Towards Quantum Gravity

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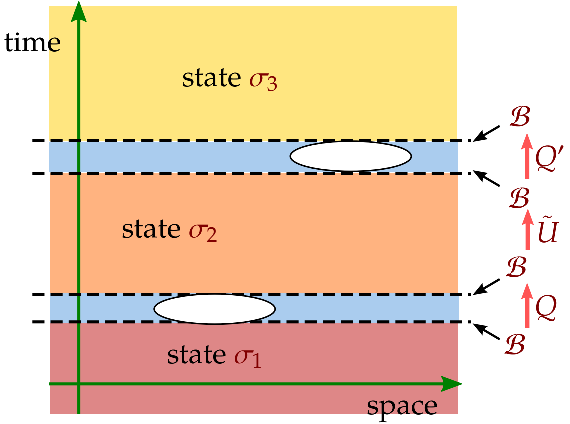
Lecture 21 – 28 May 2025

Measurement in the standard formulation and time

- **Measurements** are encoded by **quantum operations** Q, Q'.
- These are **superoperators** on the space $\mathcal{B}$ of **mixed states**.
- **Unitary dynamics** is also encoded by **superoperators** $\tilde{U}$ on $\mathcal{B}$.

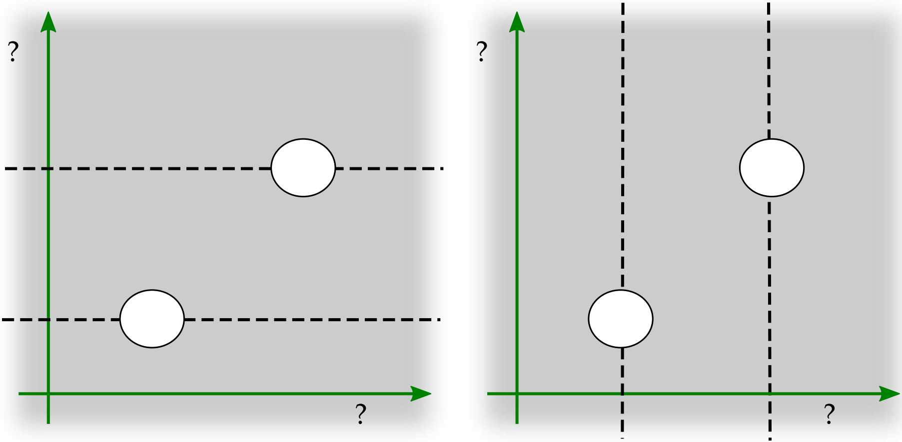


The product $Q' \circ \tilde{U} \circ Q$ encodes **joint measurement** and evolution. Its order is the **temporal order** of processes.

Time plays a special role!

Quantum theory without spacetime metric?

If spacetime is dynamical, as in a **general relativistic** setting, there is no a priori metric “separating” space and time. What do we do then?



The **standard formulation of quantum theory** breaks down.

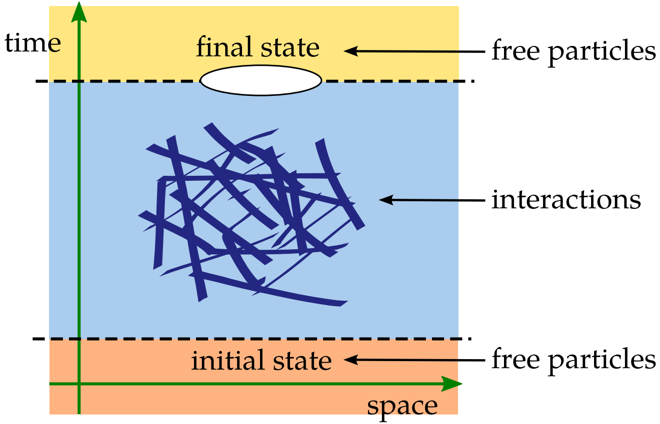
How do we do quantum gravity? (I)

Traditionally three lines of attack have been followed:

1. **Do what you can do:** Measure in classical spacetime.
2. **Just go for it:** Ignore the problems (for now).
3. **Quantum theory is wrong:** There.

Asymptotic measurement: QFT

Consider measurement only at **asymptotic infinity**, infinitely early and infinitely late time, described by **transition probabilities**. This is how the **S-matrix** in **quantum field theory** works to describe **scattering processes**. This requires **perturbation theory**.

