Introduction to the general boundary formulation of quantum theory

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Outline

Limitations of the standard formulation

Usually a quantum system is encoded through a Hilbert space $\mathcal{H}$ of states and an operator algebra $\mathcal{A}$ of observables.

This standard formulation of quantum theory has limitations that obstruct its application in a general relativistic context:

- Its operational meaning is tied to a background time.
- Its ability to describe physics locally is not manifest, but arises dynamically, depending on the background metric.

# Operational meaning tied to background time

The physical role of key ingredients of the standard formulation of quantum theory...

- A Hilbert space $\mathcal{H}$ of states.
    - A state encodes information about the system between measurements.
    - The inner product allows to extract probabilities.
- An algebra of observables $\mathcal{A}$.
    - An observable encodes a possible measurement on the system.
    - A measurement changes a state to a new state.
    - The product of $\mathcal{A}$ encodes temporal composition of measurements.
- Certain unitary operators describe evolution of the system in time.
    - Probability is conserved in time.

...makes reference to an *external notion of time*, i.e., a notion of time independent of a state.

Locality in the standard formulation

In a fundamental quantum theory a state is a priori a state of the universe. But, we cannot hope to be able to describe the universe in all its details. We need to be able to describe physics locally. In quantum field theory this is achieved dynamically, using the background metric. Causality and cluster decomposition mean that the S-matrix factorizes, $S = S_1 S_2$:
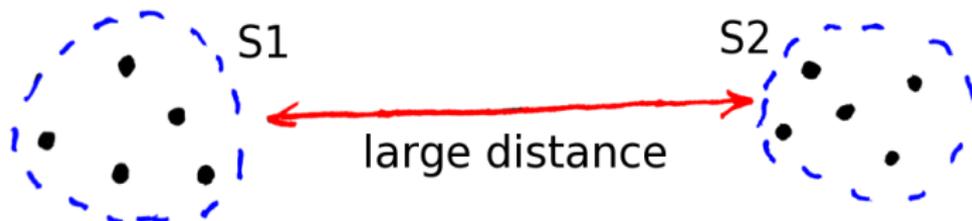
We can thus successfully describe a local system as if it was alone in an otherwise empty Minkowski universe.

Reactions

1. We keep a classical background at least in parts of spacetime, so there is no problem with quantum theory as we know it. The price is that we can only describe quantum gravitational phenomena "far away" and approximately. [Perturbative Quantum Gravity, String Theory]

2. We keep the formalism, but throw away the background metric and with it (part of) the physical interpretation. We then have to construct a new physical interpretation of the formalism. [Many canonical approaches]

3. Quantum theory as we know it is really fundamentally limited and must be replaced by something new. Known physics is modified. [Causal sets, Gravity induced collapse models]

Reactions

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OR

4. Perhaps we have formulated quantum theory in an inconvenient way. There may be a better formulation, free of these limitations.

Towards a new formulation

Can we **reformulate** what constitutes a quantum theory such that

- there is no reference to time
- locality is manifest
- what was considered a quantum theory previously is still a quantum theory
- the increase in "structural baggage" is minimal?

Towards a new formulation

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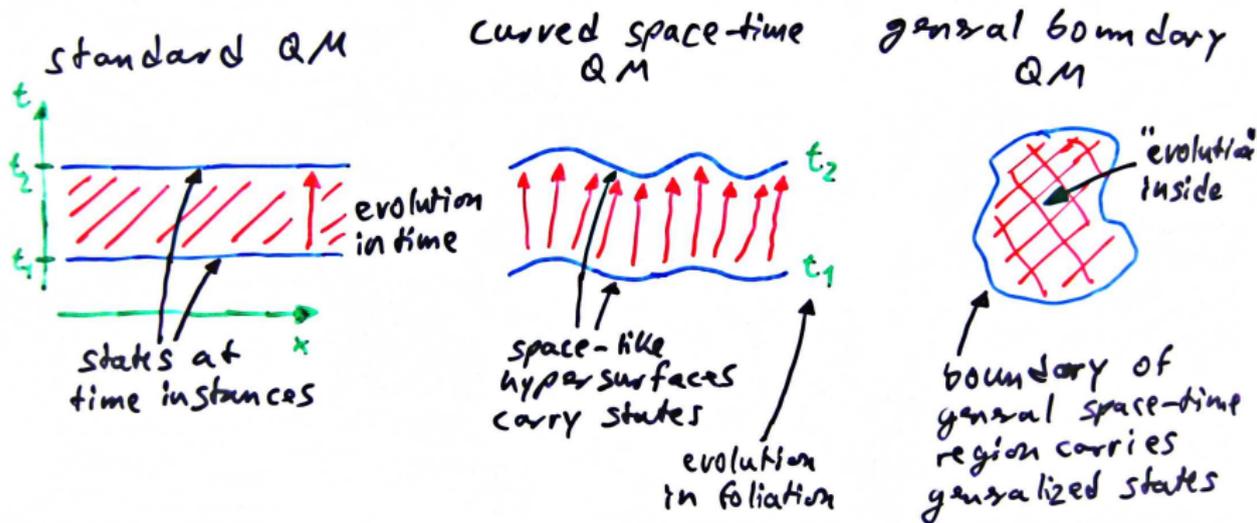- the increase in "structural baggage" is minimal?

