Holomorphic quantization in background-independent quantum field theory

Robert Oeckl

Centro de Ciencias Matemáticas
UNAM, Morelia

Cinvestav-IPN
Queretaro
19 August 2011

Outline

1. Motivation
2. The general boundary formulation
   - Basic structure
   - Probability interpretation
   - Recovering the standard formulation
3. Holomorphic Quantization
   - Ingredients
     - Lagrangian field theory
     - Geometric quantization
     - Schrödinger-Feynman quantization
   - Classical Data
   - The Quantum Theory
   - Main Result
   - Coherent states

Why a different formulation of quantum theory?

Usually a quantum system is encoded through a Hilbert space $\mathscr{H}$ of states and an operator algebra $\mathscr{A}$ of observables.

This standard formulation of quantum theory has limitations that obstruct its application in a general relativistic context:

- Its operational meaning is tied to a background time: States encode information on the system between measurements, the product of observables encodes temporal composition of measurements, probability is conserved in time etc.
- Its ability to describe physics locally is not manifest, but arises dynamically, depending on the background metric: Causality and cluster decomposition allow then a factorization of the S-matrix.

A new formulation

Can we reformulate what constitutes a quantum theory such that

- there is no reference to time
- locality is manifest
- what was considered a quantum theory previously is still a quantum theory
- extra assumptions are kept to a minimum?

A new formulation

Can we reformulate what constitutes a quantum theory such that

- there is no reference to time
- locality is manifest
- what was considered a quantum theory previously is still a quantum theory
- extra assumptions are kept to a minimum?

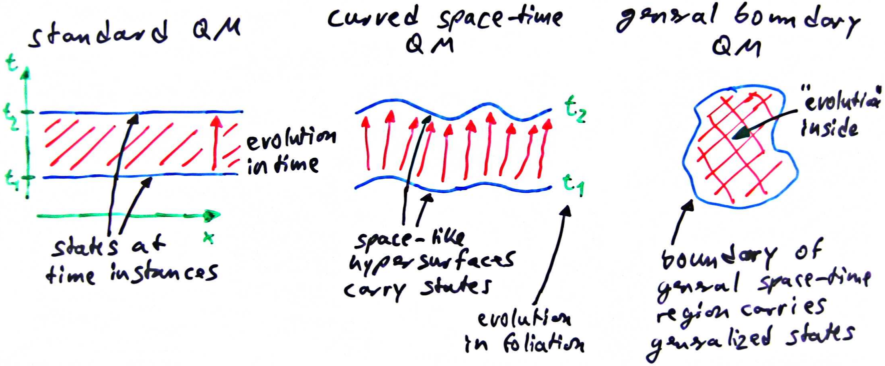
YES, using:

- The mathematical framework of **topological quantum field theory**. (A branch of modern algebraic topology.)
- A generalization of the **Born rule**.

General boundary formulation

generalizing amplitudes

A starting point is the idea to generalize transition amplitudes.

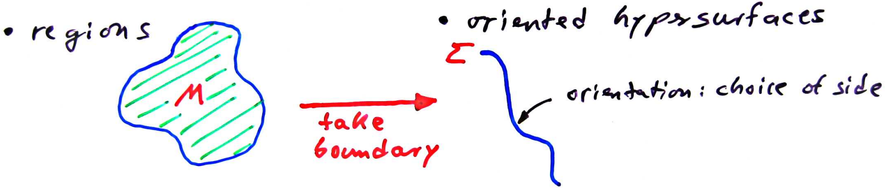


Basic structures

At the basis of the general boundary formulation lies an assignment of algebraic structures to geometric ones.

Basic geometric structures (representing pieces of spacetime):

- **hypersurfaces**: oriented manifolds of dimension $d-1$
- **regions**: oriented manifolds of dimension d with boundary



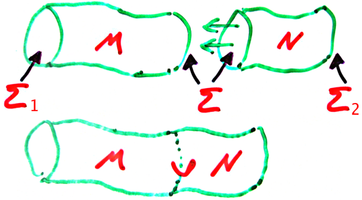
Basic algebraic structures:

- To each hypersurface Σ associate a Hilbert space $\mathcal{H}_\Sigma$ of **states**.
- To each region M with boundary ∂M associate a linear **amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$.

