The positive formalism: towards spacetime

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What is the positive formalism?

An axiomatic framework for formulating physical theories.

Accommodates:

- classical statistical mechanics
- the standard formulation of quantum theory
- quantum field theory*
- generalized probabilistic theories
- a timeless formulation of quantum theory*

Should accommodate:

- quantum gravity

What is the positive formalism?

An axiomatic framework for formulating physical theories. **(PF)**

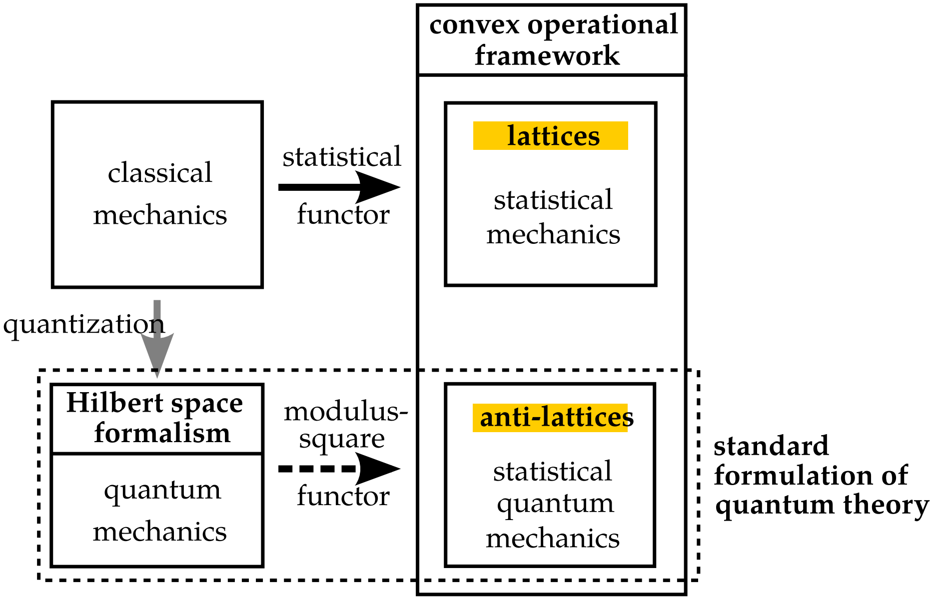
Accommodates:

- classical statistical mechanics **(PF+T+N+C)**
- the standard formulation of quantum theory **(PF+T+N+Q)**
- quantum field theory* **(PF+LOC+Q)**
- generalized probabilistic theories **(PF+T+N)**
- a timeless formulation of quantum theory* **(PF+Q)**

Should accommodate:

- quantum gravity **(PF+LOC+Q) ?**

Fundamental physics: Time-evolution frameworks



The "modulus square" functor

Hilbert space formulation to mixed state formulation (PF+T+N+Q) without operations

per system

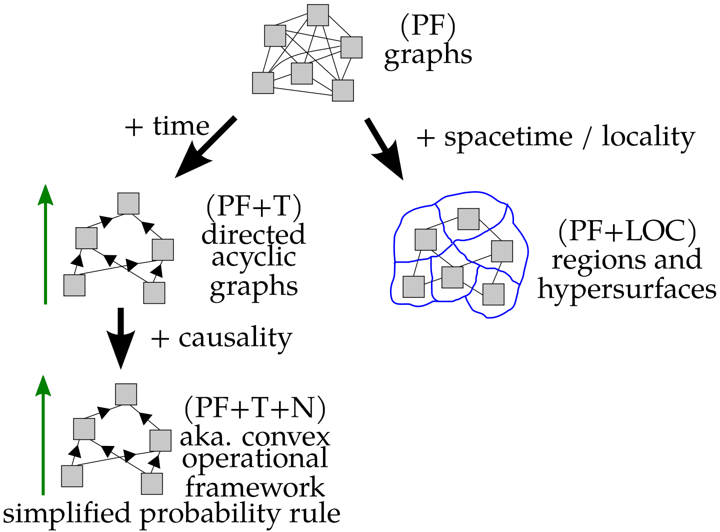
Replace $\mathcal{H}$ by $\mathcal{B} := \mathbf{B}^{\mathbb{R}}(\mathcal{H}) \approx \mathfrak{R}(\mathcal{H} \otimes \mathcal{H}^*)$.

per time interval

Replace $U : \mathcal{H} \to \mathcal{H}$ by $\tilde{U} : \mathcal{B} \to \mathcal{B}$ given by $\tilde{U}(\sigma) = U\sigma U^\dagger$.

