Klein-Gordon theory
The generalized S-matrix
Evanescent particles

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Lecture 20 part 2 – 26 May 2025

We consider a **real scalar field** theory in **Minkowski spacetime** with the action

$$S_M(\phi) = \frac{1}{2}\int \mathrm{d}^4x \left((\partial_\mu\phi)\partial^\mu\phi - m^2\phi^2\right).$$

The equations of motion are given by the **Klein-Gordon equation**:

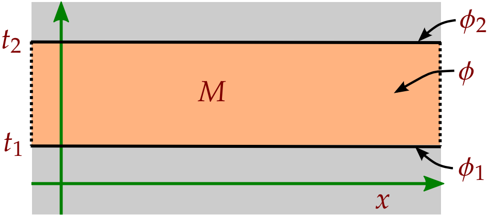
$$(\Box + m^2)\phi = 0.$$

We denote by L_M the **space of solutions** in a spacetime region M, and by L_Σ the space of **germs of solutions** on a hypersurface Σ.

Klein-Gordon Theory

Standard geometry – spacelike hypersurplanes (I)

Consider **constant-time hypersurfaces** and **time-interval regions** as in the **standard formulation**.



Consider an **constant-time hypersurface** at time t. Expanding in **Fourier modes**, elements of L_t are conveniently parametrized in terms of functions on **momentum space**,

$$\phi(t, x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left(\phi(k) e^{-\mathrm{i}(Et - kx)} + \overline{\phi(k)} e^{\mathrm{i}(Et - kx)} \right).$$

Klein-Gordon Theory

Standard geometry – spacelike hyperplanes (II)

The Lagrangian gives rise to the **symplectic form**,

$$\begin{aligned}\omega_t(\phi_1, \phi_2) &= \frac{1}{2}\int \mathrm{d}^3x\ (\phi_2(t,x)\partial_0\phi_1(t,x) - \phi_1(t,x)\partial_0\phi_2(t,x)) \\ &= \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left(\phi_2(k)\overline{\phi_1(k)} - \phi_1(k)\overline{\phi_2(k)}\right).\end{aligned}$$

The standard **complex structure** is,

$$(J(\phi))(k) = -\mathrm{i}\phi(k).$$

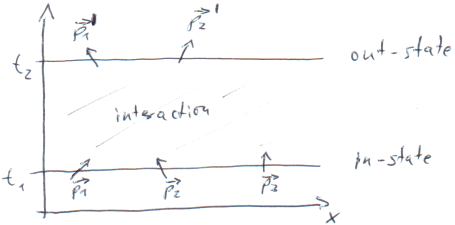
This yields the **complex inner product**,

$$\{\phi_1, \phi_2\}_t = 2\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\,\phi_1(k)\overline{\phi_2(k)}.$$

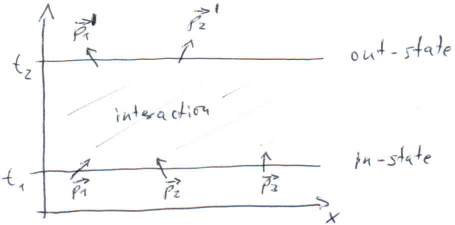
L_t becomes the **one-particle Hilbert space**.
$\mathcal{H}_t$ is the space of **wave functions** or **Fock space** over L_t.

Klein-Gordon Theory

Standard geometry – scattering



Particles, i.e., elements of L_t, can be characterized by 3 **quantum numbers**: the components p_i of the **3-momentum**. Moreover, each particle is part of either the **in-state** or the **out-state**.



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Denote $\psi(p_1, \ldots, p_n)$ the n-particle state with momenta $p_1, \ldots, p_n$ in $\mathcal{H}$.