Core axioms

The structures are subject to a number of axioms, in the spirit of **topological quantum field theory**:

- Let $\overline{\Sigma}$ denote Σ with opposite orientation. Then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_{\Sigma}^*$.
- **(Decomposition rule)** Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a disjoint union of hypersurfaces. Then $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.
- **(Gluing rule)** If M and N are adjacent regions, then

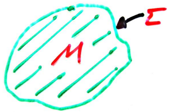


$$\rho_{M\cup N}(\psi_1 \otimes \psi_2) \cdot c_{M,N} = \sum_{i\in\mathbb{N}} \rho_M(\psi_1 \otimes \xi_i)\rho_N(\xi_i^* \otimes \psi_2)$$

Here, $\psi_1 \in \mathcal{H}_{\Sigma_1}$, $\psi_2 \in \mathcal{H}_{\Sigma_2}$ and $\{\xi_i\}_{i\in\mathbb{N}}$ is an ON-basis of $\mathcal{H}_\Sigma$. $c_{M,N}$ is the gluing anomaly.

Amplitudes and Probabilities

Consider the context of a general spacetime region M with boundary Σ.



Probabilities in quantum theory are generally conditional probabilities. They depend on two pieces of information. Here these are:

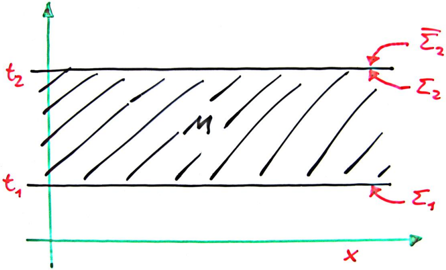
- $\mathscr{S} \subset \mathscr{H}_\Sigma$ representing **preparation** or **knowledge**
- $\mathscr{A} \subset \mathscr{H}_\Sigma$ representing **observation** or the **question**

The probability that the system is described by $\mathscr{A}$ given that it is described by $\mathscr{S}$ is:

$$P(\mathscr{A}|\mathscr{S}) = \frac{|\rho_M \circ P_{\mathscr{S}} \circ P_{\mathscr{A}}|^2}{|\rho_M \circ P_{\mathscr{S}}|^2}$$

- $P_{\mathscr{S}}$ and $P_{\mathscr{A}}$ are the orthogonal projectors onto the subspaces.

Recovering transition amplitudes and probabilities



- region: $M = [t_1, t_2] \times \mathbb{R}^3$
- boundary: $\partial M = \Sigma_1 \cup \overline{\Sigma}_2$
- state space: $\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$

Via time-translation symmetry identify $\mathcal{H}_{\Sigma_1} \cong \mathcal{H}_{\Sigma_2} \cong \mathcal{H}$. Then,

$$\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, U(t_1, t_2)\psi_1 \rangle.$$

To compute the probability of measuring ψ_2 at t_2 given that we prepared ψ_1 at t_1 we set

$$\mathcal{S} = \psi_1 \otimes \mathcal{H}^*, \quad \mathcal{A} = \mathcal{H} \otimes \psi_2^*.$$

The resulting expression yields correctly

$$P(\mathcal{A}|\mathcal{S}) = |\langle \psi_2, U(t_1, t_2)\psi_1 \rangle|^2.$$

Holomorphic Quantization

For linear field theories with certain additional data a quantization scheme can be devised that yields a quantum field theory in GBF form. This can be seen as a kind of functor from a category of classical field theories to a category of quantum field theories.

Lagrangian field theory

Formulate field theory in terms of first order Lagrangian density $\Lambda(\varphi, \partial\varphi, x)$. For a spacetime region M the **action** of a field ϕ is

$$S_M(\phi) := \int_M \Lambda(\phi(\cdot), \partial\phi(\cdot), \cdot). \tag{1}$$

Classical solutions in M are extremal points of this action.
For a hypersurface Σ the **symplectic form** is

$$(\omega_\Sigma)_\phi(X,Y) = -\frac{1}{2}\int_\Sigma \left((X^b Y^a - Y^b X^a)\, \partial_\mu \lrcorner \frac{\delta^2 \Lambda}{\delta \varphi^b \delta \partial_\mu \varphi^a}\bigg|_\phi \right.$$
$$\left. +(Y^a \partial_\nu X^b - X^a \partial_\nu Y^b)\, \partial_\mu \lrcorner \frac{\delta^2 \Lambda}{\delta \partial_\nu \varphi^b \delta \partial_\mu \varphi^a}\bigg|_\phi \right). \tag{2}$$

Geometric quantization

Let L denote the space of classical solutions with a symplectic form ω.