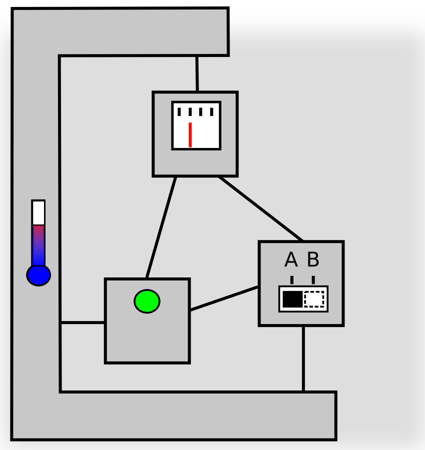
No replacement targeting **quantum operations**, no analog on the Hilbert space side.

(PF+)



(PF)
graphs
+ time
+ spacetime / locality
(PF+T)
directed
acyclic
graphs
(PF+LOC)
regions and
hypersurfaces
+ causality
(PF+T+N)
aka. convex
operational
framework
simplified probability rule

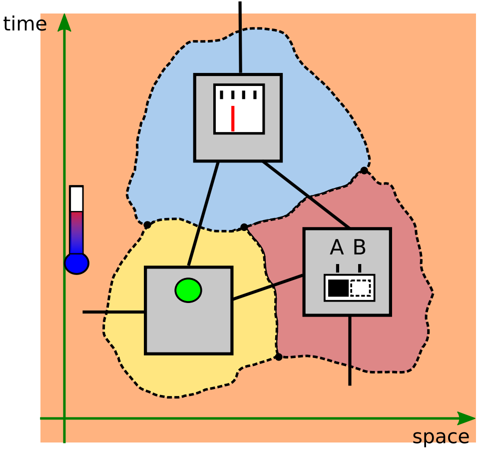
Spacetime and locality (PF+LOC)



Locality:

Real experiments happen in spacetime and interact directly only with adjacent experiments.

Spacetime and locality (PF+LOC)

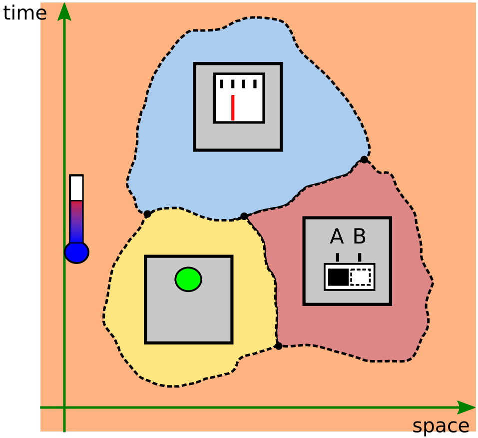


Locality:

Real experiments happen in spacetime and interact directly only with adjacent experiments.

Divide spacetime into regions and associate a **probe** to each **region**.

Spacetime and locality (PF+LOC)



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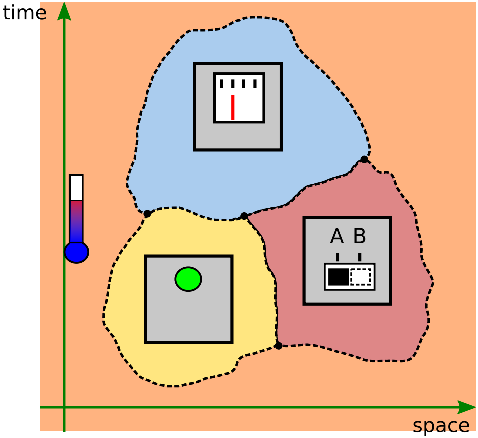
Divide spacetime into regions and associate a **probe** to each **region**.