The **probability** to find outgoing particles with momenta $p_1', \ldots, p_m'$ given incoming particles $p_1, \ldots, p_n$ is,

$$|\langle \psi(p_1', \ldots, p_m'), U\psi(p_1, \ldots, p_n)\rangle|^2$$

Evanescent waves

Wave equations generally have two types of solutions:

propagating waves

oscillating in all spatial directions:

$$\boxed{e^{\mathrm{i}(Et-p_1x_1-p_2x_2-p_3x_3)}}$$

$$E^2 = p_1^2 + p_2^2 + p_3^2 + m^2$$

evanescent waves

exponentially increasing/decaying in one or more spatial directions:

$$\boxed{e^{\mathrm{i}(Et-p_2x_2-p_3x_3)-\tilde{p}_1x_1}}$$

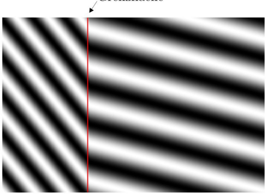
$$E^2 = -\tilde{p}_1^2 + p_2^2 + p_3^2 + m^2$$

Applications: **electromagnetic waves**, **acoustic waves**, etc.

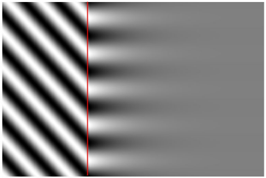
For simplicity in this lecture:
Relativistic scalar waves (Klein-Gordon equation)

Evanescent Waves

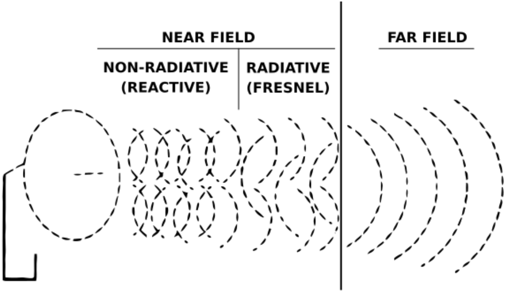
Grenzfläche



gebeugte Welle



evanescente Welle
(der totalreflektierte Anteil wurde
nicht dargestellt)



Including evanescent waves

[Carniglia and Mandel 1971]
One possibility to include electromagnetic evanescent waves in the quantization is to model media with different refractive indices in space. Then, global solutions can exist, which consist of evanescent waves in parts of space.

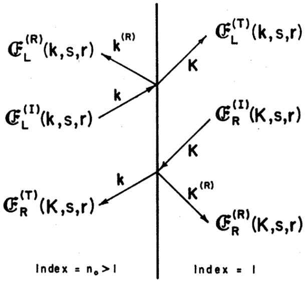


FIG. 1. Illustrating the notation for the incident, reflected, and transmitted components of each mode. All modes are labeled by the wave vector of the incident wave. For waves incident from the left the wave vector **k** is in the dielectric; for waves incident from the right the wave vector **K** is in vacuum. Although electric fields were chosen for illustration, the notation is similar for the magnetic fields.

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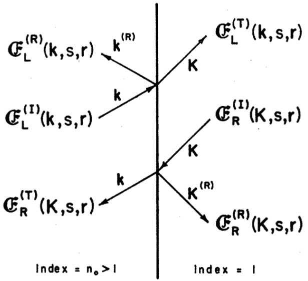

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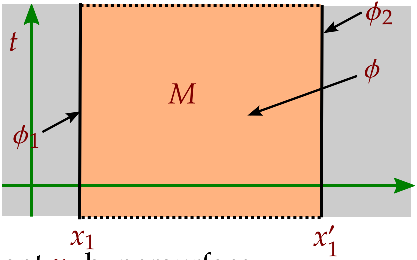
Limitations:

- Applies only in very special situations (background medium)
- No description of evanescent quanta alone

Consider hypersurfaces with constant x_1 coordinate and corresponding **space-interval regions**.



Parametrize solution near constant x_1 hypersurface,

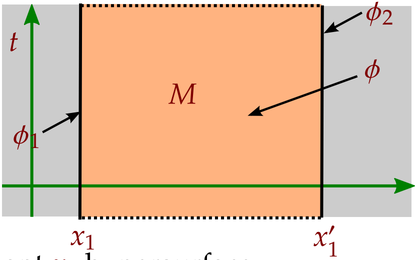
$$\phi(t, x_1, \tilde{x}) = \int_{E^2 > \tilde{k}^2 + m^2} \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1} \left(\phi(E, \tilde{k}) e^{-\mathrm{i}(Et - \tilde{k}\tilde{x} - k_1 x_1)} + \overline{\phi(E, \tilde{k})}\, e^{\mathrm{i}(Et - \tilde{k}\tilde{x} - k_1 x_1)} \right)$$

where $\tilde{x} := (x_2, x_3)$, $\tilde{k} := (k_2, k_3)$, $k_1 := \sqrt{|E^2 - \tilde{k}^2 - m^2|}$.
Note that the sign of E can be negative.

