

## FUNCTIONAL ANALYSIS – Semester 2024-2

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# 1 Topological and metric spaces

## 1.1 Basic Definitions

**Definition 1.1** (Topology). Let  $S$  be a set. A subset  $\mathcal{T}$  of the set  $\mathfrak{P}(S)$  of subsets of  $S$  is called a *topology* iff it has the following properties:

- $\emptyset \in \mathcal{T}$  and  $S \in \mathcal{T}$ .
- Let  $\{U_i\}_{i \in I}$  be a family of elements in  $\mathcal{T}$ . Then  $\bigcup_{i \in I} U_i \in \mathcal{T}$ .
- Let  $U, V \in \mathcal{T}$ . Then  $U \cap V \in \mathcal{T}$ .

A set equipped with a topology is called a *topological space*. The elements of  $\mathcal{T}$  are called the *open sets* in  $S$ . A complement of an open set in  $S$  is called a *closed set*.

**Definition 1.2.** Let  $S$  be a topological space and  $x \in S$ . Then a subset  $U \subseteq S$  is called a *neighborhood* of  $x$  iff it contains an open set which in turn contains  $x$ .

**Definition 1.3.** Let  $S$  be a topological space and  $U$  a subset. The *closure*  $\bar{U}$  of  $U$  is the smallest closed set containing  $U$ . The *interior*  $\overset{\circ}{U}$  of  $U$  is the largest open set contained in  $U$ .  $U$  is called *dense* in  $S$  iff  $\bar{U} = S$ .

**Definition 1.4** (base). Let  $\mathcal{T}$  be a topology. A subset  $\mathcal{B}$  of  $\mathcal{T}$  is called a *base* of  $\mathcal{T}$  iff the elements of  $\mathcal{T}$  are precisely the unions of elements of  $\mathcal{B}$ . It is called a *subbase* iff the elements of  $\mathcal{T}$  are precisely the finite intersections of unions of elements of  $\mathcal{B}$ .

**Proposition 1.5.** Let  $S$  be a set and  $\mathcal{B}$  a subset of  $\mathfrak{P}(S)$ .  $\mathcal{B}$  is the base of a topology on  $S$  iff it satisfies all of the following properties:

- $\emptyset \in \mathcal{B}$ .
- For every  $x \in S$  there is a set  $U \in \mathcal{B}$  such that  $x \in U$ .
- Let  $U, V \in \mathcal{B}$ . Then there exists a family  $\{W_\alpha\}_{\alpha \in A}$  of elements of  $\mathcal{B}$  such that  $U \cap V = \bigcup_{\alpha \in A} W_\alpha$ .

*Proof.* **Exercise.** □

**Definition 1.6** (Filter). Let  $S$  be a set. A subset  $\mathcal{F}$  of the set  $\mathfrak{P}(S)$  of subsets of  $S$  is called a *filter* iff it has the following properties:

- $\emptyset \notin \mathcal{F}$  and  $S \in \mathcal{F}$ .
- Let  $U, V \in \mathcal{F}$ . Then  $U \cap V \in \mathcal{F}$ .
- Let  $U \in \mathcal{F}$  and  $U \subseteq V \subseteq S$ . Then  $V \in \mathcal{F}$ .

**Definition 1.7.** Let  $\mathcal{F}$  be a filter. A subset  $\mathcal{B}$  of  $\mathcal{F}$  is called a *base* of  $\mathcal{F}$  iff every element of  $\mathcal{F}$  contains an element of  $\mathcal{B}$ .

**Proposition 1.8.** Let  $S$  be a set and  $\mathcal{B} \subseteq \mathfrak{P}(S)$ . Then  $\mathcal{B}$  is the base of a filter on  $S$  iff it satisfies the following properties:

- $\emptyset \notin \mathcal{B}$  and  $\mathcal{B} \neq \emptyset$ .
- Let  $U, V \in \mathcal{B}$ . Then there exists  $W \in \mathcal{B}$  such that  $W \subseteq U \cap V$ .

*Proof.* **Exercise.** □

Let  $S$  be a topological space and  $x \in S$ . It is easy to see that the set of neighborhoods of  $x$  forms a filter. It is called the *filter of neighborhoods* of  $x$  and denoted by  $\mathcal{N}_x$ . The family of filters of neighborhoods in turn encodes the topology:

**Proposition 1.9.** Let  $S$  be a topological space and  $\{\mathcal{N}_x\}_{x \in S}$  the family of filters of neighborhoods. Then a subset  $U$  of  $S$  is open iff for every  $x \in U$ , there is a set  $W_x \in \mathcal{N}_x$  such that  $W_x \subseteq U$ .

*Proof.* **Exercise.** □

**Proposition 1.10.** Let  $S$  be a set and  $\{\mathcal{F}_x\}_{x \in S}$  an assignment of a filter to every point in  $S$ . Then this family of filters are the filters of neighborhoods of a topology on  $S$  iff they satisfy the following properties:

1. For all  $x \in S$ , every element of  $\mathcal{F}_x$  contains  $x$ .
2. For all  $x \in S$  and  $U \in \mathcal{F}_x$ , there exists  $W \in \mathcal{F}_x$  such that  $U \in \mathcal{F}_y$  for all  $y \in W$ .

*Proof.* If  $\{\mathcal{F}_x\}_{x \in S}$  are the filters of neighborhoods of a topology it is clear that the properties are satisfied: 1. Every neighborhood of a point contains the point itself. 2. For a neighborhood  $U$  of  $x$  take  $W$  to be an open neighborhood of  $x$  contained in  $U$ . Then  $W$  is a neighborhood for each point in  $W$ .

Conversely, suppose  $\{\mathcal{F}_x\}_{x \in S}$  satisfies Properties 1 and 2. Given  $x$  we define a provisional open neighborhood of  $x$  to be an element  $U \in \mathcal{F}_x$  such that  $U \in \mathcal{F}_y$  for all  $y \in U$ . This definition is not empty since at least  $S$  itself is a provisional open neighborhood of every point  $x$  in this way. Moreover, for any  $y \in U$ , by the same definition,  $U$  is a provisional open neighborhood of  $y$ . Now take  $y \notin U$ . Then, by Property 1,  $U$  is not a provisional open neighborhood of  $y$ . We define a provisional open set as a set that is a provisional open neighborhood for one (and thus any) of its points. We also declare the empty set to be a provisional open set. Let  $\mathcal{T}$  be the set of provisional open sets.

We proceed to verify that  $\mathcal{T}$  satisfies the axioms of a topology. Property 1 of Definition 1.1 holds since  $S \in \mathcal{T}$ , and we have declared  $\emptyset \in \mathcal{T}$ . Let  $\{U_\alpha\}_{\alpha \in I}$  be a family in  $\mathcal{T}$  and consider their union  $U = \bigcup_{\alpha \in I} U_\alpha$ . Assume  $U$  is not empty (otherwise  $U \in \mathcal{T}$  trivially)

and pick  $x \in U$ . Thus, there is  $\alpha \in I$  such that  $x \in U_\alpha$ . But then  $U_\alpha \in \mathcal{F}_x$  and also  $U \in \mathcal{F}_x$ . This is true for any  $x \in U$ . Hence,  $U \in \mathcal{T}$ . Consider now  $U, V \in \mathcal{T}$ . Assume the intersection  $U \cap V$  to be non-empty (otherwise  $U \cap V \in \mathcal{T}$  trivially) and pick a point  $x$  in it. Then  $U \in \mathcal{F}_x$  and  $V \in \mathcal{F}_x$  and therefore  $U \cap V \in \mathcal{F}_x$ . The same is true for any point in  $U \cap V$ , hence  $U \cap V \in \mathcal{T}$ . We thus drop the adjective “provisional”.

It remains to show that  $\{\mathcal{F}_x\}_{x \in S}$  are the filters of neighborhoods for the topology just defined. It is already clear that any open neighborhood of a point  $x$  is contained in  $\mathcal{F}_x$ . We need to show that every element of  $\mathcal{F}_x$  contains an open neighborhood of  $x$ . Take  $U \in \mathcal{F}_x$ . We define  $V$  to be the set of points  $y$  such that  $U \in \mathcal{F}_y$ . This cannot be empty as  $x \in V$ . Moreover, Property 1 implies  $V \subseteq U$ . Let  $y \in V$ , then  $U \in \mathcal{F}_y$  and we can apply Property 2 to obtain a subset  $W \subseteq V$  with  $W \in \mathcal{F}_y$ . But this implies  $V \in \mathcal{F}_y$ . Since the same is true for any  $y \in V$  we find that  $V$  is an open neighborhood of  $x$ . This completes the proof.  $\square$

**Definition 1.11** (Continuity). Let  $S, T$  be topological spaces. A map  $f : S \rightarrow T$  is called *continuous at*  $p \in S$  iff  $f^{-1}(\mathcal{N}_{f(p)}) \subseteq \mathcal{N}_p$ .  $f$  is called *continuous* iff it is continuous at every  $p \in S$ . We denote the space of continuous maps from  $S$  to  $T$  by  $C(S, T)$ .

**Proposition 1.12.** Let  $S, T$  be topological spaces and  $f : S \rightarrow T$  a map. Then,  $f$  is continuous iff for every open set  $U \in T$  the preimage  $f^{-1}(U)$  in  $S$  is open.

*Proof.* Exercise.  $\square$

**Proposition 1.13.** Let  $S, T, U$  be topological spaces,  $f \in C(S, T)$  and  $g \in C(T, U)$ . Then, the composition  $g \circ f : S \rightarrow U$  is continuous.

*Proof.* Immediate.  $\square$

**Definition 1.14.** Let  $S, T$  be topological spaces. A bijection  $f : S \rightarrow T$  is called a *homeomorphism* iff  $f$  and  $f^{-1}$  are both continuous. If such a homeomorphism exists  $S$  and  $T$  are called *homeomorphic*.

**Definition 1.15.** Let  $\mathcal{T}_1, \mathcal{T}_2$  be topologies on the set  $S$ . Then,  $\mathcal{T}_1$  is called *finer* than  $\mathcal{T}_2$  and  $\mathcal{T}_2$  is called *coarser* than  $\mathcal{T}_1$  iff all open sets of  $\mathcal{T}_2$  are also open sets of  $\mathcal{T}_1$ .

**Definition 1.16** (Induced Topology). Let  $S$  be a topological space and  $U$  a subset. Consider the topology given on  $U$  by the intersection of each open set on  $S$  with  $U$ . This is called the *induced topology* on  $U$ .

**Definition 1.17** (Product Topology). Let  $S$  be the Cartesian product  $S = \prod_{\alpha \in I} S_\alpha$  of a family of topological spaces. Consider subsets of  $S$  of the form  $\prod_{\alpha \in I} U_\alpha$  where finitely many  $U_\alpha$  are open sets in  $S_\alpha$  and the others coincide with the whole space  $U_\alpha = S_\alpha$ . These subsets form the base of a topology on  $S$  which is called the *product topology*.

**Exercise 1.** Show that alternatively, the product topology can be characterized as the coarsest topology on  $S = \prod_{\alpha \in I} S_\alpha$  such that all projections  $S \rightarrow S_\alpha$  are continuous.

**Proposition 1.18.** Let  $S, T, X$  be topological spaces and  $f \in C(S \times T, X)$ , where  $S \times T$  carries the product topology. Then the map  $f_x : T \rightarrow X$  defined by  $f_x(y) = f(x, y)$  is continuous for every  $x \in S$ .