Links are now superfluous as they are dual to **hypersurfaces**.

Spacetime and locality (PF+LOC)



Locality:

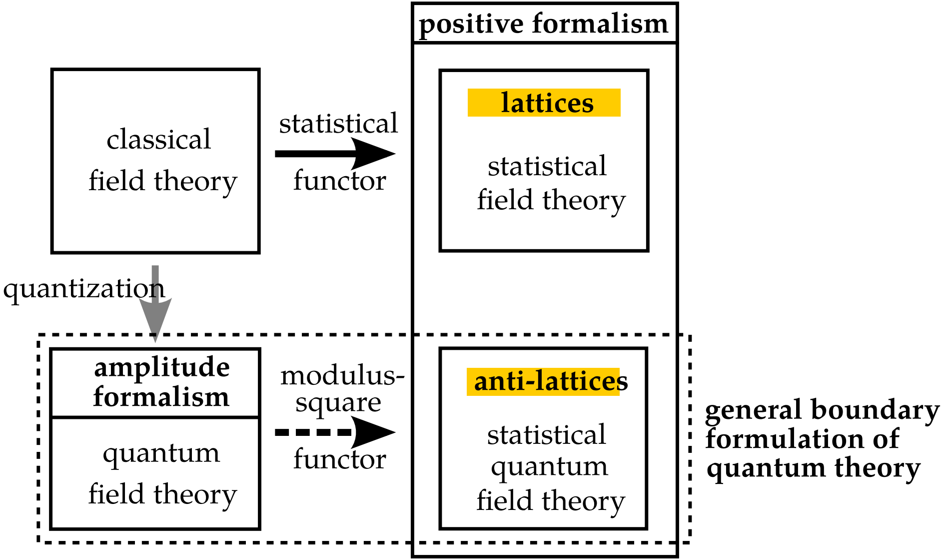
Real experiments happen in spacetime and interact directly only with adjacent experiments.

Divide spacetime into regions and associate a **probe** to each **region**.

Links are now superfluous as they are dual to **hypersurfaces**.

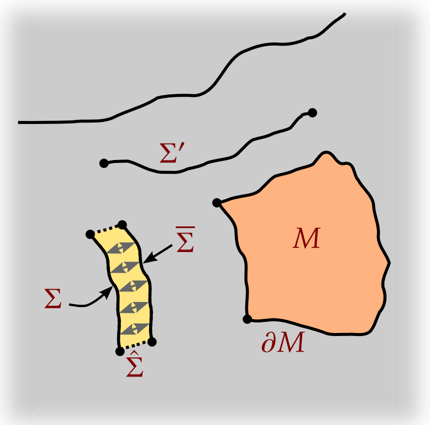
Associate the **null-probe** to regions where nothing happens.

Fundamental physics: Spacetime frameworks



geometric setting – manifolds

Fix dimension d. Manifolds are **oriented** and may carry additional structure: differentiable, metric, complex, etc.



region M

d-manifold with boundary.

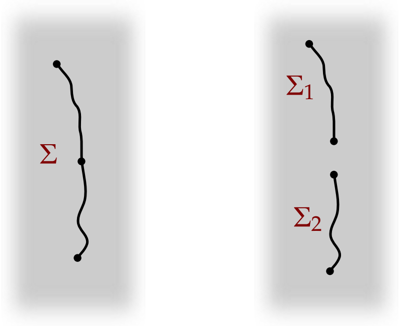
hypersurface Σ

$d-1$-manifold with boundary, with germ of d-manifold.

slice region $\hat{\Sigma}$

$d-1$-manifold with boundary, with germ of d-manifold, interpreted as "infinitely thin" region.