1. We consider a hermitian line bundle B over L with a connection ∇ that has curvature 2-form ω. Define the **prequantum** Hilbert space H as the space of square-integrable sections with inner product

$$\langle s', s \rangle = \int (s'(\eta), s(\eta))_\eta \, \mathrm{d}\mu(\eta).$$

2. This Hilbert space is too large. Choose in each complexified tangent space $(T_\phi L)^{\mathbb{C}}$ a Lagrangian subspace P_ϕ with respect to ω_ϕ. We then restrict H to those sections s of B such that

$$\nabla_{\overline{X}} s = 0, \tag{3}$$

if $X_\phi \in P_\phi$ for all $\phi \in L$. This is called a **polarization**.

Kähler polarization

We are interested in a **Kähler polarization**. Then P_ϕ is determined by a complex structure J_ϕ in $T_\phi L$ that is compatible with ω_ϕ. J_ϕ satisfies $J_\phi \circ J_\phi = -1$ and $\omega_\phi(J_\phi X, J_\phi Y) = \omega_\phi(X, Y)$. Then

$$P_\phi = \{X \in (T_\phi L)^{\mathbb{C}} : \mathrm{i}X = J_\phi X\}. \tag{4}$$

J_ϕ yields a real inner product on $T_\phi L$:

$$g_\phi(X_\phi, Y_\phi) := 2\omega_\Sigma(X_\phi, J_\phi Y_\phi). \tag{5}$$

We shall require g_ϕ to be positive definite. We also obtain a complex inner product on $T_\phi L$ viewed as a complex vector space:

$$\{X_\phi, Y_\phi\}_\phi := g_\phi(X_\phi, Y_\phi) + 2\mathrm{i}\omega_\phi(X_\phi, Y_\phi). \tag{6}$$

The Hilbert space $\mathscr{H}$ obtained from H through a Kähler polarization is also called the **holomorphic representation**.

Schrödinger-Feynman quantization

In the **Schrödinger representation** states are wave functions on instantaneous field configurations. Transition amplitudes between such wave functions can be obtained through the **Feynman path integral**:

$$\langle \psi_2, U_{[t_1,t_2]} \psi_1 \rangle = \int_{K_{[t_1,t_2]}} \psi_1(\phi|_{t_1}) \overline{\psi_2(\phi|_{t_2})} \exp\left(\mathrm{i} S_{[t_1,t_2]}(\phi)\right) \mathrm{d}\mu(\phi), \quad (7)$$

The integral is over the space of field configurations $K_{[t_1,t_2]}$ in the time interval t_1, t_2 between initial state ψ_1 and final state ψ_2. For spacetime regions M this generalizes to

$$\rho_M(\psi) = \int_{K_M} \psi(\phi|_{\partial M}) \exp\left(\mathrm{i} S_M(\phi)\right) \mathrm{d}\mu(\phi), \quad (8)$$

If the space of classical solutions L_M is linear, the integral over the space K_M can be replaced by an integral over the "much smaller" space L_M.

Classical Data

A classical linear field theory is encoded in the following data:

- For each hypersurface Σ there is a real vector space L_Σ (of classical solutions near Σ). L_Σ carries a non-degenerate symplectic form ω_Σ (from Lagrangian field theory). Moreover, L_Σ carries a compatible complex structure J_Σ (for geometric quantization). In particular, L_Σ is a real Hilbert space with g_Σ and a complex Hilbert space with $\{\cdot,\cdot\}_\Sigma$.
- For each region M there is a real vector space L_M (of classical solutions in M) and a real linear map $r_M : L_M \to L_{\partial M}$.
- The subspace $r_M(L_M) \subseteq L_{\partial M}$ is Lagrangian with respect to $\omega_{\partial M}$.
- These structures are compatible with orientation change, decomposition of hypersurfaces and gluing of regions. We also require a certain integrability condition.

It follows: $L_{\partial M} = r_M(L_M) \oplus_{\mathbb{R}} J_{\partial M} r_M(L_M)$ is an orthogonal sum.