*Proof.* Fix  $x \in S$ . Let  $U$  be an open set in  $X$ . We want to show that  $W := f_x^{-1}(U)$  is open. We do this by finding for any  $y \in W$  an open neighborhood of  $y$  contained in  $W$ . If  $W$  is empty we are done, hence assume that this is not so. Pick  $y \in W$ . Then  $(x, y) \in f^{-1}(U)$  with  $f^{-1}(U)$  open by continuity of  $f$ . Since  $S \times T$  carries the product topology there must be open sets  $V_x \subseteq S$  and  $V_y \subseteq T$  with  $x \in V_x$ ,  $y \in V_y$  and  $V_x \times V_y \subseteq f^{-1}(U)$ . But clearly  $V_y \subseteq W$  and we are done.  $\square$

**Definition 1.19** (Quotient Topology). Let  $S$  be a topological space and  $\sim$  an equivalence relation on  $S$ . Then, the *quotient topology* on  $S/\sim$  is the finest topology such that the quotient map  $S \rightarrow S/\sim$  is continuous.

**Definition 1.20.** Let  $S, T$  be topological spaces and  $f : S \rightarrow T$ . For  $a \in S$  we say that  $f$  is *open at  $a$*  iff for every neighborhood  $U$  of  $a$  the image  $f(U)$  is a neighborhood of  $f(a)$ . We say that  $f$  is *open* iff it is open at every  $a \in S$ .

**Proposition 1.21.** Let  $S, T$  be topological spaces and  $f : S \rightarrow T$ .  $f$  is open iff it maps any open set to an open set.

*Proof.* Straightforward.  $\square$

**Definition 1.22** (Ultrafilter). Let  $\mathcal{F}$  be a filter. We call  $\mathcal{F}$  an *ultrafilter* iff  $\mathcal{F}$  cannot be enlarged as a filter. That is, given a filter  $\mathcal{F}'$  such that  $\mathcal{F} \subseteq \mathcal{F}'$  we have  $\mathcal{F}' = \mathcal{F}$ .

**Lemma 1.23.** Let  $S$  be a set,  $\mathcal{F}$  an ultrafilter on  $S$  and  $U \subseteq S$  such that  $U \cap V \neq \emptyset$  for all  $V \in \mathcal{F}$ . Then  $U \in \mathcal{F}$ .

*Proof.* Let  $\mathcal{F}$  be an ultrafilter on  $S$  and  $U \subseteq S$  such that  $U \cap V \neq \emptyset$  for all  $V \in \mathcal{F}$ . Then,  $\mathcal{B} := \{U \cap V : V \in \mathcal{F}\}$  forms the base of a filter  $\mathcal{F}'$  such that  $\mathcal{F} \subseteq \mathcal{F}'$  and  $U \in \mathcal{F}'$ . But since  $\mathcal{F}$  is ultrafilter we have  $\mathcal{F} = \mathcal{F}'$  and hence  $U \in \mathcal{F}$ .  $\square$

**Proposition 1.24** (Ultrafilter lemma). Let  $\mathcal{F}$  be a filter. Then there exists an ultrafilter  $\mathcal{F}'$  such that  $\mathcal{F} \subseteq \mathcal{F}'$ .

*Proof.* **Exercise.** Use Zorn's Lemma.  $\square$

## 1.2 Some properties of topological spaces

In a topological space it is useful if two distinct points can be distinguished by the topology. A strong form of this distinguishability is the *Hausdorff property*.

**Definition 1.25** (Hausdorff). Let  $S$  be a topological space. Assume that given any two distinct points  $x, y \in S$  we can find open sets  $U, V \subset S$  such that  $x \in U$  and  $y \in V$  and  $U \cap V = \emptyset$ . Then,  $S$  is said to have the *Hausdorff property*. We also say that  $S$  is a *Hausdorff space*.

**Definition 1.26.** A topological space  $S$  is called *completely regular* iff given a closed subset  $C \subseteq S$  and a point  $p \in S \setminus C$  there exists a continuous function  $f : S \rightarrow [0, 1]$  such that  $f(C) = \{0\}$  and  $f(p) = 1$ .

**Definition 1.27.** A topological space is called *normal* iff it is Hausdorff and if given two disjoint closed sets  $A$  and  $B$  there exist disjoint open sets  $U, V$  such that  $A \subseteq U$  and  $B \subseteq V$ .

**Lemma 1.28.** Let  $S$  be a normal topological space,  $U$  an open subset and  $C$  a closed subset such that  $C \subseteq U$ . Then, there exists an open subset  $U'$  and a closed subset  $C'$  such that  $C \subseteq U' \subseteq C' \subseteq U$ .

*Proof.* **Exercise.** □

**Theorem 1.29** (Uryson's Lemma). Let  $S$  be a normal topological space and  $A, B$  disjoint closed subsets. Then, there exists a continuous function  $f : S \rightarrow [0, 1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ .

*Proof.* Let  $C_0 := A$  and  $U_1 := S \setminus B$ . Applying Lemma 1.28 we find an open subset  $U_{1/2}$  and a closed subset  $C_{1/2}$  such that

$$C_0 \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_1.$$

Performing the same operation on the pairs  $C_0 \subseteq U_{1/2}$  and  $C_{1/2} \subseteq U_1$  we obtain

$$C_0 \subseteq U_{1/4} \subseteq C_{1/4} \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_{3/4} \subseteq C_{3/4} \subseteq U_1.$$

We iterate this process, at step  $n$  replacing the pairs  $C_{(k-1)/2^n} \subseteq U_{k/2^n}$  by  $C_{(k-1)/2^n} \subseteq U_{(2k-1)/2^{n+1}} \subseteq C_{(2k-1)/2^{n+1}} \subseteq U_{k/2^n}$  for all  $k \in \{1, \dots, n\}$ .

Now define

$$f(p) := \begin{cases} 1 & \text{if } p \in B \\ \inf\{x \in (0, 1] : p \in U_x\} & \text{if } p \notin B \end{cases}$$

Obviously  $f(B) = \{1\}$  and also  $f(A) = \{0\}$ . To show that  $f$  is continuous it suffices to show that  $f^{-1}([0, a))$  and  $f^{-1}((b, 1])$  are open for  $0 < a \leq 1$  and  $0 \leq b < 1$ . But,

$$f^{-1}([0, a)) = \bigcup_{x < a} U_x, \quad f^{-1}((b, 1]) = \bigcup_{x > b} (S \setminus C_x).$$

□

**Corollary 1.30.** *Every normal space is completely regular.*

**Definition 1.31.** Let  $S$  be a topological space.  $S$  is called *first-countable* iff for each point in  $S$  there exists a countable base of its filter of neighborhoods.  $S$  is called *second-countable* iff the topology of  $S$  admits a countable base.

**Definition 1.32.** Let  $S$  be a topological space and  $U, V \subseteq S$  subsets.  $U$  is called *dense* in  $V$  iff  $V \subseteq \overline{U}$ .

**Definition 1.33** (separable). A topological space is called *separable* iff it contains a countable dense subset.

**Proposition 1.34.** *A topological space that is second-countable is separable.*

*Proof.* **Exercise.** □

**Definition 1.35** (open cover). Let  $S$  be a topological space and  $U \subseteq S$  a subset. A family of open sets  $\{U_\alpha\}_{\alpha \in A}$  is called an *open cover* of  $U$  iff  $U \subseteq \bigcup_{\alpha \in A} U_\alpha$ .

**Proposition 1.36.** *Let  $S$  be a second-countable topological space and  $U \subseteq S$  a subset. Then, every open cover of  $U$  contains a countable subcover.*

*Proof.* **Exercise.** □

**Definition 1.37** (compact). Let  $S$  be a topological space and  $U \subseteq S$  a subset.  $U$  is called *compact* iff every open cover of  $U$  contains a finite subcover.

**Definition 1.38.** Let  $S$  be a topological space and  $U \subseteq S$  a subset. Then,  $U$  is called *relatively compact* in  $S$  iff the closure of  $U$  in  $S$  is compact.

**Proposition 1.39.** *A closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed.*

*Proof.* **Exercise.** □

**Proposition 1.40.** *The image of a compact set under a continuous map is compact.*

*Proof.* **Exercise.** □

**Lemma 1.41.** *Let  $T_1$  be a compact Hausdorff space,  $T_2$  be a Hausdorff space and  $f : T_1 \rightarrow T_2$  a continuous bijective map. Then,  $f$  is a homeomorphism.*

*Proof.* The image of a compact set under  $f$  is compact and hence closed in  $T_2$ . But every closed set in  $T_1$  is compact, so  $f$  is open and hence a homeomorphism. □

**Lemma 1.42.** *Let  $T$  be a Hausdorff topological space and  $C_1, C_2$  disjoint compact subsets of  $T$ . Then, there are disjoint open subsets  $U_1, U_2$  of  $T$  such that  $C_1 \subseteq U_1$  and  $C_2 \subseteq U_2$ . In particular, if  $T$  is compact, then it is normal.*

*Proof.* We first show a weaker statement: Let  $C$  be a compact subset of  $T$  and  $p \notin C$ . Then there exist disjoint open sets  $U$  and  $V$  such that  $p \in U$  and  $C \subseteq V$ . Since  $T$  is Hausdorff, for each point  $q \in C$  there exist disjoint open sets  $U_q$  and  $V_q$  such that  $p \in U_q$  and  $q \in V_q$ . The family of sets  $\{V_q\}_{q \in C}$  defines an open covering of  $C$ . Since  $C$  is compact, there is a finite subset  $S \subseteq C$  such that the family  $\{V_q\}_{q \in S}$  already covers  $C$ . Define  $U := \bigcap_{q \in S} U_q$  and  $V := \bigcup_{q \in S} V_q$ . These are open sets with the desired properties.

We proceed to prove the first statement of the lemma. By the previous demonstration, for each point  $p \in C_1$  there are disjoint open sets  $U_p$  and  $V_p$  such that  $p \in U_p$  and  $C_2 \subseteq V_p$ . The family of sets  $\{U_p\}_{p \in C_1}$  defines an open covering of  $C_1$ . Since  $C_1$  is compact, there is a finite subset  $S \subseteq C_1$  such that the family  $\{U_p\}_{p \in S}$  already covers  $C_1$ . Define  $U_1 := \bigcup_{p \in S} U_p$  and  $U_2 := \bigcap_{p \in S} V_p$ .

For the second statement of the lemma observe that if  $T$  is compact, then every closed subset is compact.  $\square$

**Definition 1.43.** A topological space is called *locally compact* iff every point has a compact neighborhood.

**Definition 1.44.** A topological space is called  *$\sigma$ -compact* iff it is locally compact and admits a covering by countably many compact subsets.

**Definition 1.45.** Let  $T$  be a topological space. A *compact exhaustion* of  $T$  is a sequence  $\{U_i\}_{i \in \mathbb{N}}$  of open and relatively compact subsets such that  $\overline{U_i} \subseteq U_{i+1}$  for all  $i \in \mathbb{N}$  and  $\bigcup_{i \in \mathbb{N}} U_i = T$ .