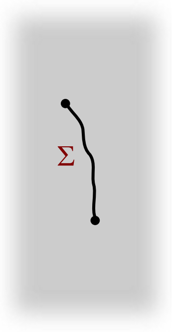
Hypersurface decomposition



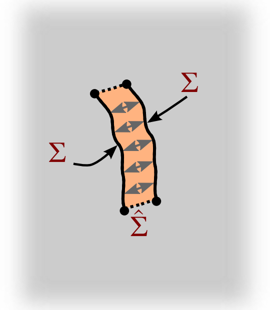
$$\mathcal{B}_{\Sigma} = \mathcal{B}_{\Sigma_1} \otimes \mathcal{B}_{\Sigma_2}$$

Spaces of boundary conditions decompose as tensor products under hypersurface decomposition.

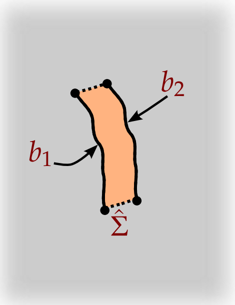
Slice regions



A **hypersurface** Σ gives rise to an infinitesimally thin **slice region** $\hat{\Sigma}$ by thickening. $\hat{\Sigma}$ has a boundary $\partial\hat{\Sigma}$ with two components, each a copy of Σ.



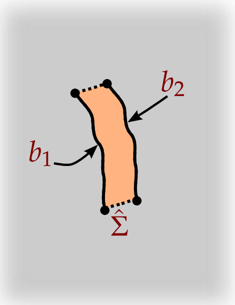
An inner product on boundary conditions



Putting **boundary conditions** on the two sides of a **slice region** allows evaluation with the **null probe**. This yields an **inner product** $\mathcal{B}_\Sigma \times \mathcal{B}_\Sigma \to \mathbb{R}$ on the space of boundary conditions.

$$(\!|b_1, b_2|\!)_\Sigma := [\![\boxslash, b_1 \otimes b_2]\!]_{\hat{\Sigma}}$$

An inner product on boundary conditions



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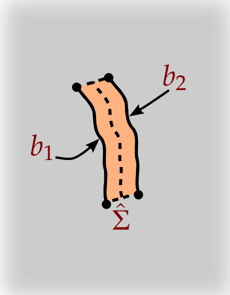
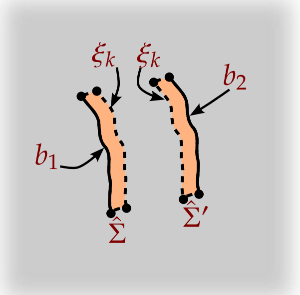
$$(\!| b_1, b_2 |\!)_\Sigma := [\![\boxslash, b_1 \otimes b_2]\!]_{\hat{\Sigma}}$$

Different boundary conditions should encode different physics of adjacent regions. This means that the inner product must be **non-degenerate**. Due to the dual role of boundary conditions the inner product should identify $\mathcal{B}_\Sigma$ with its dual $\mathcal{B}_\Sigma^*$. That is, it should be **symmetric** and **positive-definite**.

Composition of slice regions

The **completeness property** satisfied by the inner product translates into a geometric composition property of slice regions.



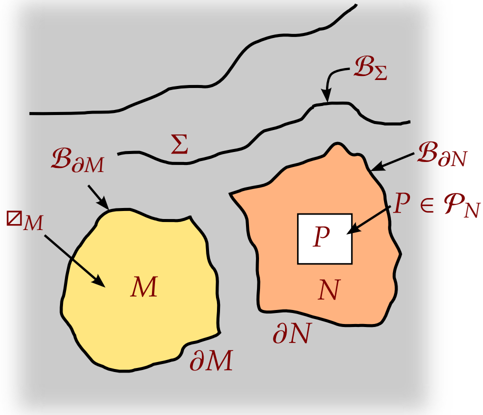


$$(\!|b_1, b_2|\!)_{\hat{\Sigma}} \quad = \quad \sum_{k \in I} (\!|b_1, \xi_k|\!)_{\hat{\Sigma}} \, (\!|\xi_k, b_2|\!)_{\hat{\Sigma}}$$

Here, $\{\xi_k\}_{k \in I}$ is an orthonormal basis of $\mathcal{B}_{\Sigma}$.