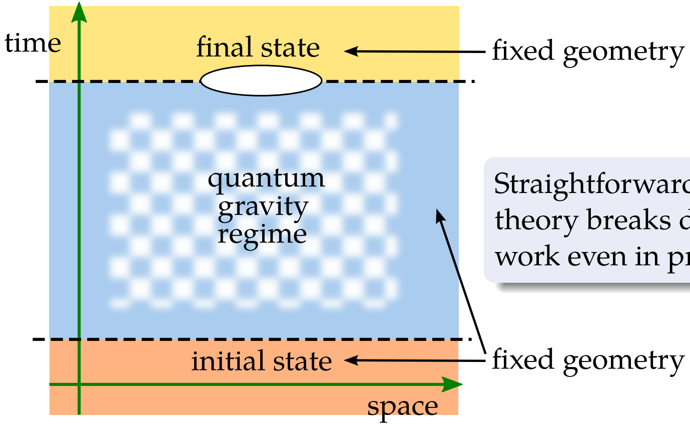


At early and late times particles are far apart and do not interact. The interesting physics happens at **intermediate times**.

Asymptotic measurement: QG

Fix an approximate **classical metric background** at **asymptotic infinity**. Observations take place exclusively in this region. This requires **perturbation theory** in the **metric**.

(Perturbative Quantum Gravity, String Theory)



Straightforward perturbation theory breaks down. Can this work even in principle?

How do we do quantum gravity? (II)

Traditionally three lines of attack have been followed:

2. Just go for it: Keep the mathematics of the **standard formulation**, but throw away the background metric and with it the physical content. Focus on the **mathematical objects**: Hilbert spaces, a Hamiltonian, observables as operators. Use **canonical quantization** to construct these. Hope that in some future an operational connection of these objects with physical reality can be established. (Quantum Geometrodynamics, Loop Quantum Gravity)

So far no such connection has been proposed. There might be none.

How do we do quantum gravity? (III)

Traditionally three lines of attack have been followed:

3. Quantum theory is wrong: Quantum theory as we know it is fundamentally limited and must be replaced by some different underlying theory. Known physics is modified. (Causal sets, Gravity induced collapse models)

There is no evidence for violations of quantum theory as we know it. Also, it is difficult to reinvent physics from scratch and still reproduce known results to high precision.

We can do better...

...by using the **positive formalism** which compared to the **standard formulation of quantum theory** is,

- **more fundamental:** the standard formulation is recovered when appropriate, known physics is not modified
- **timeless:** does not require a notion of time
- **local:** implements manifest spacetime locality without metric
- **operational:** recovers and generalizes quantum measurement theory

Quantum gravity in the GBF

1. Perturbative quantum gravity: This depends on the integration of QFT with the GBF. If finite regions can be described successfully, this might yield new insight into this approach. But there is no reason to expect improvement of the **non-renormalizability** issue.

Quantum gravity in the GBF

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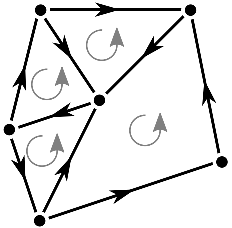
2. Spin foam quantum gravity: **Spin foam models** arise naturally from a **path integral** picture. Also, they naturally describe **finite regions** of spacetime. This suggests their interpretation as background independent quantum theories in terms of the GBF.
3. A functorial top-down approach: The mathematical structure of CQFT that is part of the GBF also suggests a top-down approach: Guided by **axiomatics**, **functoriality**, and **representation theory** and with a minimum of assumptions explore the **theory space**.

Spin foam models

Spin foam models provide a **quantization** for a range of **gauge theories**. They are based on a discretization of spacetime in terms of **cell complexes** or **triangulations**.

- **Hypersurfaces** are 3-dimensional oriented cell complexes
- **Regions** are 4-dimensional oriented cell complexes

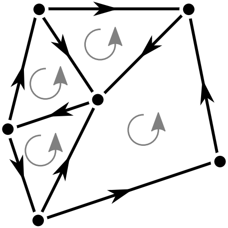


Gauge fields on hypersurfaces are encoded in terms of **holonomies** between cells. The Schrödinger quantization of this space leads to a basis in terms of **spin networks**.