**Proposition 1.46.** A topological space admits a compact exhaustion iff it is  $\sigma$ -compact.

*Proof.* Suppose the topological space  $T$  is  $\sigma$ -compact. Then there exists a sequence  $\{K_n\}_{n \in \mathbb{N}}$  of compact subsets such that  $\bigcup_{n \in \mathbb{N}} K_n = T$ . Since  $T$  is locally compact, every point possesses an open and relatively compact neighborhood. (Take an open subneighborhood of a compact neighborhood.) We cover  $K_1$  by such open and relatively compact neighborhoods around every point. By compactness a finite subset of those already covers  $K_1$ . Their union, which we call  $U_1$ , is open and relatively compact. We proceed inductively. Suppose we have constructed the open and relatively compact set  $U_n$ . Consider the compact set  $\overline{U_n} \cup K_{n+1}$ . Covering it with open and relatively compact neighborhoods and taking the union of a finite subcover we obtain the open and relatively compact set  $U_{n+1}$ . It is then clear that the sequence  $\{U_n\}_{n \in \mathbb{N}}$  obtained in this way provides a compact exhaustion of  $T$  since  $\overline{U_i} \subseteq U_{i+1}$  for all  $i \in \mathbb{N}$  and  $T = \bigcup_{n \in \mathbb{N}} K_n \subseteq \bigcup_{n \in \mathbb{N}} U_n$ .

Conversely, suppose  $T$  is a topological space and  $\{U_n\}_{n \in \mathbb{N}}$  is a compact exhaustion of  $T$ . Then, the sequence  $\{\overline{U_n}\}_{n \in \mathbb{N}}$  provides a countable covering of  $T$  by compact sets. Also, given  $p \in T$  there exists  $n \in \mathbb{N}$  such that  $p \in U_n$ . Then, the compact set  $\overline{U_n}$  is a neighborhood of  $p$ . That is,  $T$  is locally compact.  $\square$

**Proposition 1.47.** Let  $T$  be a topological space,  $K \subseteq T$  a compact subset and  $\{U_n\}_{n \in \mathbb{N}}$  a compact exhaustion of  $T$ . Then, there exists  $n \in \mathbb{N}$  such that  $K \subseteq U_n$ .

*Proof.* **Exercise.** □

**Exercise 2** (One-point compactification). Let  $S$  be a locally compact Hausdorff space. Let  $\tilde{S} := S \cup \{\infty\}$  to be the set  $S$  with an extra element  $\infty$  adjoint. Define a subset  $U$  of  $\tilde{S}$  to be open iff either  $U$  is an open subset of  $S$  or  $U$  is the complement of a compact subset of  $S$ . Show that this makes  $\tilde{S}$  into a compact Hausdorff space.

### 1.3 Sequences and convergence

**Definition 1.48** (Convergence of sequences). Let  $x := \{x_n\}_{n \in \mathbb{N}}$  be a sequence of points in a topological space  $S$ . We say that  $x$  has an *accumulation point* (or *limit point*)  $p$  iff for every neighborhood  $U$  of  $p$  we have  $x_k \in U$  for infinitely many  $k \in \mathbb{N}$ . We say that  $x$  *converges* to a point  $p$  iff for any neighborhood  $U$  of  $p$  there is a number  $n \in \mathbb{N}$  such that for all  $k \geq n : x_k \in U$ .

**Proposition 1.49.** Let  $S, T$  be topological spaces and  $f : S \rightarrow T$ . If  $f$  is continuous, then for any  $p \in S$  and sequence  $\{x_n\}_{n \in \mathbb{N}}$  converging to  $p$ , the sequence  $\{f(x_n)\}_{n \in \mathbb{N}}$  in  $T$  converges to  $f(p)$ . Conversely, if  $S$  is first-countable and for any  $p \in S$  and sequence  $\{x_n\}_{n \in \mathbb{N}}$  converging to  $p$ , the sequence  $\{f(x_n)\}_{n \in \mathbb{N}}$  in  $T$  converges to  $f(p)$ , then  $f$  is continuous.

*Proof.* **Exercise.** □

**Proposition 1.50.** Let  $S$  be Hausdorff space and  $\{x_n\}_{n \in \mathbb{N}}$  a sequence in  $S$  which converges to a point  $p \in S$ . Then,  $\{x_n\}_{n \in \mathbb{N}}$  does not converge to any other point in  $S$ .

*Proof.* **Exercise.** □

**Definition 1.51.** Let  $S$  be a topological space and  $U \subseteq S$  a subset. Consider the set  $B_U$  of sequences of elements of  $U$ . Then the set  $\overline{U}^s$  consisting of the points to which some element of  $B_U$  converges is called the *sequential closure* of  $U$ .

**Proposition 1.52.** Let  $S$  be a topological space and  $U \subseteq S$  a subset. Let  $x$  be a sequence of points in  $U$  which has an accumulation point  $p \in S$ . Then,  $p \in \overline{U}$ .

*Proof.* Suppose  $p \notin \overline{U}$ . Since  $\overline{U}$  is closed  $S \setminus \overline{U}$  is an open neighborhood of  $p$ . But  $S \setminus \overline{U}$  does not contain any point of  $x$ , so  $p$  cannot be accumulation point of  $x$ . This is a contradiction. □

**Corollary 1.53.** Let  $S$  be a topological space and  $U$  a subset. Then,  $U \subseteq \overline{U}^s \subseteq \overline{U}$ .

*Proof.* Immediate. □

**Proposition 1.54.** Let  $S$  be a first-countable topological space and  $U$  a subset. Then,  $\overline{U}^s = \overline{U}$ .

*Proof.* **Exercise.** □

**Definition 1.55.** Let  $S$  be a topological space and  $U \subseteq S$  a subset.  $U$  is said to be *limit point compact* iff every sequence in  $U$  has an accumulation point (limit point) in  $U$ .  $U$  is called *sequentially compact* iff every sequence of elements of  $U$  contains a subsequence converging to a point in  $U$ .

**Proposition 1.56.** Let  $S$  be a first-countable topological space and  $x = \{x_n\}_{n \in \mathbb{N}}$  a sequence in  $S$  with accumulation point  $p$ . Then,  $x$  has a subsequence that converges to  $p$ .

*Proof.* By first-countability choose a countable base  $\{U_n\}_{n \in \mathbb{N}}$  of the filter of neighborhoods at  $p$ . Now consider the family  $\{W_n\}_{n \in \mathbb{N}}$  of open neighborhoods  $W_n := \bigcap_{k=1}^n U_k$  at  $p$ . It is easy to see that this is again a countable neighborhood base at  $p$ . Moreover, it has the property that  $W_n \subseteq W_m$  if  $n \geq m$ . Now, Choose  $n_1 \in \mathbb{N}$  such that  $x_{n_1} \in W_1$ . Recursively, choose  $n_{k+1} > n_k$  such that  $x_{n_{k+1}} \in W_{k+1}$ . This is possible since  $W_{k+1}$  contains infinitely many points of  $x$ . Let  $V$  be a neighborhood of  $p$ . There exists some  $k \in \mathbb{N}$  such that  $U_k \subseteq V$ . By construction, then  $W_m \subseteq W_k \subseteq U_k$  for all  $m \geq k$  and hence  $x_{n_m} \in V$  for all  $m \geq k$ . Thus, the subsequence  $\{x_{n_m}\}_{m \in \mathbb{N}}$  converges to  $p$ . □

**Proposition 1.57.** Sequential compactness implies limit point compactness. In a first-countable space the converse is also true.

*Proof.* **Exercise.** □

**Proposition 1.58.** A compact set is limit point compact.

*Proof.* Consider a sequence  $x$  in a compact set  $S$ . Suppose  $x$  does not have an accumulation point. Then, for each point  $p \in S$  we can choose an open neighborhood  $U_p$  which contains only finitely many points of  $x$ . However, by compactness,  $S$  is covered by finitely many of the sets  $U_p$ . But their union can only contain a finite number of points of  $x$ , a contradiction. □

## 1.4 Filters and convergence

**Definition 1.59** (convergence of filters). Let  $S$  be a topological space and  $\mathcal{F}$  a filter on  $S$ .  $\mathcal{F}$  is said to *converge* to  $p \in S$  iff every neighborhood of  $p$  is contained in  $\mathcal{F}$ , i.e.,  $\mathcal{N}_p \subseteq \mathcal{F}$ . Then,  $x$  is said to be a *limit* of  $x$ . Also,  $p \in S$  is called *accumulation point* of  $\mathcal{F}$  iff  $p \in \bigcap_{U \in \mathcal{F}} \overline{U}$ .

**Proposition 1.60.** Let  $S$  be a topological space and  $\mathcal{F}$  a filter on  $S$  converging to  $p \in S$ . Then,  $p$  is accumulation point of  $\mathcal{F}$ .

*Proof.* **Exercise.** □

**Proposition 1.61.** *Set  $S$  be a topological space and  $\mathcal{F}, \mathcal{F}'$  filters on  $S$  such that  $\mathcal{F} \subseteq \mathcal{F}'$ . If  $p \in S$  is accumulation point of  $\mathcal{F}'$ , then it is also accumulation point of  $\mathcal{F}$ . If  $\mathcal{F}$  converges to  $p \in S$ , then so does  $\mathcal{F}'$ .*

*Proof.* Immediate. □

Let  $x = \{x_n\}_{n \in \mathbb{N}}$  be a sequence of points in a topological space  $S$ . We define the filter  $\mathcal{F}_x$  associated with this sequence as follows:  $\mathcal{F}_x$  contains all the subsets  $U$  of  $S$  such that  $U$  contains all  $x_n$ , except possibly finitely many.

**Proposition 1.62.** *Let  $x := \{x_n\}_{n \in \mathbb{N}}$  be a sequence of points in a topological space  $S$ . Then  $x$  converges to a point  $p \in S$  iff the associated filter  $\mathcal{F}_x$  converges to  $p$ . Also,  $p \in S$  is accumulation point of  $x$  iff it is accumulation point of  $\mathcal{F}_x$ .*

*Proof.* Exercise. □

**Proposition 1.63.** *Let  $S$  be a topological space and  $U \subseteq S$  a subset. Consider the set  $A_U$  of filters containing  $U$ . Then, the closure  $\overline{U}$  of  $U$  coincides with the set of points to which some element in  $A_U$  converges.*

*Proof.* If  $U = \emptyset$ , then  $A_U$  is empty and the proof is trivial. Assume the contrary. If  $x \in \overline{U}$ , then the intersection of  $U$  with any neighbourhood of  $x$  is non-empty and thus generates a filter that contains  $U$  as well as all neighborhoods of  $x$  and thus converges to  $x$ . If  $x \notin \overline{U}$ , then there exists a neighborhood  $V$  of  $x$  such that  $U \cap V = \emptyset$ . So no filter containing  $U$  can contain  $V$ . □

**Proposition 1.64.** *Let  $S, T$  be topological spaces and  $f : S \rightarrow T$ . If  $f$  is continuous, then for any  $p \in S$  and filter  $\mathcal{F}$  converging to  $p$ , the filter generated by  $f(\mathcal{F})$  in  $T$  converges to  $f(p)$ . Conversely, if for any  $p \in S$  and filter  $\mathcal{F}$  converging to  $p$ , the filter generated by  $f(\mathcal{F})$  in  $T$  converges to  $f(p)$ , then  $f$  is continuous.*

*Proof.* Exercise. □

**Proposition 1.65.** *Let  $S$  be a Hausdorff topological space,  $\mathcal{F}$  a filter on  $S$  converging to a point  $p \in S$ . Then  $\mathcal{F}$  does not converge to any other point in  $S$ .*

*Proof.* Exercise. □

**Proposition 1.66.** *Let  $S$  be a topological space and  $K \subseteq S$  a subset. Then,  $K$  is compact iff every filter containing  $K$  has at least one accumulation point in  $K$ .*

*Proof.* Let  $K \subseteq S$  be compact. We suppose that there is a filter  $\mathcal{F}$  containing  $K$  that has no accumulation point in  $K$ . For each  $U \in \mathcal{F}$  consider the open set  $O_U := S \setminus \overline{U}$ . By assumption, these open sets cover  $K$ . Since  $K$  is compact, there must be a finite subset  $\{U_1, \dots, U_n\}$  of elements of  $\mathcal{F}$  such that  $\{O_{U_1}, \dots, O_{U_n}\}$  covers  $K$ . But this implies

$K \cap \bigcap_{i=1}^n \overline{U_i} = \emptyset$  and thus, in particular, also  $K \cap \bigcap_{i=1}^n U_i = \emptyset$ , contradicting the fact that  $\mathcal{F}$  is a filter. Thus, any filter containing  $K$  must have an accumulation point in  $K$ .