YES, using:

- The mathematical framework of topological quantum field theory. (A branch of modern algebraic topology.)
- A generalization of the Born rule.

General boundary formulation
generalizing amplitudes

A starting point is the idea to generalize transition amplitudes.

Basic structures

At the basis of the general boundary formulation lies an assignment of algebraic structures to geometric ones.

Basic geometric structures (representing pieces of spacetime):

- **hypersurfaces**: oriented manifolds of dimension $d - 1$
- **regions**: oriented manifolds of dimension d with boundary

Basic algebraic structures:

- To each hypersurface Σ associate a Hilbert space $\mathcal{H}_\Sigma$ of **states**.
- To each region M with boundary ∂M associate a linear **amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$.

Core axioms
state spaces and amplitude maps

The structures are subject to a number of axioms, in the spirit of
topological quantum field theory:

- Let $\overline{\Sigma}$ denote Σ with opposite orientation. Then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_{\Sigma}^*$.
- **(Decomposition rule)** Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a disjoint union of hypersurfaces. Then $\mathcal{H}_{\Sigma} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.
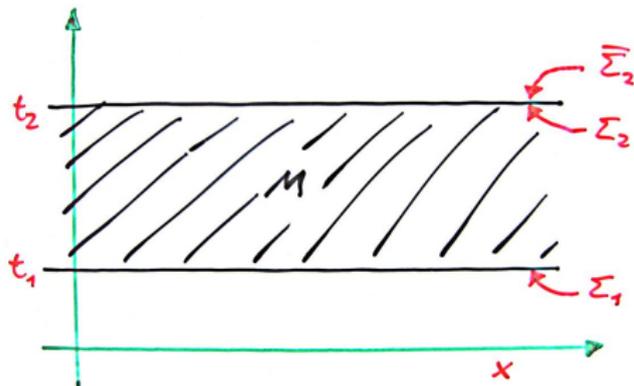- **(Gluing rule)** If M and N are adjacent regions, then

$$\rho_{M \cup N}(\psi_1 \otimes \psi_2)$$
$$= \sum_{i \in \mathbb{N}} \rho_M(\psi_1 \otimes \xi_i) \rho_N(\xi_i^* \otimes \psi_2)$$

Here, $\psi_1 \in \mathcal{H}_{\Sigma_1}$, $\psi_2 \in \mathcal{H}_{\Sigma_2}$ and
$\{\xi_i\}_{i \in \mathbb{N}}$ is an ON-basis of $\mathcal{H}_{\Sigma}$.
Notation: $\rho_{M \cup N} = \rho_M \diamond \rho_N$

Recovering transition amplitudes

Consider the geometry of a standard transition.

- region: $M = [t_1, t_2] \times \mathbb{R}^3$
- boundary: $\partial M = \Sigma_1 \cup \overline{\Sigma}_2$
- state space:
  $\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}^*$

- Via time-translation symmetry identify $\mathcal{H}_{\Sigma_1} \cong \mathcal{H}_{\Sigma_2} \cong \mathcal{H}$, where $\mathcal{H}$ is the state space of standard quantum mechanics.
- Write the amplitude map as $\rho_{[t_1, t_2]} : \mathcal{H} \otimes \mathcal{H}^* \to \mathbb{C}$.
- The relation to the standard amplitude is:

$$\rho_{[t_1, t_2]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2^*, U(t_1, t_2)\psi_1 \rangle$$