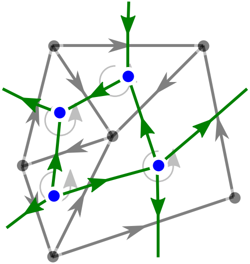
The dynamics in regions is quantized via a Feynman path integral. This yields **spin foam** amplitudes.

Discretized connections I

To construct the Hilbert space $\mathcal{H}_\Sigma$ for the hypersurface Σ, we need to **quantize** the **space of connections** on Σ. Σ is discretized in terms of a **cellular decomposition**.

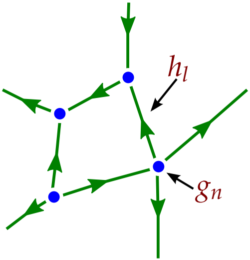


Given a "gauge" (local trivialization), connections give rise to **holonomies** along paths. We choose paths dual to the cellular decomposition. We call them **links** (green lines). Their end points are **nodes** (blue dots).

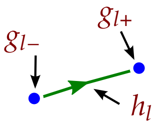


Discretized connections II

The **holonomies** associate one element h_l of the **structure group** G to each **link** l. We denote this space by $K^1_\Sigma = G^L$, where L is the number of links in Σ.



A **gauge transformation** consists of the assignment of one element g_n of G to each **node** n. The **gauge group** is thus $K^0_\Sigma = G^N$, where N is the number of nodes.



A gauge transformation $g \in K^0_\Sigma$ acts on $h \in K^1_\Sigma$ via $(g \triangleright h)_l := g_{l+} h_l g^{-1}_{l-}$. The **configuration space** is the quotient $K_\Sigma := K^1_\Sigma / K^0_\Sigma$.

State space

Supposing that G is compact for simplicity, there is a unique normalized biinvariant measure on G, the **Haar measure** μ. This allows to define a Hilbert space $\mathrm{L}^2(G)$ of complex functions on G with the inner product,

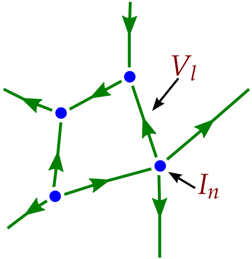
$$\langle \psi, \eta \rangle = \int_G \overline{\psi(g)} \eta(g) \, \mathrm{d}\mu(g).$$

By putting the same inner product on each copy of G, we obtain a Hilbert space $\mathcal{H}^1_\Sigma := \mathrm{L}^2(K^1_\Sigma)$. The action of the gauge group K^0_Σ on K^1_Σ induces an action on $\mathcal{H}^1_\Sigma$. The subspace $\mathcal{H}_\Sigma \subseteq \mathcal{H}^1_\Sigma$ of invariant functions on K^1_Σ can be identified with a space of functions on the configuration space K_Σ. This Hilbert space is our **state space**.

The dual picture: spin networks

The Hilbert space $\mathcal{H}_\Sigma$ on the cellular hypersurface Σ can be constructed explicitly in terms of **spin networks**.

- Associate to each **link** l a finite-dimensional irreducible representation V_l of G.
- Associate to each **node** n an intertwiner $I_n \in \mathrm{Inv}\left(\bigotimes_{l\in\partial n} V_l^\pm\right)$ between the representations of the adjacent nodes.



Spin networks yield a complete description of $\mathcal{H}_\Sigma$:

$$\mathcal{H}_\Sigma = \bigoplus_{V_l} \bigotimes_{n\in\Sigma} \mathrm{Inv}\left(\bigotimes_{l\in\partial n} V_l^\pm\right).$$

A simple model: BF-theory

We start with the Palatini action of gravity,

$$S_M^{\text{Palatini}}(e, A) = \int_M \text{tr}(e \wedge e \wedge F).$$

- A – connection with gauge group $\text{Spin}(1,3) = \text{SL}(2, \mathbb{C})$
- F – curvature 2-form of the connection A
- e – 4-bein frame field

To simplify this theory we replace $e \wedge e$ with the Lie algebra valued 2-form field B. This yields BF theory,

$$S_M^{\text{BF}}(B, A) = \int_M \text{tr}(B \wedge F).$$

This is not gravity, but becomes gravity if we add certain **constraints**.

Amplitude

The Feynman path integral determining the **amplitude** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ in the region M can be encoded in terms of the **propagator** Z_M.

