Quantization of bosonic linear field theory

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Canonical quantization in curved spacetime (review)

- L space of germs of solutions of the equations of motion (a real vector space). $L^{\mathbb{C}}$ complexification.
- $\omega : L \times L \to \mathbb{R}$ symplectic form – a bilinear antisymmetric form
- Define sesquilinear form $\boxed{(\phi, \phi') := 4\mathrm{i}\omega(\overline{\phi}, \phi')}$
- A **quantization** is determined by a complete set of "positive energy" modes $\{u_k\}_{k\in I}$:

$$(u_k, u_l) = \delta_{k,l}, \quad (\overline{u}_k, \overline{u}_l) = -\delta_{k,l}, \quad (u_k, \overline{u}_l) = 0, \quad \forall k, l \in I.$$

- $L^+ \subseteq L^{\mathbb{C}}$, $L^- \subseteq L^{\mathbb{C}}$ subspaces generated by the modes u_k, $\overline{u}_k$. Have $L^{\mathbb{C}} = L^+ \oplus L^-$ and $L^- = \overline{L^+}$.
- postulate corresponding creation and annihilation operators:

$$[a_k, a_l] = 0, \quad [a_k^\dagger, a_l^\dagger] = 0, \quad [a_k, a_l^\dagger] = \delta_{k,l}.$$

This determines the state space $\mathcal{H}$ as a **Fock space**.

Positive-definite Lagrangian subspaces

Choice of vacuum corresponds to choice of **positive-definite Lagrangian subspace** L^+:

1. $L^+ \subseteq L^{\mathbb{C}}$ is Lagrangian subspace
2. L^+ is positive-definite with respect to $(\cdot, \cdot)$

$\{u_k\}_{k \in I}$ is just an ON-basis of L^+.

Lagrangian means

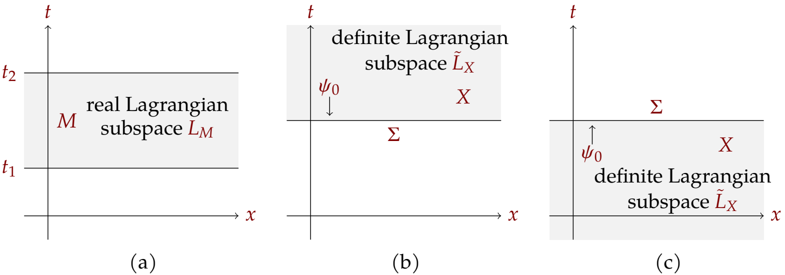
$$\begin{aligned} \textbf{isotropic}\text{, i.e.,} \quad & \omega(\phi, \eta) = 0, \quad \forall \phi, \eta \in L^+, \\ \text{and } \textbf{coisotropic}\text{, i.e.,} \quad & \omega(\phi, \eta) = 0, \quad \forall \phi \in L^+ \Longrightarrow \eta \in L^+ \end{aligned}$$

L^+ has a **complement** $L^- := \overline{L^+}$, a negative-definite Lagrangian subspace. Let J be the operator with eigenspaces L^+ and L^- and eigenvalues i and $-\mathrm{i}$. Then, J defines a **complex structure** on L with respect to which

$$\{\phi, \eta\} := (\phi^+, \eta^+) = g(\phi, \eta) + 2\mathrm{i}\omega(\phi, \eta), \qquad g(\phi, \eta) := 2\omega(\phi, J\eta)$$

is a **positive-definite inner product** on (L, J).

Extended classical axioms



The classical axioms extend to general non-compact regions.

- For **compact** regions $L_M \subseteq L^{\mathbb{C}}_{\partial M}$ is the complexification of a **real Lagrangian subspace**.
- For **non-compact** regions $L_M \subseteq L^{\mathbb{C}}_{\partial M}$ is a general **complex Lagrangian subspace**.

[D. Colosi, RO 2019]

Holomorphic quantization: State spaces

In textbook QFT, the state space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is usually taken as the **Fock space** over the Hilbert space $(L, \{\cdot,\cdot\})$. Another possibility we have seen is the **Schrödinger representation**.

Here, we construct $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ as the space of **square-integrable holomorphic** functions on L with respect to a **Gaussian measure** ν. This is the **holomorphic representation**.

$$\langle\psi',\psi\rangle := \int_{\hat{L}} \overline{\psi'(\phi)}\psi(\phi)\,\mathrm{d}\nu(\phi),$$

where ν is the **Gaussian measure**, with,

$$\mathrm{d}\nu(\phi) \approx \exp\left(-\frac{1}{2}\Re\{\phi,\phi\}\right)\mathrm{d}\mu(\phi).$$

Coherent States

The Hilbert space $\mathcal{H}$ is a reproducing kernel Hilbert space and contains **coherent states** of the form

$$K_\xi(\phi) = \exp\left(\frac{1}{2}\{\xi, \phi\}\right)$$

associated to germs of solutions $\xi \in L$. They have the **reproducing property**,

$$\langle K_\xi, \psi \rangle = \psi(\xi),$$

and satisfy the **completeness relation**

$$\langle \psi', \psi \rangle = \int_{\hat{L}} \langle \psi', K_\xi \rangle \langle K_\xi, \psi \rangle \, \mathrm{d}\nu(\xi).$$

They can be thought of as representing quantum states that approximate specific classical solutions.

Amplitudes

Consider a spacetime region M. Let L_M be the **space of solutions** in M. There is a map $L_M \to L_{\partial M}$. Denote the image by $L_{\tilde{M}}$.

Recall that $L_{\tilde{M}} \subseteq L_{\partial M}$ is a **Lagrangian subspace**. Then,

$L_{\partial M} = L_{\tilde{M}} \oplus J_{\partial M} L_{\tilde{M}}$, orthogonal with respect to $g_{\partial M}$.

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Write $\xi = \xi^{\mathrm{R}} + \xi^{\mathrm{I}}$. Then, ($k_\xi$ normalized version of K_ξ) [RO 2010]

$$\boxed{\rho_M(k_\xi) = \exp\left(-\mathrm{i}\omega_{\partial M}(\xi^{\mathrm{R}}, \xi^{\mathrm{I}}) - \frac{1}{2} g_{\partial M}(\xi^{\mathrm{I}}, \xi^{\mathrm{I}})\right)}.$$