Klein-Gordon Theory

Timelike Hyperplanes (I)

Consider hypersurfaces with constant x_1 coordinate and corresponding **space-interval regions**.



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where $\tilde{x} := (x_2, x_3)$, $\tilde{k} := (k_2, k_3)$, $k_1 := \sqrt{|E^2 - \tilde{k}^2 - m^2|}$.
Note that the sign of E can be negative.

These are the **propagating waves**: $E^2 > \tilde{k}^2 + m^2$, oscillate in space

Klein-Gordon Theory

Timelike Hyperplanes (II)

There are also **evanescent waves**: $E^2 < \tilde{k}^2 + m^2$, exponential in space

$$\phi(t, x_1, \tilde{x}) = \int_{E^2<\tilde{k}^2+m^2} \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1} \left(\phi_+(E,\tilde{k})e^{k_1 x_1} + \phi_-(E,\tilde{k})e^{-k_1 x_1}\right) e^{\mathrm{i}(Et-\tilde{k}\tilde{x})},$$

with $\phi_\pm(E,\tilde{k}) = \overline{\phi_\pm(-E,-\tilde{k})}$.

Klein-Gordon Theory

Timelike Hyperplanes (II)

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with $\phi_\pm(E, \tilde{k}) = \overline{\phi_\pm(-E, -\tilde{k})}$.

The space of solutions decomposes as $L_{x_1} = L^{\mathrm{p}}_{x_1} \oplus L^{\mathrm{e}}_{x_1}$.
The space of states is a tensor product $\mathcal{H}_{x_1} = \mathcal{H}^{\mathrm{p}}_{x_1} \otimes \mathcal{H}^{\mathrm{e}}_{x_1}$.

Klein-Gordon Theory

Timelike Hyperplanes (III)

The construction of $\mathcal{H}^{\mathrm{p}}_{x_1}$ based on $L^{\mathrm{p}}_{x_1}$ parallels the spacelike case.

The Lagrangian gives rise to the **symplectic form**,

$$\begin{aligned}\omega_{x_1}(\phi_1,\phi_2) &= -\frac{1}{2}\int \mathrm{d}^3x\ \left(\phi_2(t,x)\partial_{x_1}\phi_1(t,x) - \phi_1(t,x)\partial_{x_1}\phi_2(t,x)\right)\\ &= \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1}\left(\phi_2(E,\tilde{k})\overline{\phi_1(E,\tilde{k})} - \phi_1(E,\tilde{k})\overline{\phi_2(E,\tilde{k})}\right).\end{aligned}$$

The standard **complex structure** is,

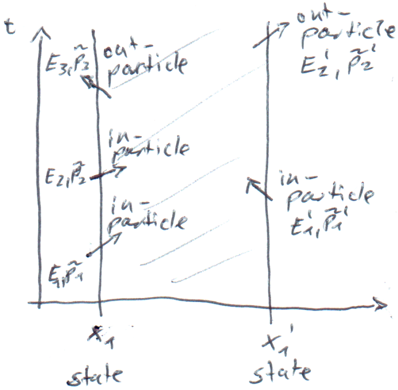
$$(J(\phi))(E,\tilde{k}) = -\mathrm{i}\phi(E,\tilde{k}).$$

This yields the **complex inner product**,

$$\{\phi_1,\phi_2\}_{x_1} = 2\int \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1}\,\phi_1(E,\tilde{k})\overline{\phi_2(E,\tilde{k})}.$$

Klein-Gordon Theory

Timelike Hypersurfaces – scattering



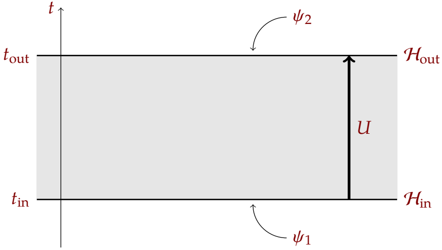
Excluding evanescent degrees of freedom:

Particles can be characterized by 3 **quantum numbers**: the momenta k_2, k_3 and the energy E. Recall that E may be negative. This yields the **same degrees of freedom** as in the spacelike case.

