Scattering and boundary measurement

Robert Oeckl
robert@matmor.unam.mx

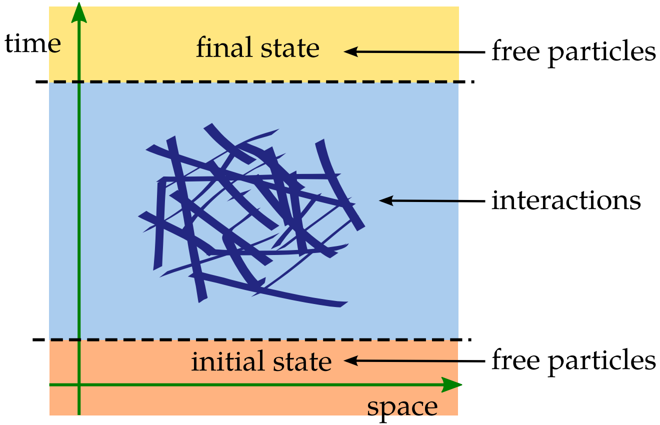
Office 4.08
Institut für Quantenoptik und Quanteninformation (IQOQI)
Österreichische Akademie de Wissenschaften (ÖAW)
Wien, Austria

Centro de Ciencias Matemáticas (CCM)
Universidad Nacional Autónoma de México (UNAM)
Morelia, Mexico

Lecture 18 – 19 May 2025

Scattering in QFT: S-matrix

Consider measurement only at **asymptotic infinity**, infinitely early and infinitely late time, described by **transition probabilities**. This is how the **S-matrix** in **quantum field theory** works to describe **scattering processes**. This requires **perturbation theory**.

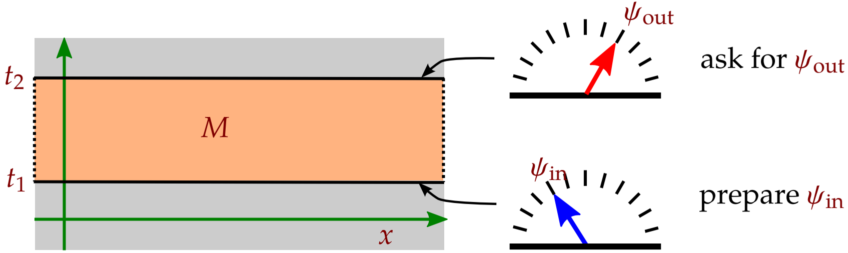


At early and late times particles are far apart and do not interact. The interesting physics happens at **intermediate times**.

Probabilities from transition amplitudes

Consider a simple measurement:

- At t_1 we **prepare** a state ψ_{in}.
- At t_2 we **ask** whether the system is in state ψ_{out}.



The **conditional probability** for this is $\boxed{P(\psi_{\text{out}}|\psi_{\text{in}}) = |\langle\psi_{\text{out}}, U\psi_{\text{in}}\rangle|^2}$

Assume: states normalized

Boundary measurement and probability

Consider a **measurement** on the **boundary** ∂M of a region M.

Probabilities in quantum theory are generally conditional probabilities. They depend on two pieces of information. Here these are encoded in terms of **positive operators** in $\mathcal{B}_{\partial M}$ that we call **quantum boundary conditions** satisfying,

$$0 \le \mathbf{A} \le \mathbf{S} \le \dot{\top}.$$

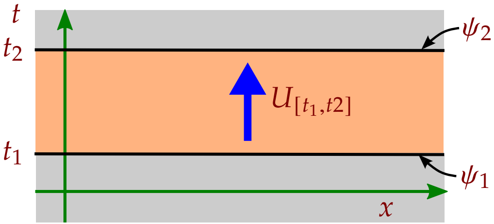
- $\mathbf{S}$ represents **preparation** or **knowledge**
- $\mathbf{A}$ represents **observation** or the **question**

Equivalently, the operators describe **probes** in the exterior X of M. The probability that the physics in M is described by $\mathbf{A}$ given that it is described by $\mathbf{S}$ is: [RO 2005, 2016]

$$\boxed{P(\mathbf{A}|\mathbf{S}) = \frac{[\![\boxslash, \mathbf{A}]\!]_M}{[\![\boxslash, \mathbf{S}]\!]_M}} \qquad [\![\boxslash, \sigma]\!]_M := \sum_{k\in I} \overline{\rho_M(\zeta_k)}\rho_M(\sigma\zeta_k)$$

Here $\{\zeta_k\}_{k\in I}$ is an ON-basis of $\mathcal{H}_{\partial M}$.

Recovering transition probabilities



$M = [t_1, t_2] \times \mathbb{R}^3$. To compute the probability of measuring ψ_2 at t_2 given that we prepared ψ_1 at t_1 we set in $\mathcal{B}_{\partial M} = \mathcal{B}_1 \otimes \mathcal{B}_2$,

$$\mathbf{S} = |\psi_1\rangle\langle\psi_1| \otimes \doteqdot, \quad \mathbf{A} = |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|.$$

The resulting expression yields correctly

$$P(\mathbf{A}|\mathbf{S}) = \frac{|\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2^*)|^2}{1} = |\langle\psi_2, U_{[t_1,t_2]}\psi_1\rangle|^2.$$

Spatially asymptotic S-matrix

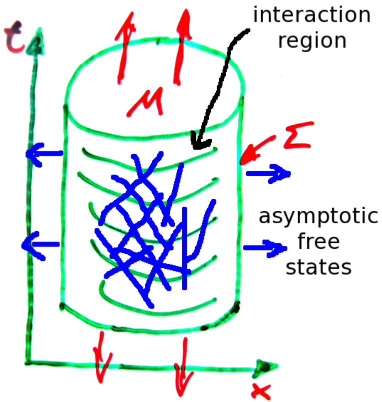
With the **PF** we are no longer bound to geometries with an **initial** and **final** hypersurface.

$$P(\mathbf{A}|\mathbf{S}) = \frac{[\![\boxslash, \mathbf{A}]\!]}{[\![\boxslash, \mathbf{S}]\!]}$$

with $0 \leq \mathbf{A} \leq \mathbf{S} \leq \overset{-}{\top}$.