Now suppose that  $K \subseteq S$  is not compact. Then, there exists a cover of  $K$  by open sets  $\{U_\alpha\}_{\alpha \in A}$  which does not admit any finite subcover. Now consider finite intersections of the sets  $C_\alpha := K \setminus U_\alpha$ . These are non-empty and form the base of a filter containing  $K$ . But this filter clearly has no accumulation point in  $K$ . Thus, if every filter containing  $K$  is to possess an accumulation point,  $K$  must be compact.  $\square$

## 1.5 Metric and pseudometric spaces

**Definition 1.67.** Let  $S$  be a set and  $d : S \times S \rightarrow \mathbb{R}_0^+$  a map with the following properties:

- $d(x, y) = d(y, x) \quad \forall x, y \in S$ . (symmetry)
- $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in S$ . (triangle inequality)
- $d(x, x) = 0 \quad \forall x \in S$ .

Then  $d$  is called a *pseudometric* on  $S$ .  $S$  is also called a *pseudometric space*. Suppose  $d$  also satisfies

- $d(x, y) = 0 \implies x = y \quad \forall x, y \in S$ . (definiteness)

Then  $d$  is called a *metric* on  $S$  and  $S$  is called a *metric space*.

**Definition 1.68.** Let  $S$  be a pseudometric space,  $x \in S$  and  $r > 0$ . Then the set  $B_r(x) := \{y \in S : d(x, y) < r\}$  is called the *open ball* of radius  $r$  centered around  $x$  in  $S$ . The set  $\overline{B}_r(x) := \{y \in S : d(x, y) \leq r\}$  is called the *closed ball* of radius  $r$  centered around  $x$  in  $S$ .

**Proposition 1.69.** Let  $S$  be a pseudometric space. Then, the open balls in  $S$  together with the empty set form the basis of a topology on  $S$ . This topology is first-countable and such that closed balls are closed. Moreover, the topology is Hausdorff iff  $S$  is metric.

*Proof.* **Exercise.**  $\square$

**Definition 1.70.** A topological space is called *(pseudo)metrizable* iff there exists a (pseudo)metric such that the open balls given by the (pseudo)metric are a basis of its topology.

**Proposition 1.71.** In a pseudometric space any open ball can be obtained as the countable union of closed balls. Similarly, any closed ball can be obtained as the countable intersection of open balls.

*Proof.* **Exercise.**  $\square$

**Proposition 1.72.** *Let  $S$  be a set equipped with two pseudometrics  $d^1$  and  $d^2$ . Then, the topology generated by  $d^2$  is finer than the topology generated by  $d^1$  iff for all  $x \in S$  and  $r_1 > 0$  there exists  $r_2 > 0$  such that  $B_{r_2}^2(x) \subseteq B_{r_1}^1(x)$ . In particular,  $d^1$  and  $d^2$  generate the same topology iff the condition holds both ways.*

*Proof.* **Exercise.** □

**Proposition 1.73** (epsilon-delta criterion). *Let  $S, T$  be pseudometric spaces and  $f : S \rightarrow T$  a map. Then,  $f$  is continuous at  $x \in S$  iff for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$ .*

*Proof.* **Exercise.** □

## 1.6 Elementary properties of pseudometric spaces

**Proposition 1.74.** *Every metric space is normal.*

*Proof.* Let  $A, B$  be disjoint closed sets in the metric space  $S$ . For each  $x \in A$  choose  $\epsilon_x > 0$  such that  $B_{\epsilon_x}(x) \cap B = \emptyset$  and for each  $y \in B$  choose  $\epsilon_y > 0$  such that  $B_{\epsilon_y}(y) \cap A = \emptyset$ . Then, for any pair  $(x, y)$  with  $x \in A$  and  $y \in B$  we have  $B_{\epsilon_x/2}(x) \cap B_{\epsilon_y/2}(y) = \emptyset$ . Consider the open sets  $U := \bigcup_{x \in A} B_{\epsilon_x/2}(x)$  and  $V := \bigcup_{y \in B} B_{\epsilon_y/2}(y)$ . Then,  $U \cap V = \emptyset$ , but  $A \subseteq U$  and  $B \subseteq V$ . So  $S$  is normal. □

**Proposition 1.75.** *Let  $S$  be a pseudometric space and  $x := \{x_n\}_{n \in \mathbb{N}}$  a sequence in  $S$ . Then  $x$  converges to  $p \in S$  iff for any  $\epsilon > 0$  there exists an  $n_0 \in \mathbb{N}$  such that  $d(x_n, p) < \epsilon$  for all  $n \geq n_0$ .*

*Proof.* Immediate. □

**Definition 1.76.** Let  $S$  be a pseudometric space and  $x := \{x_n\}_{n \in \mathbb{N}}$  a sequence in  $S$ . Then  $x$  is called a *Cauchy sequence* iff for all  $\epsilon > 0$  there exists an  $n_0 \in \mathbb{N}$  such that  $d(x_n, x_m) < \epsilon$  for all  $n, m \geq n_0$ .

**Exercise 3.** Give an example of a set  $S$ , a sequence  $x$  in  $S$  and two metrics  $d^1$  and  $d^2$  on  $S$  that generate the same topology, but such that  $x$  is Cauchy with respect to  $d^1$ , but not with respect to  $d^2$ .

**Proposition 1.77.** *Any converging sequence in a pseudometric space is a Cauchy sequence.*

*Proof.* **Exercise.** □

**Proposition 1.78.** *Suppose  $x$  is a Cauchy sequence in a pseudometric space. If  $p$  is accumulation point of  $x$  then  $x$  converges to  $p$ .*

*Proof.* **Exercise.** □

**Definition 1.79.** Let  $S$  be a pseudometric space and  $U \subseteq S$  a subset. If every Cauchy sequence in  $U$  converges to a point in  $U$ , then  $U$  is called *complete*.

**Proposition 1.80.** A complete subset of a metric space is closed. A closed subset of a complete pseudometric space is complete.

*Proof.* **Exercise.** □

**Exercise 4.** Give an example of a complete subset of a pseudometric space that is not closed.

**Definition 1.81** (Totally boundedness). Let  $S$  be a pseudometric space. A subset  $U \subseteq S$  is called *totally bounded* iff for any  $r > 0$  the set  $U$  admits a cover by finitely many open balls of radius  $r$ .

**Proposition 1.82.** A subset of a pseudometric space is compact iff it is complete and totally bounded.

*Proof.* We first show that compactness implies totally boundedness and completeness. Let  $U$  be a compact subset. Then, for  $r > 0$  cover  $U$  by open balls of radius  $r$  centered at every point of  $U$ . Since  $U$  is compact, finitely many balls will cover it. Hence,  $U$  is totally bounded. Now, consider a Cauchy sequence  $x$  in  $U$ . Since  $U$  is compact  $x$  must have an accumulation point  $p \in U$  (Proposition 1.58) and hence (Proposition 1.78) converge to  $p$ . Thus,  $U$  is complete.

We proceed to show that completeness together with totally boundedness imply compactness. Let  $U$  be a complete and totally bounded subset. Assume  $U$  is not compact and choose a covering  $\{U_\alpha\}_{\alpha \in A}$  of  $U$  that does not admit a finite subcover. On the other hand,  $U$  is totally bounded and admits a covering by finitely many open balls of radius  $1/2$ . Hence, there must be at least one such ball  $B_1$  such that  $C_1 := B_1 \cap U$  is not covered by finitely many  $U_\alpha$ . Choose a point  $x_1$  in  $C_1$ . Observe that  $C_1$  itself is totally bounded. Inductively, cover  $C_n$  by finitely many open balls of radius  $2^{-(n+1)}$ . For at least one of those, call it  $B_{n+1}$ ,  $C_{n+1} := B_{n+1} \cap C_n$  is not covered by finitely many  $U_\alpha$ . Choose a point  $x_{n+1}$  in  $C_{n+1}$ . This process yields a Cauchy sequence  $x := \{x_k\}_{k \in \mathbb{N}}$ . Since  $U$  is complete, the sequence converges to a point  $p \in U$ . There must be  $\alpha \in A$  such that  $p \in U_\alpha$ . Since  $U_\alpha$  is open, there exists  $r > 0$  such that  $B_r(p) \subseteq U_\alpha$ . This implies,  $C_n \subseteq U_\alpha$  for all  $n \in \mathbb{N}$  such that  $2^{-n+1} < r$ . However, this is a contradiction to the  $C_n$  not being finitely covered. Hence,  $U$  must be compact. □

**Proposition 1.83.** The notions of compactness, limit point compactness and sequential compactness are equivalent in a pseudometric space.

*Proof.* **Exercise.** □

**Proposition 1.84.** A totally bounded pseudometric space is second-countable.

*Proof.* **Exercise.** □

**Proposition 1.85.** *The notions of separability and second-countability are equivalent in a pseudometric space.*

*Proof.* **Exercise.** □

**Theorem 1.86** (Baire's Theorem). *Let  $S$  be a complete metric space and  $\{U_n\}_{n \in \mathbb{N}}$  a sequence of open and dense subsets of  $S$ . Then, the intersection  $\bigcap_{n \in \mathbb{N}} U_n$  is dense in  $S$ .*

*Proof.* Set  $U := \bigcap_{n \in \mathbb{N}} U_n$ . Let  $V$  be an arbitrary open set in  $S$ . It suffices to show that  $V \cap U \neq \emptyset$ . To this end we construct a sequence  $\{x_n\}_{n \in \mathbb{N}}$  of elements of  $S$  and a sequence  $\{\epsilon_n\}_{n \in \mathbb{N}}$  of positive numbers. Choose  $x_1 \in U_1 \cap V$  and then  $0 < \epsilon_1 \leq 1$  such that  $\overline{B_{\epsilon_1}(x_1)} \subseteq U_1 \cap V$ . Now, consecutively choose  $x_{n+1} \in U_{n+1} \cap B_{\epsilon_n/2}(x_n)$  and  $0 < \epsilon_{n+1} < 2^{-n}$  such that  $\overline{B_{\epsilon_{n+1}}(x_{n+1})} \subseteq U_{n+1} \cap B_{\epsilon_n}(x_n)$ . The sequence  $\{x_n\}_{n \in \mathbb{N}}$  is Cauchy since by construction  $d(x_n, x_{n+1}) < 2^{-n}$  for all  $n \in \mathbb{N}$ . So by completeness it converges to some point  $x \in S$ . Indeed,  $x \in \overline{B_{\epsilon_1}(x_1)} \subseteq V$ . On the other hand,  $x \in \overline{B_{\epsilon_n}(x_n)} \subseteq U_n$  for all  $n \in \mathbb{N}$  and hence  $x \in U$ . This completes the proof. □

**Proposition 1.87.** *Let  $S$  be equipped with a pseudometric  $d$ . Then  $p \sim q \iff d(p, q) = 0$  for  $p, q \in S$  defines an equivalence relation on  $S$ . The prescription  $\tilde{d}([p], [q]) := d(p, q)$  for  $p, q \in S$  is well-defined and yields a metric  $\tilde{d}$  on the quotient space  $S/\sim$ . The topology induced by this metric on  $S/\sim$  is the quotient topology with respect to that induced by  $d$  on  $S$ . Moreover,  $S/\sim$  is complete iff  $S$  is complete.*

*Proof.* **Exercise.** □

## 1.7 Completion of metric spaces

Often it is desirable to work with a complete metric space when one is only given a non-complete metric space. To this end one can construct the *completion* of a metric space. This is detailed in the following exercise.