But, in contrast to the spacelike case there is no notion of **in-state** or **out-state**. Rather each particle in a multi-particle state might individually be either **in-going** or **out-going**. This is what the **sign of the energy** E encodes.

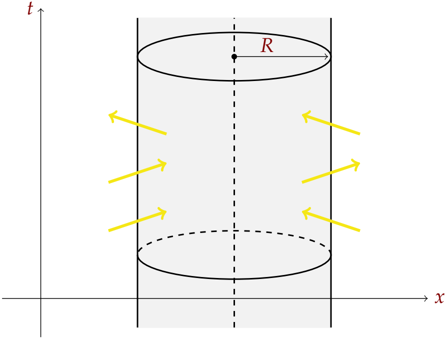
Description of scattering experiments

conventional description



evolution in terms of operator
$U : \mathcal{H}_{\text{in}} \to \mathcal{H}_{\text{out}}$
in terms of transition amplitudes
$\mathcal{H}_{\text{in}} \otimes \mathcal{H}^*_{\text{out}} \to \mathbb{C}$

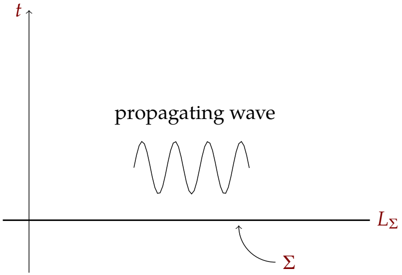
alternative description



amplitude: $\mathcal{H}_R \to \mathbb{C}$

Solutions in classical field theory

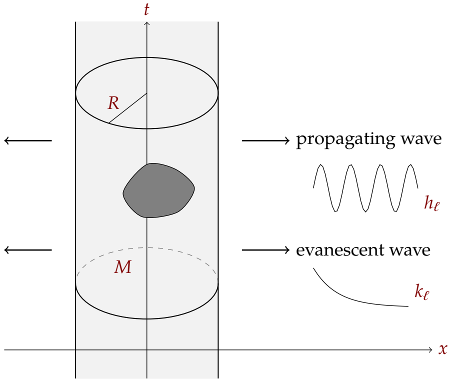
hypersurface Σ at time t



Cauchy property:
germs in $L_\Sigma \cong$ global solutions

propagating waves

hypersurface at radius R



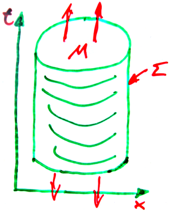
germs in $L_R >$ global solutions

propagating waves
+ evanescent waves

Klein-Gordon Theory

Timelike Hypercylinder (I)

Consider a **hypercylinder** given by a **sphere** of radius R in space, extended over all of time.



Parametrize **propagating** solutions ($E^2 > m^2$) near constant R hypersurface, ($l = 0, 1, \ldots,$ $m = -l, -l+1, \ldots, l$)

$$\phi(t, r, \Omega) = \int_{|E|>m} \mathrm{d}E \, \frac{p}{4\pi} \sum_{l,m} \Big(\phi_{l,m}(E) h_l(pr) e^{-\mathrm{i}Et} Y_l^m(\Omega) + \overline{\phi_{l,m}(E)} \, \overline{h_l(pr)} e^{\mathrm{i}Et} Y_l^{-m}(\Omega) \Big) .$$

Here Y_l^m denote the **spherical harmonics** and $p := \sqrt{|E^2 - m^2|}$. Also, $h_l = j_l + \mathrm{i} n_l$, where j_l and n_l are the **spherical Bessel functions** of the **first** and **second kind** respectively.

