbb{C}} : \hat{f}(z) = \infty\}$. Suppose that $\phi_0 \circ \hat{f}|_{D\setminus P}$ as well as $\phi_\infty \circ \hat{f}|_{D\setminus Z}$ are holomorphic. Define $f : D \setminus P \to \mathbb{C}$ by $f := \phi_0 \circ \hat{f}|_{D\setminus P}$. If $P \neq D$, then $f \in \mathcal{M}(D)$.

Definition 7.11. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $f : D \to \hat{\mathbb{C}}$ be continuous. Let $a \in D$. If $f(a) \neq \infty$, we say that f is conformal at a iff $\phi_0 \circ f$ is conformal at a. If $f(a) \neq 0$, we say that f is conformal at a iff $\phi_\infty \circ f$ is conformal at a. We say that f is conformal in D iff f is conformal at each point $a \in D$.

Definition 7.12. A conformal mapping $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that has a conformal inverse is called a *conformal automorphism* of $\hat{\mathbb{C}}$.

Proposition 7.13. *Möbius transformations are conformal automorphisms of* $\hat{\mathbb{C}}$.

Proof. **Exercise.** □

Theorem 7.14. *Suppose that* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is conformal and injective. Then,* f *is a Möbius transformation.*

Proof. (Sketch.) As in Exercise 67 we can think of f as a meromorphic function on $\hat{\mathbb{C}}$. Thus, by Proposition 7.10, f is rational, i.e., a quotient p/q of polynomials. Without loss of generality we may assume p and q not to have common divisors. Since f is injective, p can only have one zero which must be simple. Similarly, q can only have one pole which must be simple. Thus, f is a Möbius transformation. □

Corollary 7.15. $\mathrm{Aut}(\hat{\mathbb{C}}) = \boldsymbol{M\ddot{o}b}$.

Theorem 7.16. *Let* (a, b, c) *and* (a', b', c') *be triples of distinct points in* $\hat{\mathbb{C}}$. *Then, there exists exactly one Möbius transformation* f *such that* $f(a) = a'$, $f(b) = b'$, $f(c) = c'$.

Proof. **Exercise.** □