Quantum Geometry and Quantum Field Theory

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Preface

This dissertation is based on research done at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, in the period from October 1997 to August 2000. It is original except where reference is made to the work of others. Chapter 4 in its entirety and Chapter 2 partly (as specified therein) represent work done in collaboration with Shahn Majid. All other chapters represent solely my own work. No part of this dissertation nor anything substantially the same has been submitted for any qualification at any other university. Most of the material presented has been published or submitted for publication in the following papers:

- [Oec99b] R. Oeckl, *Classification of differential calculi on $U_q(b_+)$, classical limits, and duality*, J. Math. Phys. **40** (1999), 3588–3603.
- [MO99] S. Majid and R. Oeckl, *Twisting of Quantum Differentials and the Planck Scale Hopf Algebra*, Commun. Math. Phys. **205** (1999), 617–655.
- [Oec99a] R. Oeckl, *Braided Quantum Field Theory*, Preprint DAMTP-1999-82, hep-th/9906225.
- [Oec00b] R. Oeckl, *Untwisting noncommutative $\mathbb{R}^d$ and the equivalence of quantum field theories*, Nucl. Phys. **B 581** (2000), 559–574.
- [Oec00a] R. Oeckl, *The Quantum Geometry of Spin and Statistics*, Preprint DAMTP-2000-81, hep-th/0008072.

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Contents

Introduction

Reading the title of this dissertation one might ask: *What is quantum geometry?* or *What does it have to do with quantum field theory?* The first of these questions we will try to answer immediately. For the second we hope that the following chapters hold at least a partial answer.

More familiar perhaps than the term *quantum geometry* are the terms that it is meant to subsume: *noncommutative geometry* and *quantum groups*. Although both belong to the realm of mathematics, their evolution has been very much connected with developments in physics, particularly quantum physics. The story begins with the early days of quantum mechanics. Heisenberg's commutation relations

$$[X, P] = \mathrm{i}\hbar$$

imply that the geometry of classical phase space is lost. If coordinates (such as X and P) on a phase space cease to commute then there can be no such space. Instead, one viewed X and P purely as operators on a Hilbert space. Functional analysis succeeded geometry. It has been dominating quantum mechanics ever since. In spite of this, the idea, in one form or other, that this operator algebra forms some kind of "noncommutative geometric space" has accompanied quantum mechanics almost from the beginning. However, for a long time, no serious attempt has been made to develop such a generalised geometry.

Algebraic geometry is built on a correspondence between "spaces" and commutative algebras. The correspondence associates with a space the algebra of functions on it. Geometric notions are then expressed in a purely algebraic language. This principle turns out to be the right starting point for a generalised geometry. While for algebraic geometry the spaces are affine schemes, a correspondence that is closer to differential geometry is given by the Gelfand-Naimark theorem. In this case the spaces are topological spaces and the algebras commutative C^*-algebras.

Around 1980, an approach was initiated by Alain Connes which has become widely known as *noncommutative geometry* [Con80, Con85, Con94]. This approach is very much in the functional analytic tradition of quantum mechanics. It introduces notions of Dirac operator, differential structures, metric structures, vector bundles, etc. in a context of (noncommutative) operator algebras leading to many interesting developments in mathematics.

A quite different development originated also around 1980. It came out of the study of quantum integrable systems by the (then) Leningrad school of Ludwig D. Faddeev and others. They encountered generalised symmetries that were not described by groups but could be related to groups. From these emerged the q-deformations of Lie algebras and Lie groups [Jim85, Dri85, Wor87, RTF90]. It was then also realised by Vladimir G. Drinfeld that they are examples of a more general structure, called a *quantum group*. He developed the mathematical theory of quantum groups extensively in the 1980's [Dri87]. The underlying structure is that of a Hopf algebra, a concept that had appeared already much earlier in the context of group cohomology.

To connect the two developments we note that the correspondence between "spaces" and commutative algebras extends to (algebraic) groups. The extra structure provided by the group multiplication corresponds on the algebra side to the additional structure that makes a commutative algebra a commutative Hopf algebra. Thus, (noncommutative) Hopf algebras are generalisations of groups in the same way as (noncommutative) algebras are generalisations of spaces. The (algebraic) theory that encompasses both is called *quantum geometry*.[1]

Interesting mathematical developments have taken place since the early days. The representation theory of quantum groups is very rich and leads to braided monoidal categories [JS86]. This in turn has led to new insights into knot theory and invariants of 3-manifolds. Many examples of quantum spaces have been investigated since the late 80's including deformations of $\mathbb{R}^n$, of spheres, of projective spaces, etc. Differential structures on quantum groups were introduced [Wor89] as well as quantum principal bundles and connections [BM93].