In the massive case $m > 0$, there are also **evanescent** solutions for $E^2 < m^2$, with exponential behaviour in space.

$$\phi(t, r, \Omega) = \int_{-m}^{m} \mathrm{d}E \, \frac{p}{4\pi} e^{-\mathrm{i}Et} \sum_{l,m} Y_l^m(\Omega) \left(\phi^{\mathrm{x}}_{l,m}(E) k_l(pr) + \phi^{\mathrm{i}}_{l,m}(E) \tilde{k}_l(pr) \right) . \tag{1}$$

Here Y_l^m denote the **spherical harmonics** and $p := \sqrt{|E^2 - m^2|}$. Also, $k_l(z) = -\mathrm{i}^l \pi h_l(\mathrm{i}z)/2$ and $\tilde{k}_l(z) = k_l(-z)$ are **modified spherical Bessel functions** that are real on $\mathbb{R}$.

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The space of solutions decomposes as $L_R = L_R^{\mathrm{p}} \oplus L_R^{\mathrm{e}}$.
The space of states is a tensor product $\mathcal{H}_R = \mathcal{H}_R^{\mathrm{p}} \otimes \mathcal{H}_R^{\mathrm{e}}$.

Klein-Gordon Theory

Timelike Hypercylinder (III)

Consider only **propagating solutions** at first.

The Lagrangian gives rise to the **symplectic form**,

$$\begin{aligned}\omega_R(\phi,\xi) &= \frac{R^2}{2}\int \mathrm{d}t\,\mathrm{d}\Omega\ \left(\xi(t,R,\Omega)\partial_r\phi(t,R,\Omega) - \phi(t,R,\Omega)\partial_r\xi(t,R,\Omega)\right)\\ &= \int \mathrm{d}E\,\frac{\mathrm{i}p}{8\pi}\sum_{l,m}\left(\phi_{l,m}(E)\overline{\xi_{l,m}(E)} - \overline{\phi_{l,m}(E)}\xi_{l,m}(E)\right).\end{aligned}$$

The standard **complex structure** is,

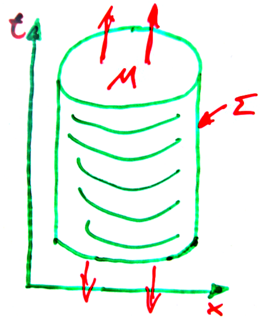
$$(J(\phi))_{l,m}(E) = \mathrm{i}\phi_{l,m}(E).$$

This yields the **complex inner product**,

$$\{\phi,\xi\}_R = \int \mathrm{d}E\,\frac{p}{2\pi}\sum_{l,m}\overline{\phi_{l,m}(E)}\xi_{l,m}(E).$$

Klein-Gordon Theory

Timelike Hypercylinder



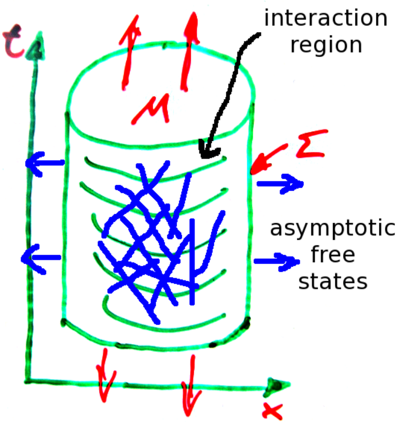
To go beyond standard transition amplitudes, consider an example with a connected boundary. [RO 2005]

- $M = \mathbb{R} \times B^3_R$.
- $\partial M = \Sigma_R = \mathbb{R} \times S^2_R$.

(Consider propagating waves only.)

- The state space $\mathcal{H}^{\mathrm{p}}_R$ is again a **Fock space**.
- A particle can be characterized by three quantum numbers: **energy** E and **angular momentum** l, m.
- The sign of the energy determines if a particle is **in-going** or **out-going**. The state space decomposes as $\mathcal{H}^{\mathrm{p}}_R = \mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_{\mathrm{out}}$.
- This decomposition is neither geometrical nor temporal.