Spatially asymptotic S-matrix

With the **PF** we are no longer bound to geometries with an **initial** and **final** hypersurface. For scattering problems a **hypercylinder geometry** makes more sense!



$$P(\mathbf{A}|\mathbf{S}) = \frac{[\![\boxslash, \mathbf{A}]\!]}{[\![\boxslash, \mathbf{S}]\!]}$$

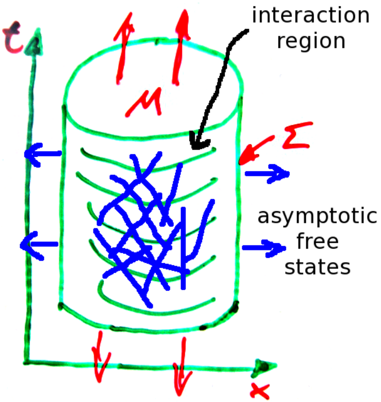
with $0 \leq \mathbf{A} \leq \mathbf{S} \leq \mp$.

- Consider the hypercylinder of radius R and let $R \to \infty$.
- Asymptotically, $\mathcal{H}_R \approx \mathcal{H}_{\text{in}} \otimes \mathcal{H}^*_{\text{out}}$ and the hypercylinder S-matrix is equivalent to the usual S-matrix.

[Colosi, RO 2007,2008]

Spatially asymptotic S-matrix

With the **PF** we are no longer bound to geometries with an **initial** and **final** hypersurface.



beyond the S-matrix:

- this works for spacetimes that are **not globally hyperbolic**, e.g. S-matrix in **AdS** [Dohse, RO 2013]
- at finite R, $\mathcal{H}_R$ contains additional **evanescent modes** that carry **finite-size effects** and **near field dynamics** [Colosi, RO 2021; RO 2021]

states as POVM

In many cases state spaces can be organized in terms of **positive operator valued measures (POVM)**:

Measure space $\mathcal{X}$, positive measure μ, family of positive operators $Q : \mathcal{X} \to \mathcal{B}$ satisfying $\int_{\mathcal{X}} Q(x) \mathrm{d}\mu(x) = \mathbb{1}$ (completeness)

Parametrize (mixed) states by positive functions $f : \mathcal{X} \to \mathbb{R}_0^+$ via

$$\boxed{f \mapsto \hat{f} := \int_{\mathcal{X}} f(x) Q(x) \mathrm{d}\mu(x)}$$

states as POVM

In many cases state spaces can be organized in terms of **positive operator valued measures (POVM)**:

Measure space $\mathcal{X}$, positive measure μ, family of positive operators $Q : \mathcal{X} \to \mathcal{B}$ satisfying $\int_{\mathcal{X}} Q(x) \mathrm{d}\mu(x) = \mathbb{1}$ (completeness)

Parametrize (mixed) states by positive functions $f : \mathcal{X} \to \mathbb{R}_0^+$ via

$$f \mapsto \hat{f} := \int_{\mathcal{X}} f(x) Q(x) \mathrm{d}\mu(x)$$

Simplest example: alternative inputs/outcomes indexed by discrete set $\mathcal{X}$ with counting measure.

$$\hat{f} = \sum_{k \in \mathcal{X}} f(k) Q(k)$$

to select outcome j set $f(k) = \delta_{j,k}$, get $Q(j)$

- choose ON-basis $\{\zeta_k\}_{k \in \mathcal{X}}$ and set $Q(k) = |\zeta_k\rangle\langle\zeta_k|$

states as POVM – further examples

- single particle in non-relativistic QM: $\mathcal{X}$ momentum space

$$\hat{f} = \int_{\mathcal{X}} \mathrm{d}^3k \, |k\rangle f(k) \langle k|$$

- particle picture in QFT: M 1-particle momentum space
$\mathcal{X} = \bigcup_{n=0}^{\infty} M^n$

$$\hat{f} = \sum_{n=0}^{\infty} \int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_1} \cdots \frac{\mathrm{d}^3k_n}{(2\pi)^3 2E_n} |k_1, \ldots, k_n\rangle f_n(k_1, \ldots, k_n) \langle k_1, \ldots, k_n|$$

- coherent states (e.g. quantum optics): $\mathcal{X}$ classical phase space, K_ξ coherent state associated to $\xi \in \mathcal{X}$.

$$\hat{f} = \int_{\mathcal{X}} |K_\xi\rangle f(\xi) \langle K_\xi| \mathrm{d}\mu(\xi)$$

POVM on the boundary

Consider a region M. Recall the POVM context (for $\mathcal{H}_{\partial M}$): $\mathcal{X}, \mu, Q$. Choose subsets,

$$\emptyset \subseteq \mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{X}$$

Consider the corresponding characteristic functions,

$$0 \leq \chi_{\mathcal{A}} \leq \chi_{\mathcal{S}} \leq \mathbf{1}$$

and their quantizations

$$0 \leq \hat{\chi}_{\mathcal{A}} \leq \hat{\chi}_{\mathcal{S}} \leq \overset{=}{\top}.$$

Then,

$$P(\mathcal{A}|\mathcal{S}) = \frac{[\![\boxslash, \hat{\chi}_{\mathcal{A}}]\!]_M}{[\![\boxslash, \hat{\chi}_{\mathcal{S}}]\!]_M}.$$