Amplitude map

$\rho_M(\psi) = \int_{K^1_\Sigma} \psi(h)\, Z_M(h^{-1})\, \mathrm{d}\mu(h)$

Here, it is simpler to think of the **propagator** as a function $Z_M : K^1_{\partial M} \to \mathbb{C}$ rather than a function $K_{\partial M} \to \mathbb{C}$.

For BF theory the propagator turns out to be,

$$\tilde{Z}^{\mathrm{BF}}_M(h) = \prod_{l \in \partial M} \delta(h_l).$$

In gauge invariant form this is,

$$Z^{\mathrm{BF}}_M(h) = \int_{K^0_{\partial M}} \prod_{l \in \partial M} \delta(g_{l-} h_l g_{l+}^{-1})\, \mathrm{d}\mu(g).$$

Regions and spin foams

For a **topological theory** like BF-theory, the amplitude for a region M is simple. In general, we take advantage of the CQFT **gluing rule** to compute amplitudes. As M is composed of many elementary regions (cells) we need only know the amplitude ρ_C for an elementary cell C, also called **vertex amplitude**. Taking a basis consisting of **spin networks**, we obtain a **spin foam**.

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A famous model for implementing the constraints is the **Barrett-Crane model**. In this model $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and we write $g = (g^{\mathrm{L}}, g^{\mathrm{R}})$. The cell propagator for (a version of) this model is,

$$Z_C^{\mathrm{BC}}(h) = \int_{K^0_{\partial C}} \prod_{l \in \partial C} \left(\int_{\mathrm{SU}(2)\times\mathrm{SU}(2)} \delta(g^{\mathrm{L}}_{l-} k h^{\mathrm{L}}_l k' (g^{\mathrm{L}}_{l+})^{-1}) \delta(g^{\mathrm{R}}_{l-} k h^{\mathrm{R}}_l k' (g^{\mathrm{R}}_{l+})^{-1}) \, \mathrm{d}\mu(k) \mathrm{d}\mu(k') \right) \mathrm{d}\mu(g).$$

Spin foam summary (I)

- Spacetime **hypersurfaces** are 3-dimensional cell complexes
- Spacetime **regions** are 4-dimensional cell complexes
- **Gauge fields** on hypersurfaces are encoded in terms of **holonomies** between cells
- The **state spaces** on hypersurfaces can be described in terms of **spin networks** (with ends!)
- A simple spin foam model is completely determined by its **cell (vertex) amplitudes**
- Spin foam **partition functions**, **amplitudes** etc. then follow from the **gluing rules**

Operators

Denote $\mathcal{B}_\Sigma$ the space of operators on $\mathcal{H}_\Sigma$. Essentially,

$$\mathcal{B}_\Sigma = \mathcal{H}_\Sigma \otimes \mathcal{H}_\Sigma^* = \mathcal{H}_\Sigma \otimes \mathcal{H}_{\overline{\Sigma}} = \mathrm{L}^2(K_\Sigma) \otimes \mathrm{L}^2(K_{\overline{\Sigma}}) = \mathrm{L}^2(K_\Sigma \times K_{\overline{\Sigma}})$$

In the **connection representation**:

- $\mathcal{H}_\Sigma$ consists of gauge-invariant functions on $K_\Sigma^1 = G^L$.
- $\mathcal{B}_\Sigma$ consists of gauge-invariant functions on $K_\Sigma^1 \times K_{\overline{\Sigma}}^1 = G^{2L}$.

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$\Psi \in \mathcal{B}_\Sigma$ acts on $\tau \in \mathcal{H}_\Sigma$ as,

$$(\Psi \triangleright \tau)(g) = \int_{K_\Sigma^1} \Psi(g,h)\tau(h^{-1})\,\mathrm{d}\mu(h).$$

The **identity operator** is $\mathbf{1}(g,h) = \delta(gh^{-1})$. The operator product is,

$$(\Psi'\Psi)(g,h) = \int_{K_\Sigma^1} \Psi'(g,k)\Psi(k^{-1},h)\,\mathrm{d}\mu(k).$$

Operators

The **adjoint** $\Psi^\dagger$ of an operator Ψ is,

$$\Psi^\dagger(g,h) = \overline{\Psi(h^{-1},g^{-1})}$$

The **trace** of an operator is,

$$\mathrm{tr}(\Psi) = \int_{K^1_\Sigma} \Psi(h,h^{-1})\,\mathrm{d}\mu(h).$$

The **inner product** in $\mathcal{B}_\Sigma$ is the **Hilbert-Schmidt** inner product,

$$(\!|\Psi',\Psi|\!)_\Sigma := \mathrm{tr}(\Psi'^\dagger\Psi) = \int_{K^1_\Sigma\times K^1_\Sigma} \overline{\Psi'(g,h)}\Psi(g,h)\,\mathrm{d}\mu(g)\mathrm{d}\mu(h).$$