Spatially asymptotic S-matrix (I)



Similarly, we can describe interacting QFT via a spatially asymptotic amplitude. Assume interaction is relevant only within a radius R from the origin in space (but at all times). Consider then the asymptotic limit of the amplitude of a free state on the hypercylinder when the radius goes to infinity:

$$\boxed{S(\psi) = \lim_{R\to\infty} \rho_R(\psi)}$$

[D. Colosi, RO 2007–2008]

The evanescent sector vanishes in the limit.

Spatially asymptotic S-matrix (II)

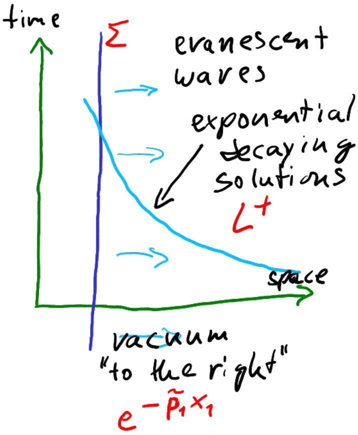
Results:

- The **perturbative description of interactions** works as in the standard path integral and S-matrix picture. Technically, the interactions are introduced via **sources**. In the hypercylinder geometry, this involves **evanescent modes** in an essential way, even if they vanish asymptotically.
- The S-matrices are equivalent when the interaction is confined in space and time. This equivalence is realized through an **isomorphism of** the asymptotic state spaces.
- In the standard formulation, **crossing symmetry** is an emergent feature of the S-matrix. In the hypercylinder setting of CQFT crossing symmetry is manifest.

Vacuum and asymptotic boundary conditions

How to choose L^+? [RO, D. Colosi 2019]

L^+ determines <u>asymptotic boundary conditions</u>



Vacuum and asymptotic boundary conditions

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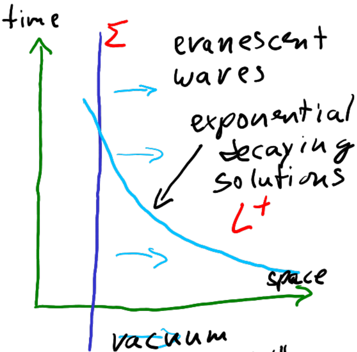
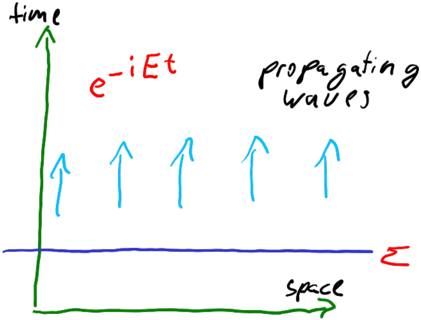


vacuum
"to the right"

$e^{-\tilde{p}_1 x_1}$



vacuum
"to the future"

e^{-iEt} $\boxed{\text{wick rotate}}$ $e^{-iE(-i\tau)} = e^{-E\tau}$

Kähler quantization and its limits

$L^+ \subseteq L^{\mathbb{C}}$ is always a **Lagrangian subspace**.

propagating waves

For propagating waves, L^+ is always **positive-definite**, both for spacelike and timelike hypersurfaces. Quantization proceeds as usual.

evanescent waves

For evanescent waves, L^+ is **null**. No positive-definite inner product suitable for quantization is obtained.

α-Kähler quantization

[D. Colosi, RO 2020]
$L^{\pm} \subseteq L^{\mathbb{C}}$ complementary Lagrangian subspaces. $L^{\mathbb{C}} = L^{+} \oplus L^{-}$.