Amplitudes and Probabilities

Consider the context of a general spacetime
region M with boundary Σ.
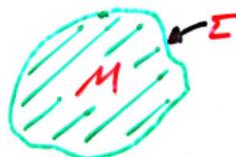
Probabilities in quantum theory are generally conditional probabilities.
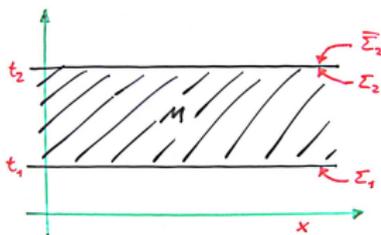They depend on two pieces of information. Here these are:

- $\mathcal{S} \subset \mathcal{H}_\Sigma$ representing preparation or knowledge
- $\mathcal{A} \subset \mathcal{H}_\Sigma$ representing observation or the question

The probability that the system is described by $\mathcal{A}$ given that it is
described by $\mathcal{S}$ is:

$$P(\mathcal{A}|\mathcal{S}) = \frac{|\rho_M \circ P_\mathcal{S} \circ P_\mathcal{A}|^2}{|\rho_M \circ P_\mathcal{S}|^2}$$

- $P_\mathcal{S}$ and $P_\mathcal{A}$ are the orthogonal projectors onto the subspaces.

Recovering standard probabilities

To compute the probability of measuring ψ_2 at t_2 given that we prepared ψ_1 at t_1 we set

$$\mathcal{S} = \psi_1 \otimes \mathcal{H}^*, \quad \mathcal{A} = \mathcal{H} \otimes \psi_2^*.$$

The resulting expression yields correctly

$$P(\mathcal{A}|\mathcal{S}) = \frac{|\rho_{[t_1,t_2]} \circ P_{\mathcal{S}} \circ P_{\mathcal{A}}|^2}{|\rho_{[t_1,t_2]} \circ P_{\mathcal{S}}|^2} = \frac{|\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2^*)|^2}{1} = |\langle \psi_2^*, U(t_1,t_2)\psi_1 \rangle|^2.$$
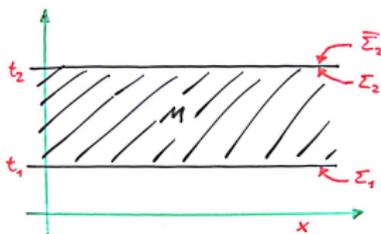
Observables

Observables are associated to regions M and encoded through **observable maps** $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$, similar to the amplitude map.

Observables can be glued in the same way as amplitudes. Suppose M and N are adjacent regions.

- If O is an observable in M represented by ρ_M^O, then there corresponds to it an observable in the region $M \cup N$ represented by $\rho_{M \cup N}^O = \rho_M^O \diamond \rho_N$.
- If O is an observable in M and P an observable in N, then there is a product observable $O \cdot P$ in $M \cup N$ represented by $\rho_{M \cup N}^{O \cdot P} = \rho_M^O \diamond \rho_N^P$.

Recovering standard observables

In the limiting case $t_2 = t_1 = t$ there is a correspondence between observable maps $\rho_{[t,t]}^O : \mathcal{H} \otimes \mathcal{H}^* \to \mathbb{C}$ and standard observables $\hat{O} : \mathcal{H} \to \mathcal{H}$ via matrix elements:

$$\rho_{[t,t]}^O(\psi_1 \otimes \psi_2^*) = \langle \psi_2, \hat{O}\psi_1 \rangle \quad \forall \psi_1, \psi_2 \in \mathcal{H}.$$

Observables and expectation values

Consider the context of a general spacetime
region M with boundary Σ.

The expectation value of the observable O conditional on the system
being prepared in the subspace $\mathcal{S} \subset \mathcal{H}_\Sigma$ can be represented as
follows:

$$\langle O \rangle_{\mathcal{S}} = \frac{\langle \rho_M^{\mathcal{S}}, \rho_M^O \rangle}{|\rho_M^{\mathcal{S}}|^2}$$

Here we write $\rho_M^{\mathcal{S}} := \rho_M \circ P_{\mathcal{S}}$.

(We also use a certain simplifying condition which in the standard formalism is always
satisfied.)