**Exercise 5.** Let  $S$  be a metric space.

- Let  $x := \{x_n\}_{n \in \mathbb{N}}$  and  $y := \{y_n\}_{n \in \mathbb{N}}$  be Cauchy sequences in  $S$ . Show that the limit  $\lim_{n \rightarrow \infty} d(x_n, y_n)$  exists.
- Let  $T$  be the set of Cauchy sequences in  $S$ . Define the function  $\tilde{d} : T \times T \rightarrow \mathbb{R}_0^+$  by  $\tilde{d}(x, y) := \lim_{n \rightarrow \infty} d(x_n, y_n)$ . Show that  $\tilde{d}$  defines a pseudometric on  $T$ .
- Show that  $T$  is complete.
- Define  $\overline{S}$  as the metric quotient  $T/\sim$  as in Proposition 1.87. Then,  $\overline{S}$  is complete.

- Show that there is a natural isometric embedding (i.e., a map that preserves the metric)  $i_S : S \rightarrow \bar{S}$ . Furthermore, show that this is a bijection iff  $S$  is complete.

**Definition 1.88.** The metric space  $\bar{S}$  constructed above is called the *completion* of the metric space  $S$ .

**Proposition 1.89** (Universal property of completion). *Let  $S$  be a metric space,  $T$  a complete metric space and  $f : S \rightarrow T$  an isometric map. Then, there is a unique isometric map  $\bar{f} : \bar{S} \rightarrow T$  such that  $f = \bar{f} \circ i_S$ . Furthermore, the closure of  $f(S)$  in  $T$  is equal to  $\bar{f}(\bar{S})$ .*

*Proof.* **Exercise.**

□



## 2 Vector spaces with additional structure

In the following  $\mathbb{K}$  denotes a field which might be either  $\mathbb{R}$  or  $\mathbb{C}$ .

**Definition 2.1.** Let  $V$  be a vector space over  $\mathbb{K}$ . A subset  $A$  of  $V$  is called *balanced* iff for all  $v \in A$  and all  $\lambda \in \mathbb{K}$  with  $|\lambda| \leq 1$  the vector  $\lambda v$  is contained in  $A$ . A subset  $A$  of  $V$  is called *convex* iff for all  $x, y \in A$  and  $t \in [0, 1]$  the vector  $(1 - t)x + ty$  is in  $A$ . Let  $A$  be a subset of  $V$ . Consider the smallest subset of  $V$  which is convex and which contains  $A$ . This is called the *convex hull* of  $A$ , denoted  $\text{conv}(A)$ .

**Proposition 2.2.** (a) Intersections of balanced sets are balanced. (b) The sum of two balanced sets is balanced. (c) A scalar multiple of a balanced set is balanced.

*Proof.* Exercise. □

**Proposition 2.3.** Let  $V$  be vector space and  $A$  a subset. Then

$$\text{conv}(A) = \left\{ \sum_{i=1}^n \lambda_i x_i : n \in \mathbb{N}, \lambda_i \in [0, 1], x_i \in A, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

*Proof.* Exercise. □

We denote the space of linear maps between a vector space  $V$  and a vector space  $W$  by  $L(V, W)$ .

### 2.1 Topological vector spaces

**Definition 2.4.** A set  $V$  that is equipped both with a vector space structure over  $\mathbb{K}$  and a topology is called a *topological vector space (tvs)* iff the vector addition  $+: V \times V \rightarrow V$  and the scalar multiplication  $\cdot: \mathbb{K} \times V \rightarrow V$  are both continuous. (Here the topology on  $\mathbb{K}$  is the standard one.)

**Proposition 2.5.** Let  $V$  be a tvs,  $\lambda \in \mathbb{K} \setminus 0$ ,  $w \in V$ . The maps  $V \rightarrow V : v \mapsto \lambda v$  and  $V \rightarrow V : v \mapsto v + w$  are automorphisms of  $V$  as a tvs. In particular, the topology  $\mathcal{T}$  of  $V$  is invariant under rescalings and translations:  $\lambda\mathcal{T} = \mathcal{T}$  and  $\mathcal{T} + w = \mathcal{T}$ . In terms of filters of neighborhoods,  $\lambda\mathcal{N}_v = \mathcal{N}_{\lambda v}$  and  $\mathcal{N}_v + w = \mathcal{N}_{v+w}$  for all  $v \in V$ .

*Proof.* It is clear that non-zero scalar multiplication and translation are vector space automorphisms. To see that they are also continuous use Proposition 1.18. The inverse maps are of the same type hence also continuous. Thus, we have homeomorphisms. The scale- and translation invariance of the topology follows. □

Note that this implies that the topology of a tvs is completely determined by the filter of neighborhoods of one of its points, say 0.

**Definition 2.6.** Let  $V$  be a tvs and  $U$  a subset.  $U$  is called *bounded* iff for every neighborhood  $W$  of  $0$  there exists  $\lambda \in \mathbb{R}^+$  such that  $U \subseteq \lambda W$ .

Remark: Changing the allowed range of  $\lambda$  in the definition of boundedness from  $\mathbb{R}^+$  to  $\mathbb{K}$  leads to an equivalent definition, i.e., is not weaker. However, the choice of  $\mathbb{R}^+$  over  $\mathbb{K}$  is more convenient in certain applications.

**Proposition 2.7.** Let  $V$  be a tvs. Then:

1. Every point set is bounded.
2. Every neighborhood of  $0$  contains a balanced subneighborhood of  $0$ .
3. Let  $U$  be a neighborhood of  $0$ . Then there exists a subneighborhood  $W$  of  $0$  such that  $W + W \subseteq U$ .

*Proof.* We start by demonstrating Property 1. Let  $x \in V$  and  $U$  some open neighborhood of  $0$ . Then  $Z := \{(\lambda, y) \in \mathbb{K} \times V : \lambda y \in U\}$  is open by continuity of multiplication. Also  $(0, x) \in Z$  so that by the product topology there exists an  $\epsilon > 0$  and an open neighborhood  $W$  of  $x$  in  $V$  such that  $B_\epsilon(0) \times W \subseteq Z$ . In particular, there exists  $\mu > 0$  such that  $\mu x \in U$ , i.e.,  $\{x\} \subseteq \mu^{-1}U$  as desired.

We proceed to Property 2. Let  $U$  be an open neighborhood of  $0$ . By continuity  $Z := \{(\lambda, x) \in \mathbb{K} \times V : \lambda x \in U\}$  is open. By the product topology, there are open neighborhoods  $X$  of  $0 \in \mathbb{K}$  and  $W$  of  $0 \in V$  such that  $X \times W \subseteq Z$ . Thus,  $X \cdot W \subseteq U$ . Now  $X$  contains an open ball of some radius  $\epsilon > 0$  around  $0$  in  $\mathbb{K}$ . Set  $Y := B_\epsilon(0) \cdot W$ . This is an (open) neighborhood of  $0$  in  $V$ , it is contained in  $U$  and it is balanced.

We end with Property 3. Let  $U$  be an open neighborhood of  $0$ . By continuity  $Z := \{(x, y) \in V \times V : x + y \in U\}$  is open. By the product topology, there are open neighborhoods  $W_1$  and  $W_2$  of  $0$  such that  $W_1 \times W_2 \subseteq Z$ . This means  $W_1 + W_2 \subseteq U$ . Now define  $W := W_1 \cap W_2$ .  $\square$

**Proposition 2.8.** Let  $V$  be a vector space and  $\mathcal{F}$  a filter on  $V$ . Then  $\mathcal{F}$  is the filter of neighborhoods of  $0$  for a compatible topology on  $V$  iff  $0$  is contained in every element of  $\mathcal{F}$  and  $\lambda\mathcal{F} = \mathcal{F}$  for all  $\lambda \in \mathbb{K} \setminus \{0\}$  and  $\mathcal{F}$  satisfies the properties of Proposition 2.7.

*Proof.* It is already clear that the properties in question are necessary for  $\mathcal{F}$  to be the filter of neighborhoods of  $0$  of  $V$ . It remains to show that they are sufficient. If  $\mathcal{F}$  is to be the filter of neighborhoods of  $0$  then, by translation invariance,  $\mathcal{F}_x := \mathcal{F} + x$  must be the filter of neighborhoods of the point  $x$ . We show that the family of filters  $\{\mathcal{F}_x\}_{x \in V}$  does indeed define a topology on  $V$ . To this end we will use Proposition 1.10. Property 1 is satisfied by assumption. It remains to show Property 2. By translation invariance it will be enough to consider  $x = 0$ . Suppose  $U \in \mathcal{F}$ . Using Property 3 of Proposition 2.7 there is  $W \in \mathcal{F}$  such that  $W + W \subseteq U$ . We claim that Property 2 of Proposition 1.10 is now satisfied with this choice of  $W$ . Indeed, let  $y \in W$  then  $y + W \in \mathcal{F}_y$  and  $y + W \subseteq U$  so  $U \in \mathcal{F}_y$  as required.

We proceed to show that the topology defined in this way is compatible with the vector space structure. Take an open set  $U \subseteq V$  and consider its preimage  $Z = \{(x, y) \in V \times V : x + y \in U\}$  under vector addition. Take some point  $(x, y) \in Z$ .  $U - x - y$  is an open neighborhood of 0. By Property 3 of Proposition 2.7 there is an open neighborhood  $W$  of 0 such that  $W + W \subseteq U - x - y$ , i.e.,  $(x + W) + (y + W) \subseteq U$ . But  $x + W$  is an open neighborhood of  $x$  and  $y + W$  is an open neighborhood of  $y$  so  $(x + W) \times (y + W)$  is an open neighborhood of  $(x, y)$  in  $V \times V$  contained in  $Z$ . Hence vector addition is continuous.