(PF+LOC) – axioms I

Manifolds need not be oriented.



(P1) per hypersurface Σ

A partially ordered vector space $\mathcal{B}_\Sigma$.

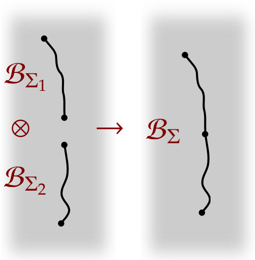
$\mathcal{B}_\emptyset = \mathbb{R}$

(P4) per region M

A partially ordered vector space $\mathcal{P}_M$ of linear maps (probes) $\mathcal{B}_{\partial M} \to \mathbb{R}$.
$\mathcal{P}_M^+ \subset \mathcal{P}_M$ are **positive maps**. A **unit** $\boxslash_M \in \mathcal{P}_M^+$.

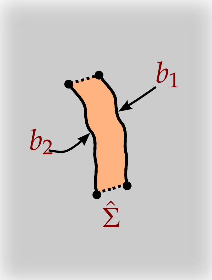
The choice of an element of $\mathcal{P}_M$ for a region M is indicated by a **label**.

(PF+LOC) – axioms II



(P2) per hypersurface decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$

A positive vector space isomorphism $\tau : \mathcal{B}_{\Sigma_1} \otimes \mathcal{B}_{\Sigma_2} \rightarrow \mathcal{B}_{\Sigma}$.



(P3x) per hypersurface Σ

The null probe gives rise to a **positive-definite sharply positive inner product** $(\!|b_1, b_2|\!)_{\Sigma} := [\![\boxslash, \tau(b_1 \otimes b_2)]\!]_{\hat{\Sigma}}$.

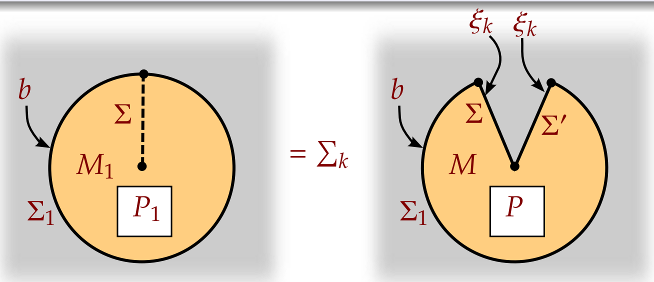
(PF+LOC) – axioms III

(P5a) per disjoint composition of regions $M = M_1 \sqcup M_2$

$[\![A, \tau(b_1 \otimes b_2)]\!]_M = [\![A_1, b_1]\!]_{M_1} [\![A_2, b_2]\!]_{M_2}$. Write $A = A_1 \diamond A_2$. ($\boxslash = \boxslash \diamond \boxslash$.)

(P5b) per self-composition of region M to M_1 along Σ

$[\![P_1, b]\!]_{M_1} \cdot c_{M,\Sigma} = \sum_k [\![P, \tau(b \otimes \xi_k \otimes \xi_k)]\!]_M$. Write $P_1 = \diamond P$. (Have $\boxslash = \diamond \boxslash$.)



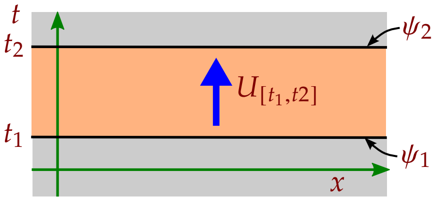
$\{\xi_k\}_{k \in I}$ ON-basis of $\mathcal{B}_\Sigma$. $c_{M,\Sigma}$ **gluing anomaly**.

And now for something completely different...

Quantum theory: states and evolution

States describe system at an instant, are elements of a **Hilbert space** $\mathcal{H}$.