Besides the purely mathematical attraction of quantum geometry there are very good reasons to expect it to play an important role in physics. This goes far beyond the initial application of quantum groups as symmetries of certain quantum integrable systems. One particularly intriguing idea is that quantum geometry might lend a far better description of space-time at the Planck scale than ordinary differential geometry does. Based on the persistent inability to unite gravity with quantum field theory it has long been conjectured that space-time at short distances might have a discrete or foam-like structure. It seems that, with the emergence of quantum geometry, for the first time the tools for such a description are at hand.

As early as the 1940's it was proposed that space-time coordinates might be noncommuting

[1] It is worth mentioning that the term *quantum geometry* has appeared in recent years also in other contexts, notably in loop quantum gravity and string theory. This is not necessarily related to the meaning we attach to it here.

observables [Sny47]. This was then motivated by the hope that the infinities arising at very short distances in quantum field theory might be regularised in this way. It took several decades – until the emergence of quantum groups and noncommutative geometry – for further development of such ideas to occur. Shortly after their introduction, Shahn Majid suggested a role for quantum groups in Planck scale physics as a unified description of the observable algebra and the gravitational curvature [Maj88]. (See Chapter 4 for further developments of this idea.) Around 1990 Alain Connes and collaborators initiated a reformulation of the standard model using noncommutative geometry to describe its internal degrees of freedom [CL90, Con94]. At about the same time quantum deformations of Minkowski space and its symmetries were introduced [PW90, CSSW90, OSWZ92] with the motivation to obtain new physical models by deformation. Braided categories were found to describe generalised particle statistics in the context of algebraic quantum field theory [FRS89, FG90], indicating quantum group symmetries (see Chapter 5). The interest in noncommutative geometry was boosted in 1997 with the emergence of noncommutative spaces in string theory [CDS98, SW99]. (See Chapter 8 for a development on this.) Quantum groups have also appeared in the loop approach at quantum gravity [MS96]. Both indicate an important role for quantum geometry in fundamental physics. The idea of noncommutativity as a regulator has found a successful application in "fuzzy physics", where function algebras are approximated by finite dimensional algebras [Mad92, Mad95, GKP96]. This is similar to a lattice approximation, but without the breaking of space-time symmetries. In contrast, the more ambitious aim of a continuous regularisation of quantum field theory by quantum deformation of symmetries (as proposed in [Maj90b]) has been open for some time. It has found its first realisation only recently in the work presented here (see Chapter 9).

This dissertation aims at contributing to some of the mentioned developments as well as introducing new approaches. Its most central development is *braided quantum field theory*, a generalisation of quantum field theory to braided spaces (Chapter 6). This allows the construction of quantum field theories with quantum group symmetries and with general braid statistics. It forms the basis for much of the other material (Chapters 7–9) or is connected to it (Chapter 5). At this point we would like to emphasise that our approach departs considerably from the functional analytic tradition of quantum mechanics (as it seems any framework must do that allows for radical generalisations of the space-time concept). In particular, our path integral no longer corresponds in general to a canonical description with field operators on a Hilbert space. Whether this is a weakness or a strength remains to be seen. The anyonic example of Chapter 7 appears to point towards the latter.

Overview

Chapter 1 serves as a review for material that underlies much of this dissertation. In particular, some facts about the representation theory of Hopf algebras are recalled which are used in most of the following chapters. Furthermore, aspects of quantum differential calculi are reviewed which are used in Chapters 2–4.

Chapter 2 presents an extension of Drinfeld's twisting theory: A twist relating two Hopf algebras gives rise to an equivalence between the module categories of the two Hopf algebras. We show that such an equivalence holds for many other representation categories as well. Furthermore this gives rise to a 1-1 correspondence between differential calculi on quantum spaces related by twist.

Differential calculi on the quantum group $\mathsf{U}_q(\mathfrak{b}_+)$ are investigated in Chapter 3. We give a complete classification and detailed examination of the classical limit $q \to 1$ and its dual. At the end we comment on the relation to κ-Minkowski space.

The quantum differential geometry of the *Planck scale Hopf algebra* [Maj88] is investigated in Chapter 4. This is a toy model for physics at the Planck scale. The differential calculi and exterior algebra on it are constructed using results from both previous chapters. The quantum geometry gives rise to a *quantum Poisson bracket*, suggesting a deviation from standard quantum mechanics at strong curvature. A Fourier transform is developed implementing a T-duality-like self-duality of the model.

Chapter 5 examines the quantum group symmetries behind spin and statistics. We show that these symmetries can be unified in the presence of a spin-statistics theorem. The Bose-Fermi case as well as the more general anyonic case are considered.