What if this is not a Kähler polarization?
(i.e., L^{+} is not positive-definite and $L^{-} \neq \overline{L^{+}}$)

α-Kähler quantization

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What if this is not a Kähler polarization?
(i.e., L^{+} is not positive-definite and $L^{-} \neq \overline{L^{+}}$)

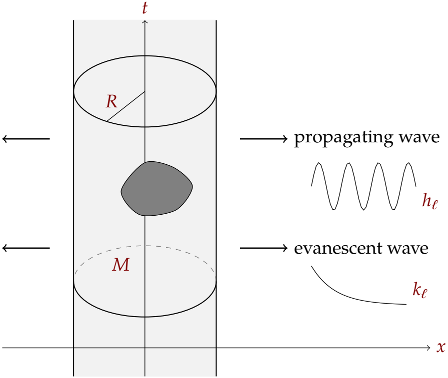
Positive-definite real structure α

- $\alpha : L^{\mathbb{C}} \to L^{\mathbb{C}}$ anti-linear involution.
- $\alpha(L^{\pm}) = \alpha(L^{\mp})$
- $\omega(\alpha(\phi), \alpha(\eta)) = \overline{\omega(\phi, \eta)}$
- $(\phi, \eta)^{\alpha} := 4\mathrm{i}\omega(\alpha(\phi), \eta)$ positive-definite on L^{+}.

- α has the interpretation of a modified **complex conjugation**
- induces a **twisted $*$-structure** on the **algebra of slice observables**
- generalizes **reflection-positivity** in Euclidean QFT

Application: Scattering at finite distance

experiment confined at $r < R$



Probe interior with both **propagating** and **evanescent** waves.

Examples:

- refractive medium
- Unruh-deWitt detector (next slides)
- black hole (ongoing project)
- ...

Emission of evanescent particles by UDW detector

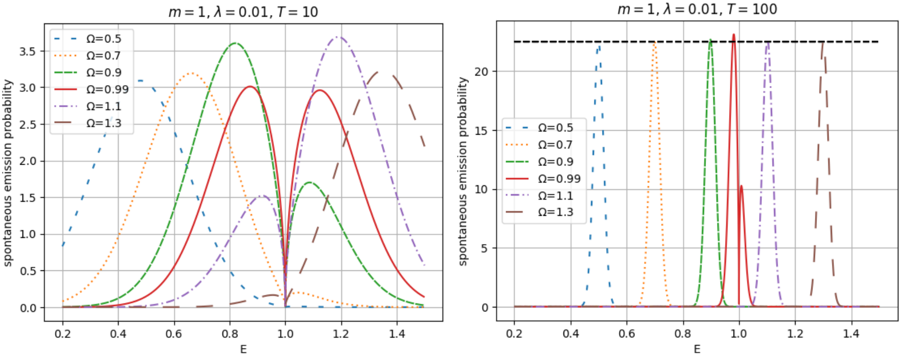


Figure: Emission spectrum, namely probability per unit energy, for different values of the detector energy gap Ω expressed in units of the mass of the field. In the left-hand (right-hand) plot the time T takes the value 10 (100). The coupling constant λ has been set equal to 0.01. [D. Colosi, RO 2023]

Emission of evanescent particles by UDW detector

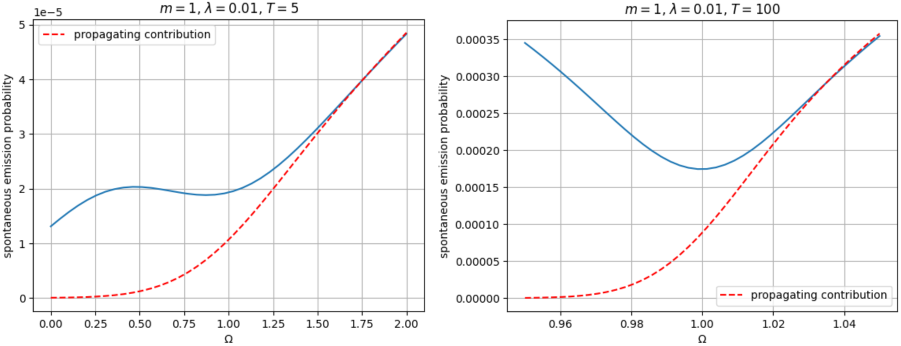


Figure: Spontaneous emission probability as a function of the detector energy gap Ω, at $\lambda = 0.01$. In addition to the emission probability for the radial picture (solid line), the emission probability for the temporal picture (dashed line) is also indicated. The characteristic time T is 5 (left-hand plot) and is 100 (right-hand plot). [D. Colosi, RO 2023]