Dynamics: **Evolution operator** $U_{[t_1,t_2]} : \mathcal{H} \to \mathcal{H}$
or transition amplitude $\langle \psi_2, U_{[t_1,t_2]} \psi_1 \rangle$.



The transition amplitude and be calculated through the **path integral** [Feynman 1948].

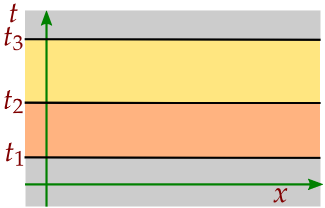
In quantum field theory this is an integral over the space $K_{[t_1,t_2]}$ of field configurations in the region $[t_1, t_2] \times \mathbb{R}^3$.

$$\langle \psi_2, U_{[t_1,t_2]} \psi_1 \rangle = \int_{K_{[t_1,t_2]}} \mathcal{D}\phi \; \psi_1(\phi|_{t_1}) \overline{\psi_2(\phi|_{t_2})} \, e^{\mathrm{i}S(\phi)}$$

Temporal composition

Composition of temporal evolutions:

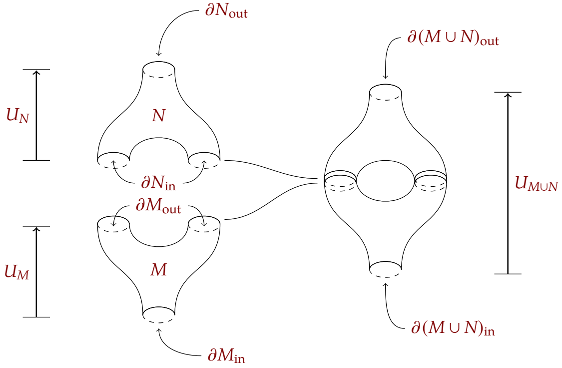
- in terms of operators: $U_{[t_1,t_3]} = U_{[t_2,t_3]} \circ U_{[t_1,t_2]}$
- in terms of matrix elements:
  $\langle \psi_3, U_{[t_1,t_3]} \psi_1 \rangle = \sum_{i \in N} \langle \psi_3, U_{[t_2,t_3]} \zeta_i \rangle \langle \zeta_i, U_{[t_1,t_2]} \psi_1 \rangle$



This temporal **composition property** is reflected in the path integral.

Composition in spacetime

The path integral has a spacetime **composition property**. This suggests:



$$U_{M \cup N} = U_N \circ U_M$$

TQFT

We come now to the promised axioms. A topological quantum field theory (QFT), in dimension *d* defined over a ground ring Λ, consists of the following data:

(A) A finitely generated Λ-module $Z(\Sigma)$ associated to each oriented closed smooth *d*-dimensional manifold Σ,

(B) An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth $(d+1)$-dimensional manifold (with boundary) M.

These data are subject to the following axioms, which we state briefly and expand upon below:

(1) Z is *functorial* with respect to orientation preserving diffeomorphisms of Σ and M,

(2) Z is *involutory*, i.e. $Z(\Sigma^*) = Z(\Sigma)^*$ where Σ^* is Σ with opposite orientation and $Z(\Sigma)^*$ denotes the dual module (see below),

(3) Z is *multiplicative*.

M. Atiyah, Topological quantum field theories, Inst. Hautes Études Sci. Publ. Math. 68 (1988), 175–186

TQFT

The mathematical framework of **Topological Quantum Field Theory (TQFT)** originating with works of Witten, Segal and Atiyah in the 1980s was inspired by quantum field theory and specifically by the path integral. It has had an enormous impact in various branches of **mathematics**, specifically low dimensional topology, knot theory, monoidal category theory, quantum groups, operator algebras and general algebraic topology.

