Axiomatization of classical field theory

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Lagrangian field theory

Formulate field theory in terms of first order Lagrangian density $\Lambda(\varphi, \partial\varphi, x)$. For a spacetime region M the **action** of a field ϕ is

$$S_M(\phi) := \int_M \Lambda(\phi(\cdot), \partial\phi(\cdot), \cdot).$$

Classical solutions in M are extremal points of this action. These are obtained by setting to zero the first variation of the action,

$$(\mathrm{d}S_M)_\phi(X) = \int_M X^a \left(\frac{\delta\Lambda}{\delta\varphi^a} - \partial_\mu \frac{\delta\Lambda}{\delta\, \partial_\mu\varphi^a}\right)(\phi) + \int_{\partial M} X^a \partial_\mu \lrcorner \frac{\delta\Lambda}{\delta\partial_\mu\varphi^a}(\phi)$$

under the condition that the infinitesimal field X vanishes on ∂M. This yields the **Euler-Lagrange equations**,

$$\left(\frac{\delta\Lambda}{\delta\varphi^a} - \partial_\mu \frac{\delta\Lambda}{\delta\, \partial_\mu\varphi^a}\right)(\phi) = 0.$$

The symplectic form

The boundary term can be defined for an arbitrary hypersurface Σ.

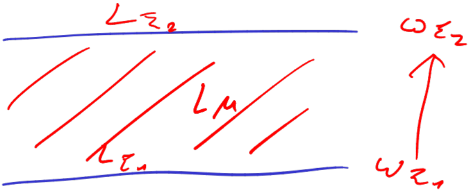
$$(\theta_\Sigma)_\phi(X) = -\int_\Sigma X^a \partial_\mu \lrcorner \frac{\delta\Lambda}{\delta\partial_\mu\varphi^a}(\phi)$$

This 1-form is called the **symplectic potential**. Its exterior derivative is the **symplectic 2-form**,

$$(\omega_\Sigma)_\phi(X,Y) = (\mathrm{d}\theta_\Sigma)_\phi(X,Y) = -\frac{1}{2}\int_\Sigma \Bigg((X^b Y^a - Y^b X^a)\, \partial_\mu \lrcorner \frac{\delta^2\Lambda}{\delta\varphi^b \delta\, \partial_\mu\varphi^a}(\phi) + (Y^a \partial_\nu X^b - X^a \partial_\nu Y^b)\, \partial_\mu \lrcorner \frac{\delta^2\Lambda}{\delta\, \partial_\nu\varphi^b \delta\, \partial_\mu\varphi^a}(\phi) \Bigg).$$

We denote the space of solutions in M by L_M and the space of germs of solutions on a hypersurface Σ by L_Σ.

Conservation of the symplectic form



globally hyp. s.!
spacelike hs.

$L_M \to L_{\Sigma_1} \times L_{\Sigma_2}$

$\omega_{\Sigma_1} + \omega_{\overline{\Sigma}_2}$
$= \omega_{\Sigma_1} - \omega_{\Sigma_2}$

conservation
$\Leftrightarrow \quad L_M \subseteq L_{\Sigma_1} \times L_{\overline{\Sigma}_2}$
Lagrangian subspace

Lagrangian submanifolds

Let M be a region and $\phi \in L_{\partial M}$. Then ϕ may or may not be induced from a solution in M. If ϕ arises from a solution in M and X, Y arise from infinitesimal solutions in M, then,

$$(\omega_{\partial M})_\phi(X, Y) = (\mathrm{d}\theta_{\partial M})_\phi(X, Y) = -(\mathrm{dd}S_M)_\phi(X, Y) = 0.$$

This means, L_M induces an **isotropic** submanifold of $L_{\partial M}$.

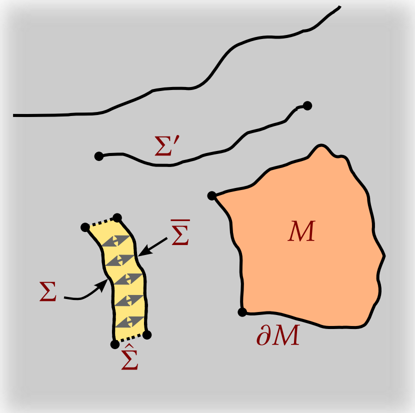
It is natural to require that the symplectic form is **non-degenerate**. We are then led to the converse statement: If given X we have $(\omega_{\partial M})_\phi(X, Y) = 0$ for all induced Y, then X itself must be induced. This means, L_M induces a **coisotropic** submanifold of $L_{\partial M}$.

L_M induces a **Lagrangian** submanifold of $L_{\partial M}$.

[Kijowski, Tulczyjew 1979]

Geometric setting – manifolds

Fix dimension d. Manifolds are **oriented** and may carry additional structure: differentiable, metric, complex, etc.



region M

d-manifold with boundary.

hypersurface Σ

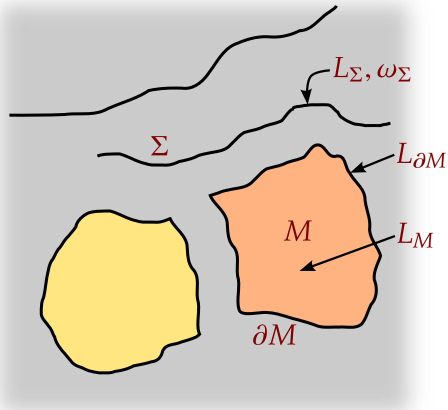
$d-1$-manifold with boundary, with germ of d-manifold.

slice region $\hat{\Sigma}$

$d-1$-manifold with boundary, with germ of d-manifold, interpreted as "infinitely thin" region.

Axiomatic classical field theory

[RO 2010]



per hypersurface Σ :

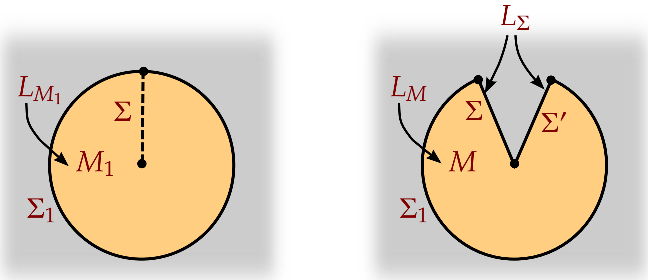
The **space of germs of solutions** near Σ. This is a **symplectic manifold** $(L_\Sigma, \omega_\Sigma)$.

per region M :

The **space of solutions** in M. Forgetting the interior yields a map $L_M \to L_{\partial M}$. Under this map L_M is a **Lagrangian submanifold** $L_M \subseteq L_{\partial M}$.

Axiomatic classical field theory

There are additional axioms related to gluing etc. like this one:



$$L_{M_1} \hookrightarrow L_M \rightrightarrows L_\Sigma$$

Axiomatic classical field theory

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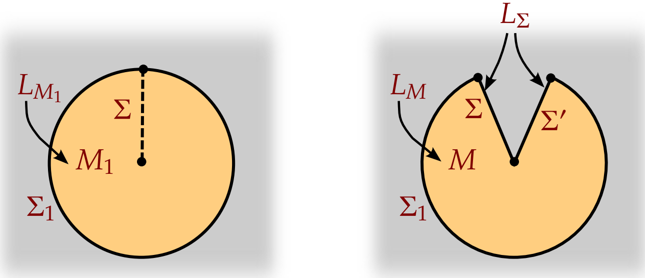

$$L_{M_1} \hookrightarrow L_M \rightrightarrows L_\Sigma$$

BUT, this does not work for many **non-compact regions**.