We proceed to show continuity of scalar multiplication. Consider an open set  $U \subseteq V$  and consider its preimage  $Z = \{(\lambda, x) \in \mathbb{K} \times V : \lambda x \in U\}$  under scalar multiplication. Take some point  $(\lambda, x) \in Z$ .  $U - \lambda x$  is an open neighborhood of 0 in  $V$ . By Property 3 of Proposition 2.7 there is an open neighborhood  $W$  of 0 such that  $W + W \subseteq U - \lambda x$ . By Property 2 of Proposition 2.7 there exists a balanced subneighborhood  $X$  of  $W$ . By Property 1 of Proposition 2.7 (boundedness of points) there exists  $\epsilon > 0$  such that  $\epsilon x \in X$ . Now define  $Y := (\epsilon + |\lambda|)^{-1}X$ . Note that scalar multiples of (open) neighborhoods of 0 are (open) neighborhoods of 0 by assumption. Hence  $Y$  is open since  $X$  is. Thus  $B_\epsilon(\lambda) \times (x + Y)$  is an open neighborhood of  $(\lambda, x)$  in  $\mathbb{K} \times V$ . We claim that it is contained in  $Z$ . First observe that since  $X$  is balanced,  $B_\epsilon(0) \cdot x \subseteq X$ . Similarly, we have  $B_\epsilon(\lambda) \cdot Y \subseteq B_{\epsilon+|\lambda|}(0) \cdot Y = B_1(0) \cdot X \subseteq X$ . Thus we have  $B_\epsilon(0) \cdot x + B_\epsilon(\lambda) \cdot Y \subseteq X + X \subseteq W + W \subseteq U - \lambda x$ . But this implies  $B_\epsilon(\lambda) \cdot (x + Y) \subseteq U$  as required.  $\square$

**Proposition 2.9.** (a) *The interior of a balanced set is balanced.* (b) *The closure of a balanced set is balanced.*

*Proof.* Let  $U$  be balanced and let  $\lambda \in \mathbb{K}$  with  $0 < |\lambda| \leq 1$ . It is then enough to observe that for (a)  $\lambda \overset{\circ}{U} = \lambda \overset{\circ}{U} \subseteq \overset{\circ}{U}$  and for (b)  $\lambda \overline{U} = \overline{\lambda U} \subseteq \overline{U}$ .  $\square$

**Proposition 2.10.** *In a tvs every neighborhood of 0 contains a closed and balanced subneighborhood.*

*Proof.* Let  $U$  be a neighborhood of 0. By Proposition 2.7.3 there exists a subneighborhood  $W \subseteq U$  such that  $W + W \subseteq U$ . By Proposition 2.7.2 there exists a balanced subneighborhood  $X \subseteq W$ . Let  $Y := \overline{X}$ . Then,  $Y$  is obviously a closed neighborhood of 0. Also  $Y$  is balanced by Proposition 2.9. Finally, let  $y \in Y = \overline{X}$ . Any neighborhood of  $y$  must intersect  $X$ . In particular,  $y + X$  is such a neighborhood. Thus, there exist  $x \in X, z \in X$  such that  $x = y + z$ , i.e.,  $y = x - z \in X - X = X + X \subseteq U$ . So,  $Y \subseteq U$ .  $\square$

**Proposition 2.11.** (a) *Subsets of bounded sets are bounded.* (b) *Finite unions of bounded sets are bounded.* (c) *The closure of a bounded set is bounded.* (d) *The sum of two bounded sets is bounded.* (e) *A scalar multiple of a bounded set is bounded.*

*Proof.* **Exercise.**  $\square$

**Definition 2.12.** Let  $V$  be a tvs and  $C \subseteq V$  a subset. Then,  $C$  is called *totally bounded* iff for each neighborhood  $U$  of 0 in  $V$  there exists a finite subset  $F \subseteq C$  such that  $C \subseteq F + U$ .

**Proposition 2.13.** (a) Subsets of totally bounded sets are totally bounded. (b) Finite unions of totally bounded sets are totally bounded. (c) The closure of a totally bounded set is totally bounded. (d) The sum of two totally bounded sets is totally bounded. (e) A scalar multiple of a totally bounded set is totally bounded.

*Proof.* **Exercise.** □

**Proposition 2.14.** Compact sets are totally bounded. Totally bounded sets are bounded.

*Proof.* **Exercise.** □

Let  $A, B$  be topological vector spaces. We denote the space of maps from  $A$  to  $B$  that are linear and continuous by  $\text{CL}(A, B)$ .

**Definition 2.15.** Let  $A, B$  be tvs. A linear map  $f : A \rightarrow B$  is called *bounded* iff there exists a neighborhood  $U$  of 0 in  $A$  such that  $f(U)$  is bounded. A linear map  $f : A \rightarrow B$  is called *compact* iff there exists a neighborhood  $U$  of 0 in  $A$  such that  $\overline{f(U)}$  is compact.

Let  $A, B$  be tvs. We denote the space of maps from  $A$  to  $B$  that are linear and bounded by  $\text{BL}(A, B)$ . We denote the space of maps from  $A$  to  $B$  that are linear and compact by  $\text{KL}(A, B)$ .

**Proposition 2.16.** Let  $A, B$  be tvs and  $f \in \text{L}(A, B)$ . (a)  $f$  is continuous iff the preimage of any neighborhood of 0 in  $B$  is a neighborhood of 0 in  $A$ . (b) If  $f$  is continuous it maps bounded sets to bounded sets. (c) If  $f$  is bounded then  $f$  is continuous, i.e.,  $\text{BL}(A, B) \subseteq \text{CL}(A, B)$ . (d) If  $f$  is compact then  $f$  is bounded.

*Proof.* **Exercise.** □

A useful property for a topological space is the Hausdorff property, i.e., the possibility to separate points by open sets. It is not the case that a tvs is automatically Hausdorff. However, the way in which a tvs may be non-Hausdorff is severely restricted. Indeed, we shall see in the following that a tvs may be split into a part that is Hausdorff and another one that is maximally non-Hausdorff in the sense of carrying the trivial topology.

**Proposition 2.17.** Let  $V$  be a tvs and  $C \subseteq V$  a vector subspace. Then, the closure  $\overline{C}$  of  $C$  is also a vector subspace of  $V$ .

*Proof.* **Exercise.** [Hint: Use Proposition 1.63.] □

**Proposition 2.18.** Let  $V$  be a tvs. The closure of  $\{0\}$  in  $V$  coincides with the intersection of all neighborhoods of 0. Moreover,  $V$  is Hausdorff iff  $\overline{\{0\}} = \{0\}$ .

*Proof.* **Exercise.** □

**Proposition 2.19.** Let  $V$  be a tvs and  $C \subseteq V$  a vector subspace.

1. The quotient space  $V/C$  is a tvs.
2.  $V/C$  is Hausdorff iff  $C$  is closed in  $V$ .
3. The quotient map  $q : V \rightarrow V/C$  is linear, continuous and open. Moreover, the quotient topology on  $V/C$  is the only topology such that  $q$  is continuous and open.
4. The image of a base of the filter of neighborhoods of 0 in  $V$  is a base of the filter of neighborhoods of 0 in  $V/C$ .

*Proof.* **Exercise.** □

Thus, for a tvs  $V$  the exact sequence

$$0 \rightarrow \overline{\{0\}} \rightarrow V \rightarrow V/\overline{\{0\}} \rightarrow 0$$

describes how  $V$  is composed of a Hausdorff piece  $V/\overline{\{0\}}$  and a piece  $\overline{\{0\}}$  with trivial topology. We can express this decomposition also in terms of a direct sum, as we shall see in the following.

A (vector) subspace of a tvs is a tvs with the subset topology. Let  $A$  and  $B$  be tvs. Then the direct sum  $A \oplus B$  is a tvs with the product topology. Note that as subsets of  $A \oplus B$ , both  $A$  and  $B$  are closed.

**Definition 2.20.** Let  $V$  be a tvs and  $A$  a subspace. Then another subspace  $B$  of  $A$  in  $V$  is called a *topological complement* iff  $V = A \oplus B$  as tvs (i.e., as vector spaces and as topological spaces).  $A$  is called *topologically complemented* if such a topological complement  $B$  exists.

Note that algebraic complements (i.e., complements merely with respect to the vector space structure) always exist (using the Axiom of Choice). However, an algebraic complement is not necessarily a topological one. Indeed, there are examples of subspaces of tvs that have no topological complement.

**Proposition 2.21** (Structure Theorem for tvs). *Let  $V$  be a tvs and  $B$  an algebraic complement of  $\overline{\{0\}}$  in  $V$ . Then  $B$  is also a topological complement of  $\overline{\{0\}}$  in  $V$ . Moreover,  $B$  is canonically isomorphic to  $V/\overline{\{0\}}$  as a tvs.*

*Proof.* **Exercise.** □

We conclude that every tvs is a direct sum of a Hausdorff tvs and a tvs with the trivial topology.

## 2.2 Metrizable and pseudometrizable vector spaces

In this section we consider *(pseudo)metrizable vector spaces* (mvs), i.e., tvs that admit a (pseudo)metric compatible with the topology.

**Definition 2.22.** A pseudometric on a vector space  $V$  is called *translation-invariant* iff  $d(x + a, y + a) = d(x, y)$  for all  $x, y, a \in V$ . A translation-invariant pseudometric on a vector space  $V$  is called *balanced* iff its open balls around the origin are balanced.

As we shall see it will be possible to limit ourselves to balanced translation-invariant pseudometrics on mvs. Moreover, these can be conveniently described by pseudo-seminorms.

**Definition 2.23.** Let  $V$  be a vector space over  $\mathbb{K}$ . Then a map  $V \rightarrow \mathbb{R}_0^+ : x \mapsto \|x\|$  is called a *pseudo-seminorm* iff it satisfies the following properties:

1.  $\|0\| = 0$ .
2. For all  $\lambda \in \mathbb{K}$ ,  $|\lambda| \leq 1$  implies  $\|\lambda x\| \leq \|x\|$  for all  $x \in V$ .
3. For all  $x, y \in V$  :  $\|x + y\| \leq \|x\| + \|y\|$ .

$\|\cdot\|$  is called a *pseudo-norm* iff it satisfies in addition the following property.

4.  $\|x\| = 0$  implies  $x = 0$ .

**Proposition 2.24.** *There is a one-to-one correspondence between pseudo-seminorms and balanced translation invariant pseudometrics on a vector space via  $d(x, y) := \|x - y\|$ . This specializes to a correspondence between pseudo-norms and balanced translation invariant metrics.*

*Proof.* **Exercise.** □

**Proposition 2.25.** *Let  $V$  be a vector space. The topology generated by a pseudo-seminorm on  $V$  is compatible with the vector space structure iff for every  $x \in V$  and  $\epsilon > 0$  there exists  $\lambda \in \mathbb{R}^+$  such that  $x \in \lambda B_\epsilon(0)$ .*

*Proof.* Assume we are given a pseudo-seminorm on  $V$  that induces a compatible topology. It is easy to see that the stated property of the pseudo-seminorm then follows from Property 1 in Proposition 2.7 (boundedness of points).

Conversely, suppose we are given a pseudo-seminorm on  $V$  with the stated property. We show that the filter  $\mathcal{N}_0$  of neighborhoods of 0 defined by the pseudo-seminorm has the properties required by Proposition 2.8 and hence defines a compatible topology on  $V$ . Firstly, it is already clear that every  $U \in \mathcal{N}_0$  contains 0. We proceed to show that  $\mathcal{N}_0$  is scale invariant. It is enough to show that for  $\epsilon > 0$  and  $\lambda \in \mathbb{K} \setminus \{0\}$  the scaled ball  $\lambda B_\epsilon(0)$  is open. Choose a point  $\lambda x \in \lambda B_\epsilon(0)$ . Take  $\delta > 0$  such that  $\|x\| < \epsilon - \delta$ . Then  $B_\delta(0) + x \subseteq B_\epsilon(0)$ . Choose  $n \in \mathbb{N}$  such that  $2^{-n} \leq |\lambda|$ . Observe that the triangle

inequality implies  $B_{2^{-n}\delta}(0) \subseteq 2^{-n}B_\delta(0)$  (for arbitrary  $\delta$  and  $n$  in fact). Hence  $B_{2^{-n}\delta}(\lambda x) = B_{2^{-n}\delta}(0) + \lambda x \subseteq \lambda B_\delta(0) + \lambda x \subseteq \lambda B_\epsilon(0)$  showing that  $\lambda B_\epsilon(0)$  is open.