TQFT and realistic QFT

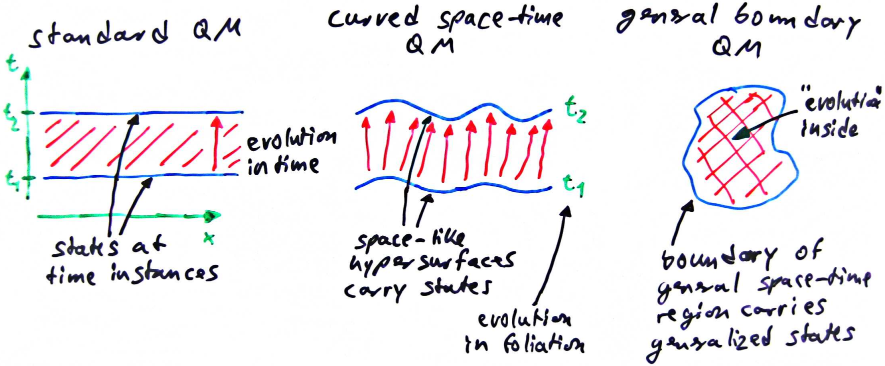
It was initially thought to provide a path to a rigorous axiomatic formulation of **quantum field theory (QFT)**. This has not been realized. Certain characteristics of TQFT do not fit well with requirements from physics.

- TQFT is limited to finite-dimensional vector spaces. QFT requires infinite-dimensional vector spaces.
- The directionality of cobordisms can be identified in quantum mechanics with the time direction. This is not sensible in QFT.
- The vector spaces need an inner product. This does not seem to have a natural origin in TQFT.

A new hope – CQFT

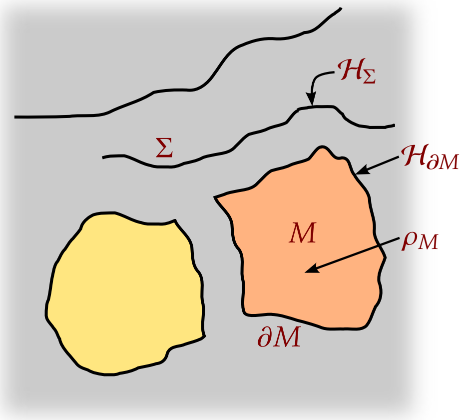
Compositional Quantum Field Theory (CQFT), also known as General Boundary Quantum Field Theory (GBQFT), is a new **axiomatic formulation** being developed since 2003. Inspired by TQFT, but avoiding some of its problems, it has been successfully applied to **realistic QFTs** in a number of contexts. It also comes with a **quantization prescription** for free QFT. It is used as the underlying framework for **loop quantum gravity** (spin foam models). It is still under development.

Amplitudes in spacetime regions



CQFT – axioms I

Assignment of algebraic structures to geometric ones.



(T1) per hypersurface Σ

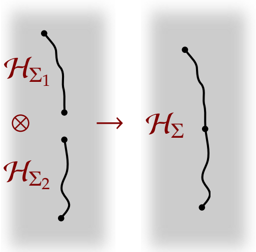
A complex vector space $\mathcal{H}_\Sigma$.
(**state space**)

$\mathcal{H}_\emptyset = \mathbb{C}$.

(T4) per region M

A linear map $\mathcal{H}_{\partial M} \to \mathbb{C}$.
(**amplitude map**)

CQFT – axioms II

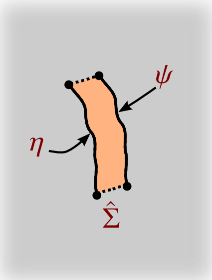


(T1b) per hypersurface Σ

A conjugate linear involution $\iota_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_{\overline{\Sigma}}$.

(T2) per hypersurface decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$

A partial isometry $\tau : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \to \mathcal{H}_\Sigma$.



(T3x) per hypersurface Σ

The amplitude map gives rise to a **positive-definite inner product** $\langle \iota_{\overline{\Sigma}}(\psi), \eta \rangle_\Sigma := \rho_{\hat{\Sigma}} \circ \tau(\psi \otimes \eta)$.