An operator $\Psi \in \mathcal{B}_\Sigma$ is **positive** if for any $\eta \in \mathcal{H}_\Sigma$,

$$\langle\eta,\Psi\eta\rangle = \int_{K^1_\Sigma\times K^1_\Sigma} \Psi(g,h)\,\overline{\eta(g)}\,\eta(h^{-1})\,\mathrm{d}\mu(g)\mathrm{d}\mu(h) \geq 0.$$

General states and null probes

General states

(**density matrices** or **density operators**) are **positive** operators in $\mathcal{B}_\Sigma$.

(In the standard formulation they also need to have unit trace.)

Pure states

are states of the form $\Psi(g,h) = \psi(g)\overline{\psi(h^{-1})}$ with $\psi \in \mathcal{H}_\Sigma$.

Null probe

$$[\![\boxslash, \Psi]\!]_M = \int_{K^1_\Sigma \times K^1_\Sigma} \Psi(g,h)\, Z_M(g^{-1})\, \overline{Z_M(h)}\, \mathrm{d}\mu(g)\mathrm{d}\mu(h)$$

- If Ψ is positive, then $[\![\boxslash, \Psi]\!]_M \geq 0$.
- If Ψ is a pure state given by $\psi \in \mathcal{H}_\Sigma$, then $[\![\boxslash, \Psi]\!]_M = |\rho_M(\psi)|^2$.

From observables to general probes

To work in the positive formalism we have to convert **observables** into **quantum operations / probes** via their **spectral decomposition**.

Example: Area operator in loop quantum gravity

$G = SU(2)$, representations are labeled by half-integer spins j. Spin network state are **eigenstates**.

The area operator a_τ is associated to a 2-dimensional surface τ within the 3-dimensional hypersurface Σ. For a spin network state ψ its **eigenvalue** is,

$$a_\tau \psi = c\, L_{\text{Planck}}^2 \sum_{l \in \tau} \sqrt{j_l (j_{l+1})}\, \psi .$$

To construct the corresponding operation we have to extract the projectors onto the subspaces of spin network states with given area a, $a_\tau = \sum_a a\, P_a$.

Spin foam summary (II)

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- A simple spin foam model is completely determined by its **cell (vertex) amplitudes**
- Spin foam **partition functions**, **amplitudes** etc. then follow from the **gluing rules**
- (NEW) Adequate notions of **operators**, **mixed state spaces** and **null probes** exist
- (NEW) General **probes** are to be constructed...

A top-down approach to quantum gravity

List properties expected of a quantum theory of gravity and **construct**/**classify** models with these properties.

- In GR the metric is dynamical, but **differentiable** or **topological** structure may be fixed. Need ball-shaped regions for local physics.

A top-down approach to quantum gravity

List properties expected of a quantum theory of gravity and **construct**/**classify** models with these properties.

- In GR the metric is dynamical, but **differentiable** or **topological** structure may be fixed. Need ball-shaped regions for local physics.

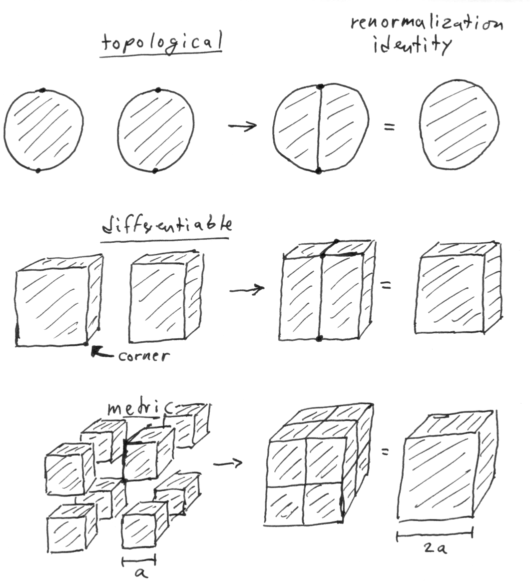
→ Consider a class of oriented compact **topological/differentiable 4-manifolds** with boundary as **regions**. Must include 4-balls and be closed under gluing.