Recovering standard expectation values
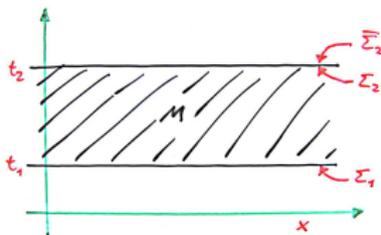
To compute the expectation value of observable O at time t given by

$$\rho^O_{[t,t]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, \hat{O}\psi_1 \rangle$$

in the state ψ we set

$$\mathcal{S} = \psi \otimes \mathcal{H}^*.$$

The expectation value is then correctly obtained as

$$\langle O \rangle_{\mathcal{S}} = \frac{\langle \rho^{\mathcal{S}}_{[t,t]}, \rho^O_{[t,t]} \rangle}{|\rho^{\mathcal{S}}_{[t,t]}|^2} = \frac{\rho^O_{[t,t]}(\psi \otimes \psi^*)}{1} = \langle \psi, \hat{O}\psi \rangle.$$

GBF and quantum field theory

The main motivation for the general boundary formulation (GBF) is to provide a suitable framework for a quantum theory of gravity. However, the GBF is a general framework that also embraces standard quantum field theory.

Conjecture

Standard quantum field theories can be formulated within the GBF.

Some applications of the GBF to QFT

- Description of quantum states on timelike hypersurfaces. [RO] This permits the quantization of evanescent waves that are "invisible" in traditional quantization prescriptions. [RO, paper out today]
- Description of free theories in a bounded region of space. [RO]
- Description of a free Euclidean theory in a bounded region of spacetime [D. Colosi, RO]
- Description of new types of asymptotic amplitudes, generalizing the S-matrix framework. [D. Colosi, RO]
- Application of this to de Sitter space. [D. Colosi, paper to appear in next few days]
- Rigorous (holomorphic) quantization of linear field theories without need for metric background. [RO, paper out today]
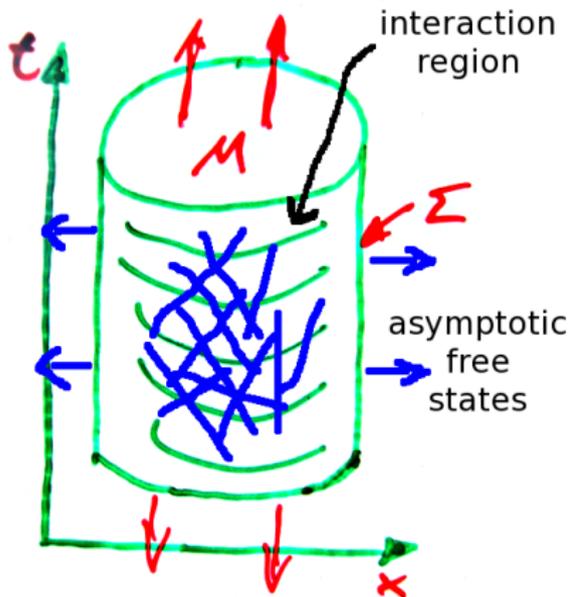
S-matrix

Usually, interacting QFT is described via the S-matrix:
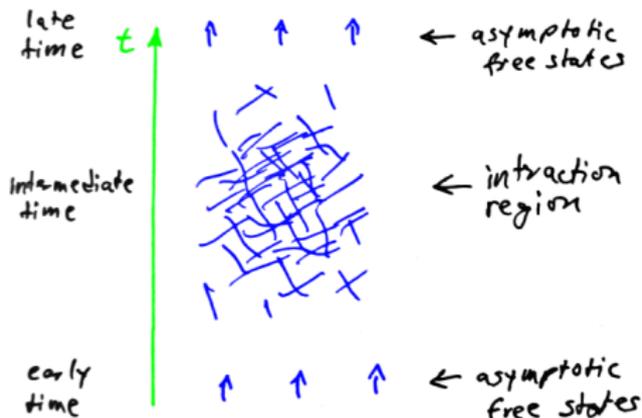
Assume interaction is relevant only after the initial time t_1 and before the final time t_2. The S-matrix is the asymptotic limit of the amplitude between free states at early and at late time:

$$\langle\psi_2|S|\psi_1\rangle = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \langle\psi_2|U_{\text{int}}(t_1, t_2)|\psi_1\rangle$$

Spatially asymptotic S-matrix

Similarly, we can describe interacting QFT via a *spatially* asymptotic amplitude. Assume interaction is relevant only within a radius R from the origin in space (but at all times). Consider then the asymptotic limit of the amplitude of a free state on the hypercylinder when the radius goes to infinity:

$$S(\psi) = \lim_{R \to \infty} \rho_R(\psi)$$

Result

The S-matrices are equivalent when both are valid.

GBF and quantum gravity

Three dimensional quantum gravity is already formulated as a TQFT and fits thus "automatically" into the GBF.

The GBF is heavily used in spin foam approaches to quantum gravity. [C. Rovelli and his group in Marseille]

The GBF suggests also an intrinsic top-down approach to quantum gravity.