State spaces

For each hypersurface Σ we define a Hilbert space of states $\mathcal{H}_\Sigma$ as follows. The polarization induced by the complex structure J_Σ yields a global trivialization of the prequantum bundle B_Σ. Polarized sections become **holomorphic** functions on L_Σ that are square-integrable with respect to a **Gaussian measure** ν_Σ, depending on g_Σ,

$$\langle\psi',\psi\rangle_\Sigma = \int_{\hat{L}_\Sigma} \psi(\phi)\overline{\psi'(\phi)}\mathrm{d}\nu_\Sigma(\phi).$$

If L_Σ is infinite-dimensional no Gaussian measure on L_Σ exists. However, ν_Σ does exists on the larger space $\hat{L}_\Sigma$ that is the algebraic dual of the topological dual of L_Σ. So, wave functions $\psi \in \mathcal{H}_\Sigma$ are really functions on $\hat{L}_\Sigma$. However, they turn out to be completely determined by their values on L_Σ.

Amplitudes

For each region M we define the linear amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ by

$$\rho_M(\psi) := \int_{\hat{L}_M} \psi(r(\phi))\, \mathrm{d}\nu_M(\phi).$$

Here $\hat{L}_M$ is an extension of L_M and ν_M is a Gaussian measure on $\hat{L}_M$, depending on $g_{\partial M}$. ν_M arises by combining three ingredients:

- A translation-invariant measure in the Feynman path integral.
- The factor $\exp(\mathrm{i} S_M(\phi))$ in the Feynman path integral.
- The transformation between the Schrödinger and the holomorphic representations.

Main Result

Theorem

The GBF core axioms are satisfied by this quantization prescription.

Coherent States

The Hilbert spaces $\mathcal{H}_\Sigma$ are reproducing kernel Hilbert spaces and contain coherent states of the form

$$K_\xi(\phi) = \exp\left(\frac{1}{2}\{\xi,\phi\}_\Sigma\right)$$

associated to classical solutions $\xi \in L_\Sigma$. They have the reproducing property,

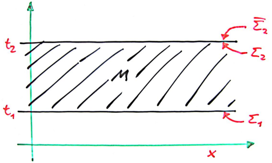
$$\langle K_\xi, \psi\rangle_\Sigma = \psi(\xi),$$

and satisfy the completeness relation

$$\langle \psi', \psi\rangle_\Sigma = \int_{\hat{L}_\Sigma} \langle \psi', K_\xi\rangle_\Sigma \langle K_\xi, \psi\rangle_\Sigma \,\mathrm{d}\nu_\Sigma(\xi).$$

They can be thought of as representing quantum states that approximate a specific classical solutions.

Evolution Picture



Suppose M is a region with $\partial M = \Sigma_1 \cup \overline{\Sigma_2}$. If there is a unitary map $T : L_{\Sigma_1} \to L_{\Sigma_2}$ such that $r_M(L_M) = \{(\phi, T\phi) : \phi \in L_{\Sigma_1}\}$, then there is a unitary map $U : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$ such that

$$\rho_M(\psi_1 \otimes \psi_2^*) = \langle \psi_2, U\psi_1 \rangle_{\Sigma_2}$$

where

$$(U\psi)(\phi) = \psi(T^{-1}\phi) \quad \text{and} \quad UK_\xi = K_{T\xi}.$$

Amplitudes of coherent states

Remarkably, the amplitude of a coherent state can be calculated explicitly. Let $\xi \in L_{\partial M}$ and set $\xi = \xi^{\mathrm{R}} + J_{\partial M}\xi^{\mathrm{I}} \in r_M(L_M) \oplus_R J_{\partial M} r_M(L_M)$. Let $\tilde{K}_\xi$ denote the normalized coherent state associated with ξ. Then,

$$\rho_M(\tilde{K}_\xi) = \exp\left(-\frac{1}{2} g_{\partial M}(\xi^{\mathrm{I}}, \xi^{\mathrm{I}}) - \frac{\mathrm{i}}{2} g_{\partial M}(\xi^{\mathrm{R}}, \xi^{\mathrm{I}})\right)$$

This has a compelling physical interpretation ...

Selected references

Short overview of the GBF:

R. O., *Probabilities in the general boundary formulation*, J. Phys.: Conf. Ser. **67** (2007) 012049, arXiv:hep-th/0612076.

- R. O., *General boundary quantum field theory: Foundations and probability interpretation*, Adv. Theor. Math. Phys. **12** (2008) 319-352, arXiv:hep-th/0509122.
- R. O., *Holomorphic quantization of linear field theory in the general boundary formulation*, arXiv:1009.5615.