It now remains to show the properties of  $\mathcal{N}_0$  listed in Proposition 2.7. As for Property 3, we may take  $U$  to be an open ball of radius  $\epsilon$  around 0 for some  $\epsilon > 0$ . Define  $W := B_{\epsilon/2}(0)$ . Then  $W + W \subseteq U$  follows from the triangle inequality. Concerning Property 2 we simply notice that open balls are balanced by construction. The only property that is not automatic for a pseudo-seminorm and does require the stated condition is Property 1 (boundedness of points). The equivalence of the two is easy to see.  $\square$

**Theorem 2.26.** *A tvs  $V$  is pseudometrizable iff it is first-countable, i.e., iff there exists a countable base for the filter of neighborhoods of 0. Moreover, if  $V$  is pseudometrizable it admits a compatible pseudo-seminorm.*

*Proof.* It is clear that pseudometrizability implies the existence of a countable base of  $\mathcal{N}_0$ . For example, the sequence of balls  $\{B_{1/n}(0)\}_{n \in \mathbb{N}}$  provides such a base. Conversely, suppose that  $\{U_n\}_{n \in \mathbb{N}}$  is a base of the filter of neighborhoods of 0 such that all  $U_n$  are balanced and  $U_{n+1} + U_{n+1} \subseteq U_n$ . (Given an arbitrary countable base of  $\mathcal{N}_0$  we can always produce another one with the desired properties.) Now for each finite subset  $H$  of  $\mathbb{N}$  define  $U_H := \sum_{n \in H} U_n$  and  $\lambda_H := \sum_{n \in H} 2^{-n}$ . Note that each  $U_H$  is a balanced neighborhood of 0. Define now the function  $V \rightarrow \mathbb{R}_0^+ : x \mapsto \|x\|$  by

$$\|x\| := \inf_H \{\lambda_H | x \in U_H\}$$

if  $x \in U_H$  for some  $H$  and  $\|x\| = 1$  otherwise. We proceed to show that  $\|\cdot\|$  defines a pseudo-seminorm and generates the topology of  $V$ .

Fix  $x \in V$  and  $\lambda \in \mathbb{K}$  with  $|\lambda| \leq 1$ . Since  $U_H$  is balanced for each  $H$ ,  $\lambda x$  is contained at least in the same sets  $U_H$  as  $x$ . Because the definition of  $\|\cdot\|$  uses an infimum,  $\|\lambda x\| \leq \|x\|$ . This confirms Property 1 of Definition 2.23.

To show the triangle inequality (Property 3 of Definition 2.23) we first note that for finite subsets  $H, K$  of  $\mathbb{N}$  with the property  $\lambda_H + \lambda_K < 1$  there is another unique finite subset  $L$  of  $\mathbb{N}$  such that  $\lambda_L = \lambda_H + \lambda_K$ . Furthermore,  $U_H + U_K \subseteq U_L$  in this situation. Now, fix  $x, y \in V$ . If  $\|x\| + \|y\| \geq 1$  the triangle inequality is trivial. Otherwise, we can find  $\epsilon > 0$  such that  $\|x\| + \|y\| + 2\epsilon < 1$ . We now fix finite subsets  $H, K$  of  $\mathbb{N}$  such that  $x \in U_H$ ,  $y \in U_K$  while  $\lambda_H < \|x\| + \epsilon$  and  $\lambda_K < \|y\| + \epsilon$ . Let  $L$  be the finite subset of  $\mathbb{N}$  such that  $\lambda_L = \lambda_H + \lambda_K$ . Then  $x + y \in U_L$  and hence  $\|x + y\| \leq \lambda_L = \lambda_H + \lambda_K < \|x\| + \|y\| + 2\epsilon$ . Since the resulting inequality holds for any  $\epsilon > 0$  we must have  $\|x + y\| \leq \|x\| + \|y\|$  as desired.

It remains to show that the pseudo-seminorm generates the topology of the tvs. Since the topology generated by the pseudo-seminorm as well as that of the tvs are translation invariant, it is enough to show that the open balls around 0 of the pseudo-seminorm form a base of the filter of neighborhoods of 0 in the topology of the tvs. Let  $n \in \mathbb{N}$ . Clearly  $B_{2^{-n}}(0) \subseteq U_n \subseteq B_{2^{-(n-1)}}(0)$ . But this shows that  $\{B_{2^{-n}}(0)\}_{n \in \mathbb{N}}$  generates the same filter as  $\{U_n\}_{n \in \mathbb{N}}$ . This completes the proof.  $\square$

**Exercise 6.** Show that for a tvs with a balanced translation-invariant pseudometric the concepts of totally boundedness of Definitions 1.81 and 2.12 coincide.

**Proposition 2.27.** Let  $V$  be a mvs with pseudo-seminorm. Let  $r > 0$  and  $0 < \mu \leq 1$ . Then,  $B_{\mu r}(0) \subseteq \mu B_r(0)$ .

*Proof.* **Exercise.** □

**Proposition 2.28.** Let  $V, W$  be mvs with compatible metrics and  $f \in L(V, W)$ . (a)  $f$  is continuous iff for all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $f(B_\delta^V(0)) \subseteq B_\epsilon^W(0)$ . (b)  $f$  is bounded iff there exists  $\delta > 0$  such that for all  $\epsilon > 0$  there is  $\mu > 0$  such that  $f(\mu B_\delta^V(0)) \subseteq B_\epsilon^W(0)$ .

*Proof.* **Exercise.** □

**Proposition 2.29.** Let  $V$  be a mvs and  $C$  a subspace. Then, the quotient space  $V/C$  is a mvs.

*Proof.* **Exercise.** □

### 2.3 Locally convex tvs

**Definition 2.30.** A tvs is called *locally convex* iff every neighborhood of 0 contains a convex neighborhood of 0.

**Definition 2.31.** Let  $V$  be a vector space over  $\mathbb{K}$ . Then a map  $V \rightarrow \mathbb{R}_0^+ : x \mapsto \|x\|$  is called a *seminorm* iff it satisfies the following properties:

1.  $\|\lambda x\| = |\lambda| \|x\|$  for all  $\lambda \in \mathbb{K}, x \in V$ .
2. For all  $x, y \in V : \|x + y\| \leq \|x\| + \|y\|$ . (triangle inequality)

A seminorm is called a *norm* iff it satisfies in addition the following property:

3.  $\|x\| = 0 \implies x = 0$ .

**Proposition 2.32.** A seminorm induces a balanced translation-invariant pseudometric via  $d(x, y) := \|x - y\|$ . Moreover, the open balls of this metric are convex.

*Proof.* **Exercise.** □

**Proposition 2.33.** Let  $V$  be a vector space and  $\{\|\cdot\|_\alpha\}_{\alpha \in A}$  a set of seminorms on  $V$ . For any finite subset  $I \subseteq A$  and any  $\epsilon > 0$  define

$$U_{I, \epsilon} := \{x \in V : \|x\|_\alpha < \epsilon \ \forall \alpha \in I\}.$$

Then, the sets  $U_{I, \epsilon}$  form the base of the filter of neighborhoods of 0 in a topology on  $V$  that makes it into a locally convex tvs. If  $A$  is countable, then  $V$  is pseudometrizable. Moreover, the topology is Hausdorff iff for any  $x \in V \setminus \{0\}$  there exists  $\alpha \in A$  such that  $\|x\|_\alpha > 0$ .

*Proof.* Let  $I, I' \subseteq A$  be finite and  $\epsilon, \epsilon' > 0$ . Set  $I'' := I \cup I'$  and  $\epsilon'' := \min(\epsilon, \epsilon')$ . Then,  $U_{I'', \epsilon''} \subseteq U_{I, \epsilon} \cap U_{I', \epsilon'}$ . So the  $U_{I, \epsilon}$  really form the basis of a filter  $\mathcal{F}$ . We proceed to verify that  $\mathcal{F}$  satisfies the properties required by Proposition 2.8. Clearly,  $0 \in U$  for all  $U \in \mathcal{F}$  since  $\|0\|_\alpha = 0$  and so  $0 \in U_{I, \epsilon}$  for all  $I \subseteq A$  finite and  $\epsilon > 0$ . Also  $\lambda\mathcal{F} = \mathcal{F}$  since  $\lambda U_{I, \epsilon} = U_{I, |\lambda|\epsilon}$  for all  $I \subseteq A$  finite and  $\epsilon > 0$  by linearity of seminorms. As for property 1 of Proposition 2.7 consider  $x \in V$ ,  $I \subseteq A$  finite and  $\epsilon > 0$  arbitrary. Set  $\mu := \max_{\alpha \in I} \{\|x\|_\alpha\}$ . Then,  $x \in \frac{\mu+1}{\epsilon} U_{I, \epsilon}$ . Property 2 of Proposition 2.7 is satisfied since open balls of a seminorm are balanced and the sets  $U_{I, \epsilon}$  are finite intersections of such open balls and hence also balanced. Property 3 of Proposition 2.7 is sufficient to satisfy for a base. Observe then,  $U_{I, \epsilon/2} + U_{I, \epsilon/2} \subseteq U_{I, \epsilon}$  for all  $I \subseteq A$  finite and  $\epsilon > 0$  due to the triangle inequality. Thus, the so defined topology makes  $V$  into a tvs.

Observe that the sets  $U_{I, \epsilon}$  are convex, being finite intersections of open balls which are convex by Proposition 2.32. Thus,  $V$  is locally convex. If  $A$  is countable, then there is an enumeration  $I_1, I_2, \dots$  of the finite subsets of  $A$ . It is easy to see that  $U_{I_j, 1/n}$  with  $j \in \{1, \dots\}$  and  $n \in \mathbb{N}$  provides then a countable basis of the filter of neighborhoods of 0. That is,  $V$  is pseudometrizable. Concerning the Hausdorff property suppose that for any  $x \in V \setminus \{0\}$  there exists  $\alpha \in A$  such that  $\|x\|_\alpha > 0$ . Then, for this  $x$  we have  $x \notin U_{\{\alpha\}, \|x\|_\alpha}$ . So  $V$  is Hausdorff. Conversely, suppose  $V$  is Hausdorff. Given  $x \in V \setminus \{0\}$  there exist thus  $I \subseteq A$  finite and  $\epsilon > 0$  such that  $x \notin U_{I, \epsilon}$ . In particular, there exists  $\alpha \in I$  such that  $\|x\|_\alpha \geq \epsilon > 0$ .  $\square$

**Exercise 7.** In the context of Proposition 2.33 show that the topology is the coarsest such that all seminorms  $\|\cdot\|_\alpha$  are continuous.

**Definition 2.34.** Let  $V$  be a tvs and  $W \subseteq V$  a neighborhood of 0. The map  $\|\cdot\|_W : V \rightarrow \mathbb{R}_0^+$  defined as

$$\|x\|_W := \inf\{\lambda \in \mathbb{R}_0^+ : x \in \lambda W\}$$

is called the *Minkowski functional* associated to  $W$ .