(T4a) per region M

$\rho_{\overline{M}}(\psi) = \overline{\rho_M(\iota_{\overline{\partial M}}(\psi))}$

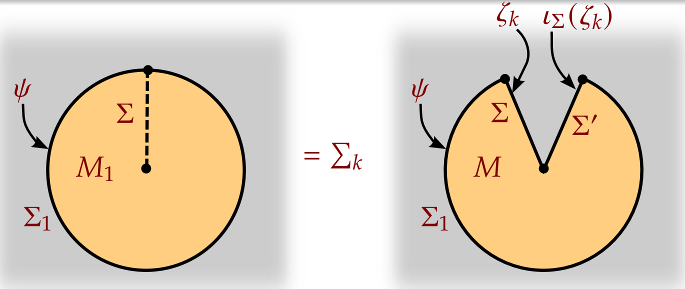
CQFT – axioms III

(T5a) per disjoint composition of regions $M = M_1 \sqcup M_2$

$\rho_M(\tau(\psi_1 \otimes \psi_2)) = \rho_{M_1}(\psi_1)\rho_{M_2}(\psi_2)$. We write $\rho_M = \rho_{M_1} \diamond \rho_{M_2}$.

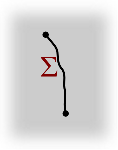
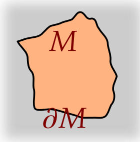
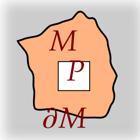
(T5b) per self-composition of region M to M_1 along Σ

$\rho_{M_1}(\psi) \cdot c_{M,\Sigma} = \sum_k \rho_M(\tau(\psi \otimes \zeta_k \otimes \iota_\Sigma(\zeta_k)))$. We write $\rho_{M_1} = \diamond\rho_M$.

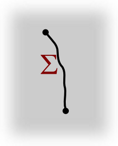
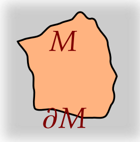
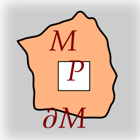


$\{\zeta_k\}_{k\in I}$ ON-basis of $\mathcal{H}_\Sigma$. $c_{M,\Sigma}$ **gluing anomaly**.

Assignments

spacetime object			**positive formalism**
			ordered vector space $\mathcal{B}_\Sigma$
			null probe $\boxslash : \mathcal{B}_{\partial M} \to \mathbb{R}$ (positive!)
			probe $P : \mathcal{B}_{\partial M} \to \mathbb{R}$ (positive!)

Assignments in quantum theory

spacetime object	**amplitude formalism**	$\longrightarrow$ functor $\longrightarrow$	**positive formalism**
	Hilbert space $\mathcal{H}_\Sigma$	self-adjoint operators	ordered vector space $\mathcal{B}_\Sigma$
	amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$	$\boxslash(\sigma) = \sum_i \overline{\rho(\zeta_i)}\rho(\sigma\zeta_i)$	null probe $\boxslash : \mathcal{B}_{\partial M} \to \mathbb{R}$ (positive!)
			probe $P : \mathcal{B}_{\partial M} \to \mathbb{R}$ (positive!)

The "modulus square" functor

Converts **CQFT** to **(PF+LOC+Q)** (without probes)

per hypersurface Σ

Replace $\mathcal{H}_\Sigma$ by $\mathcal{B}_\Sigma := \mathbf{B}^{\mathbb{R}}(\mathcal{H}_\Sigma) \approx \mathfrak{R}(\mathcal{H}_\Sigma \otimes \mathcal{H}_{\overline{\Sigma}})$.

per region M

Replace $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ by $\boxslash_M : \mathcal{B}_{\partial M} = \mathbf{B}^{\mathbb{R}}(\mathcal{H}_{\partial M}) \to \mathbb{R}$ via

$$[\![\boxslash, \sigma]\!]_M := \sum_k \overline{\rho_M(\zeta_k)} \rho_M(\sigma \zeta_k) \quad \text{or} \quad [\![\boxslash, \psi \otimes \eta]\!]_M = \overline{\rho_M(\iota_{\overline{\partial M}}(\eta))} \rho_M(\psi)$$

No replacement targeting **probes**, no analog on the Hilbert space side.