→ Admissible **hypersurfaces** are boundaries of regions and their connected components. (These hypersurfaces carry in addition the structure of an "infinitesimal 4-manifold neighborhood".)

→ To each hypersurface Σ associate a **Hilbert space** $\mathcal{H}_\Sigma$, to each region M an **amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$.

→ These structures have to satisfy the **axioms**. Gluings have to be compatible with the extra structure of the 3-manifolds.

Renormalization identities



topological

Relates regions of the same type. There is only one elementary region.

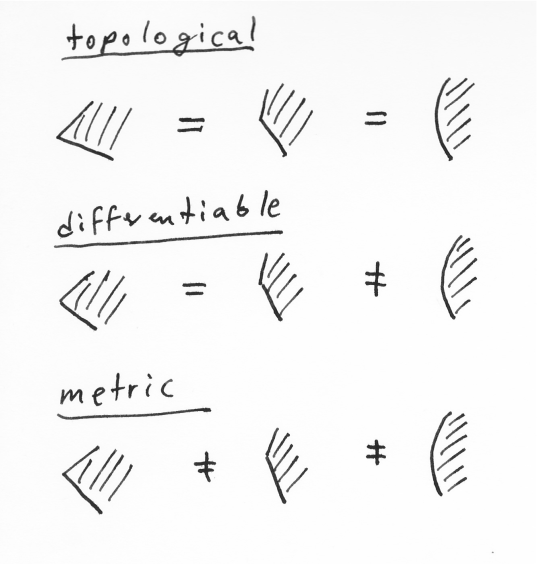
differentiable

Relates regions of the same type. Regions have **corners**.

metric

Relates regions of different sizes. Link to coupling constant renormalization.

Corners



topological

Corners are homeomorphic to smooth hypersurfaces.

differentiable

Corners of different angles are diffeomorphic, but distinct from smooth hypersurfaces.

metric

Corners of different angles are all distinct.

Symmetry

assume differentiable setting

- **Diffeomorphisms** are **gauge symmetries** of GR.

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Symmetry

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- **Diffeomorphisms** are **gauge symmetries** of GR.
- On each **region** M acts its group of orientation preserving diffeomorphisms G_M.
- On each **hypersurface** Σ acts its group of orientation preserving diffeomorphisms G_Σ. This induces $i_M : G_M \to G_{\partial M}$.
- Let $G_M^{\text{int}} \subseteq G_M$ be the subgroup that acts identically on the boundary. We have the exact sequence

$$G_M^{\text{int}} \to G_M \to G_{\partial M}$$

- For each hypersurface Σ, G_Σ must act on $\mathcal{H}_\Sigma$ by unitary transformations, i.e., $\mathcal{H}_\Sigma$ is a **unitary representation** of G_Σ.
- For each region M, ρ_M must be **invariant** under $i_M(G_M)$. That is, $\rho_M(g \triangleright \psi) = \rho_M(\psi)$ for any $\psi \in \mathcal{H}_{\partial M}$ and $g \in i_M(G_M) \subseteq G_{\partial M}$.

Representation theory of diffeomorphism groups is crucial ingredient.

Refinement: Projectivity

It is well known that representations of symmetry groups on the Hilbert space in quantum mechanics only have to be **projective representations**. This is related to the fact that what has to be preserved under symmetries are only measurable quantities like probabilities and expectation values. The same is true in the general boundary formulation. In light of this the previously mentioned implementation of symmetries may be relaxed accordingly.

Measurement in quantum gravity

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- As in QFT: scattering matrix is the main object of interest.
- But: is this justified in quantum gravity?

Measurement in quantum gravity

- So far: measurements only on the boundary.
- As in QFT: scattering matrix is the main object of interest.
- But: is this justified in quantum gravity?
- QFT scattering theory relies on perturbation theory. This does not work in quantum gravity.
- In QFT we also understand how to encode general measurements, but this requires recurring to a non-relativistic picture.

The **positive formalism** allows to implement **local measurements** into quantum theory, even in the absence of a spacetime metric.

Quantum gravity in the positive formalism

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The **top-down approach** can be directly applied at the level of the positive formalism. The **representation theory** of diffeomorphisms again would play a key role. **Projectivity** of representations would be automatic. The structure would be enriched by **probes** representing local measurements in spacetime.