**Proposition 2.35.** Let  $V$  be a tvs and  $W \subseteq V$  a neighborhood of 0.

1.  $\|\mu x\|_W = \mu \|x\|_W$  for all  $\mu \in \mathbb{R}_0^+$  and  $x \in V$ .
2. If  $W$  is balanced, then  $\|cx\|_W = |c| \|x\|_W$  for all  $c \in \mathbb{K}$  and  $x \in V$ .
3. If  $W$  is convex, then  $\|x + y\|_W \leq \|x\|_W + \|y\|_W$  for all  $x, y \in V$ .
4. If  $V$  is Hausdorff and  $W$  is bounded, then  $\|x\|_W = 0$  implies  $x = 0$ .

*Proof.* **Exercise.**  $\square$

**Theorem 2.36.** Let  $V$  be a tvs. Then,  $V$  is locally convex iff there exists a set of seminorms inducing its topology as in Proposition 2.33. Also,  $V$  is locally convex and pseudometrizable iff there exists a countable such set.

*Proof.* Given a locally convex tvs  $V$ , let  $\{U_\alpha\}_{\alpha \in A}$  be a base of the filter of neighborhoods such that  $U_\alpha$  is balanced and convex for all  $\alpha \in A$ . (**Exercise.** How can this be achieved?) In case that  $V$  is pseudometrizable we choose the base such that  $A$  is countable. Let  $\|\cdot\|_\alpha$  be the Minkowski functional associated to  $U_\alpha$ . Then, by Proposition 2.35,  $\|\cdot\|_\alpha$  is a seminorm for each  $\alpha \in A$ . We claim that the topology generated by the seminorms is precisely the topology of  $V$ . **Exercise.** Complete the proof.  $\square$

**Exercise 8.** Let  $V$  be a locally convex tvs and  $W$  a balanced and convex neighborhood of 0. Show that the Minkowski functional associated to  $W$  is continuous on  $V$ .

**Exercise 9.** Let  $V$  be a vector space and  $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$  a sequence of seminorms on  $V$ . Define the function  $q : V \rightarrow \mathbb{R}_0^+$  via

$$q(x) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|x\|_n}{\|x\|_n + 1}.$$

(a) Show that  $q$  is a pseudo-seminorm on  $V$ . (b) Show that the topology generated on  $V$  by  $q$  is the same as that generated by the sequence  $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ .

## 2.4 Normed and seminormed vector spaces

**Definition 2.37.** A tvs is called *locally bounded* iff it contains a bounded neighborhood of 0.

**Proposition 2.38.** A locally bounded tvs is pseudometrizable.

*Proof.* Let  $V$  be a locally bounded tvs and  $U$  a bounded and balanced neighborhood of 0 in  $V$ . The sequence  $\{U_n\}_{n \in \mathbb{N}}$  with  $U_n := \frac{1}{n}U$  is the base of a filter  $\mathcal{F}$  on  $V$ . Take a neighborhood  $W$  of 0. By boundedness of  $U$  there exists  $\lambda \in \mathbb{R}^+$  such that  $U \subseteq \lambda W$ . Choosing  $n \in \mathbb{N}$  with  $n \geq \lambda$  we find  $U_n \subseteq W$ , i.e.,  $W \in \mathcal{F}$ . Hence  $\mathcal{F}$  is the filter of neighborhoods of 0 and we have presented a countable base for it. By Theorem 2.26,  $V$  is pseudometrizable.  $\square$

**Proposition 2.39.** Let  $A, B$  be a tvs and  $f \in \text{CL}(A, B)$ . If  $A$  or  $B$  is locally bounded then  $f$  is bounded. Hence,  $\text{CL}(A, B) = \text{BL}(A, B)$  in this case.

*Proof.* **Exercise.**  $\square$

**Definition 2.40.** A tvs  $V$  is called *(semi)normable* iff the topology of  $V$  is induced by a (semi)norm.

**Theorem 2.41.** A tvs  $V$  is seminormable iff  $V$  is locally bounded and locally convex.

*Proof.* Suppose  $V$  is a seminormed vector space. Then, every ball is bounded and also convex, so in particular,  $V$  is locally bounded and locally convex.

Conversely, suppose  $V$  is a tvs that is locally bounded and locally convex. Take a bounded neighborhood  $U_1$  of 0 and a convex subneighborhood  $U_2$  of  $U_1$ . Now take a balanced subneighborhood  $U_3$  of  $U_2$  and its convex hull  $W = \text{conv}(U_3)$ . Then  $W$  is a balanced, convex and bounded (since  $W \subseteq U_2 \subseteq U_1$ ) neighborhood of 0 in  $V$ . Thus, by Proposition 2.35 the Minkowski functional  $\|\cdot\|_W$  defines a seminorm on  $V$ . It remains to show that the topology generated by this seminorm coincides with the topology of  $V$ . Let  $U$  be an open set in the topology of  $V$  and  $x \in U$ . The ball  $B_1(0)$  defined by the seminorm is bounded since  $B_1(0) \subseteq W$  and  $W$  is bounded. Hence there exists  $\lambda \in \mathbb{R}^+$  such that  $B_1(0) \subseteq \lambda(U - x)$ , i.e.,  $\lambda^{-1}B_1(0) \subseteq U - x$ . But  $\lambda^{-1}B_1(0) = B_{\lambda^{-1}}(0)$  by linearity and thus  $B_{\lambda^{-1}}(x) \subseteq U$ . Hence,  $U$  is open in the seminorm topology as well. Conversely, consider a ball  $B_\epsilon(0)$  defined by the seminorm for some  $\epsilon > 0$  and take  $x \in B_\epsilon(0)$ . Choose  $\delta > 0$  such that  $\|x\|_W < \epsilon - \delta$ . Observe that  $\frac{1}{2}W \subseteq B_1(0)$  and thus by linearity  $\frac{\delta}{2}W \subseteq B_\delta(0)$ . It follows that  $\frac{\delta}{2}W + x \subseteq B_\epsilon(0)$ . But  $\frac{\delta}{2}W + x$  is a neighborhood of  $x$  so it follows that  $B_\epsilon(0)$  is open. This completes the proof.  $\square$

**Exercise 10.** Let  $V$  be locally convex tvs with its topology generated by a finite family of seminorms. Show that  $V$  is seminormable.

**Proposition 2.42.** Let  $V$  be a seminormed vector space and  $U \subseteq V$  a subset. Then,  $U$  is bounded iff there exists  $c \in \mathbb{R}^+$  such that  $\|x\| \leq c$  for all  $x \in U$ .

*Proof.* **Exercise.**  $\square$

**Proposition 2.43.** Let  $A, B$  be seminormed vector spaces and  $f \in L(A, B)$ .  $f$  is bounded iff there exists  $c \in \mathbb{R}^+$  such that  $\|f(x)\| \leq c\|x\|$  for all  $x \in A$ .

*Proof.* **Exercise.**  $\square$

**Proposition 2.44.** Let  $V$  be a tvs and  $C$  a vector subspace. If  $V$  is locally convex, then so is  $V/C$ . If  $V$  is locally bounded, then so is  $V/C$ .

*Proof.* **Exercise.**  $\square$

## 2.5 Inner product spaces

As before  $\mathbb{K}$  stands for a field that is either  $\mathbb{R}$  or  $\mathbb{C}$ .

**Definition 2.45.** Let  $V$  be a vector space over  $\mathbb{K}$  and  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{K}$  a map.  $\langle \cdot, \cdot \rangle$  is called a *bilinear* (if  $\mathbb{K} = \mathbb{R}$ ) or *sesquilinear* (if  $\mathbb{K} = \mathbb{C}$ ) form iff it satisfies the following properties:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$  and  $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$  for all  $u, v, w \in V$ .

- $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$  and  $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$  for all  $\lambda \in \mathbb{K}$  and  $v \in V$ .

$\langle \cdot, \cdot \rangle$  is called *symmetric* (if  $\mathbb{K} = \mathbb{R}$ ) or *hermitian* (if  $\mathbb{K} = \mathbb{C}$ ) iff it satisfies in addition the following property:

- $\langle u, v \rangle = \overline{\langle v, u \rangle}$  for all  $u, v \in V$ .

$\langle \cdot, \cdot \rangle$  is called *positive* iff it satisfies in addition the following property:

- $\langle v, v \rangle \geq 0$  for all  $v \in V$ .

$\langle \cdot, \cdot \rangle$  is called *definite* iff it satisfies in addition the following property:

- If  $\langle v, v \rangle = 0$  then  $v = 0$  for all  $v \in V$ .

A map with all these properties is also called a *scalar product* or an *inner product*.  $V$  equipped with such a structure is called an *inner product space* or a *pre-Hilbert space*.

**Theorem 2.46** (Schwarz Inequality). *Let  $V$  be a vector space over  $\mathbb{K}$  with a scalar product  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{K}$ . Then, the following inequality is satisfied:*

$$|\langle v, w \rangle|^2 \leq \langle v, v \rangle \langle w, w \rangle \quad \forall v, w \in V.$$

*Proof.* By definiteness  $\alpha := \langle v, v \rangle \neq 0$  and we set  $\beta := -\langle w, v \rangle$ . By positivity we have,

$$0 \leq \langle \beta v + \alpha w, \beta v + \alpha w \rangle.$$

Using bilinearity and symmetry (if  $\mathbb{K} = \mathbb{R}$ ) or sesquilinearity and hermiticity (if  $\mathbb{K} = \mathbb{C}$ ) on the right hand side this yields,

$$0 \leq |\langle v, v \rangle|^2 \langle w, w \rangle - \langle v, v \rangle |\langle v, w \rangle|^2.$$

(**Exercise.** Show this.) Since  $\langle v, v \rangle \neq 0$  we can divide by it and arrive at the required inequality.  $\square$

**Proposition 2.47.** *Let  $V$  be a vector space over  $\mathbb{K}$  with a scalar product  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{K}$ . Then,  $V$  is a normed vector space with norm given by  $\|v\| := \sqrt{\langle v, v \rangle}$ .*

*Proof.* **Exercise.** Hint: To prove the triangle inequality, show that  $\|v+w\|^2 \leq (\|v\| + \|w\|)^2$  can be derived from the Schwarz inequality (Theorem 2.46).  $\square$

**Proposition 2.48.** *Let  $V$  be an inner product space. Then,  $\forall v, w \in V$ ,*

$$\begin{aligned} \langle v, w \rangle &= \frac{1}{4} \left( \|v+w\|^2 - \|v-w\|^2 \right) \quad \text{if } \mathbb{K} = \mathbb{R}, \\ \langle v, w \rangle &= \frac{1}{4} \left( \|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 - i\|v-iw\|^2 \right) \quad \text{if } \mathbb{K} = \mathbb{C} \end{aligned}$$

*Proof.* **Exercise.** □

**Proposition 2.49.** *Let  $V$  be an inner product space. Then, its scalar product  $V \times V \rightarrow \mathbb{K}$  is continuous.*

*Proof.* **Exercise.** □

**Theorem 2.50.** *Let  $V$  be a normed vector space. Then, there exists a scalar product on  $V$  inducing the norm iff the parallelogram equality holds,*

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2 \quad \forall v, w \in V.$$

*Proof.* **Exercise.** □

**Example 2.51.** The spaces  $\mathbb{R}^n$  and  $\mathbb{C}^n$  are inner product spaces via

$$\langle v, w \rangle := \sum_{i=1}^n v_i \overline{w_i},$$

where  $v_i, w_i$  are the coefficients with respect to the standard basis.