Braided quantum field theory is introduced in Chapter 6. It employs a path integral formalism based on Gaussian integration in braided categories which we review. We obtain a generalisation of Wick's theorem. Based on this we develop perturbation theory leading to *braided Feynman diagrams*, which generalise ordinary Feynman diagrams. Bosonic and Fermionic path integrals and Feynman rules are recovered as special cases.

The remaining chapters concern applications of braided quantum field theory. In Chapter 7 special cases are considered. First, the *quons* studied by Greenberg [Gre91] are investigated as an example of anyonic statistics. We show that braided quantum field theory provides the path integral counterpart to the usual canonical approach to quons. Then, we show that for symmetric braiding a correspondence between braided and ordinary Feynman diagrams can be established. In particular, this applies to Bosons and Fermions. Finally, methods are developed for quantum field theory on quantum homogeneous spaces and on compact quantum spaces.

In Chapter 8, quantum field theory on the noncommutative $\mathbb{R}^d$ arising in a certain limit of string theory is investigated. We show how this space is related to ordinary $\mathbb{R}^d$ by a Drinfeld twist. Using the twisting theory in conjunction with braided quantum field theory leads to an equivalence relating quantum field theories on ordinary and noncommutative $\mathbb{R}^d$. The equivalence exchanges commutativity with noncommutativity and ordinary with a momentum-dependent statistics.

We construct ϕ^4-theory on the quantum 2-sphere in Chapter 9 using the results about homogeneous and compact quantum spaces of Chapter 7. The free propagator is obtained as well as the tadpole diagram. The latter is responsible for making the ordinary ϕ^4-theory divergent. We show that the divergence is regularised in the noncommutative regime $q > 1$. The formalism suggests that divergences of any degree could be regularised in this way. A diagrammatic interpretation of renormalisation of ϕ^4-theory is obtained.

Chapter 1

Basics

In this chapter we review some basic concepts of quantum group theory that underly much of this work.

Section 1.1 recalls definitions of Hopf algebra and (co)quasitriangular structures. This is followed by a brief explanation of our use of the term *quantum group*. Section 1.2 discusses general aspects of the representation theory of Hopf algebras. Some of this is relevant to all of the following chapters. Section 1.3 introduces the notion of differential calculus and exterior algebra for quantum groups which is needed for Chapters 2, 3, and 4.

To fix our notation, we use the symbols Δ, ϵ, and S respectively to denote the coproduct, counit, and antipode of a Hopf algebra, see Section 1.1. For the coproduct we frequently employ Sweedler's component notation $\Delta h = h_{(1)} \otimes h_{(2)}$ etc., with summation implied. We adopt a similar notation $v \mapsto v_{(1)} \otimes v_{(\underline{2})}$ and $v \mapsto v_{(\underline{1})} \otimes v_{(2)}$ for left and right coactions, see Section 1.2.2. A braiding will generally be denoted by ψ, see Section 1.2.1. In this chapter as well as Chapter 2 certain tensor products are denoted by $\odot$. In the other chapters only the symbol $\otimes$ is used. We write a circle $\circ$ for the composition of maps. $\Bbbk$ denotes a general field.

Standard references on Hopf algebras are the books by Sweedler [Swe69] and Abe [Abe80]. For modern accounts of quantum group theory see, e.g., the text-books by Majid [Maj95b], Chari and Pressley [CP94], and Klimyk and Schmüdgen [KS97].

1.1 Hopf algebras

We recall the definitions of a Hopf algebra and those of (co)quasitriangular structures to be found in any text-book on quantum groups.

Recall that an associative *algebra* A is a vector space together with a map $\cdot : A \otimes A \to A$ so that $\cdot \circ (\cdot \otimes \mathrm{id}) = \cdot \circ (\mathrm{id} \otimes \cdot)$ as maps $A \otimes A \otimes A \to A$. A *unit* is an element $\mathbf{1} \in A$ so that $\mathbf{1} \cdot a = a \cdot \mathbf{1} = a$ for all $a \in A$. Equivalently, a unit is a map $\eta : \Bbbk \to A$ so that

$\cdot \circ (\mathrm{id} \otimes \eta) = \cdot \circ (\eta \otimes \mathrm{id}) = \mathrm{id}$ as maps $A \to A$.

Dually, a coassociative *coalgebra* C is a vector space together with a map $\Delta : C \to C \otimes C$ so that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ as maps $C \to C \otimes C \otimes C$. A *counit* is a map $\epsilon : C \to \Bbbk$ so that $(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta$ as maps $C \to C$. We also use Sweedler's notation for coproducts $\Delta c = c_{(1)} \otimes c_{(2)}$, with summation implied. Due to coassociativity, this notation can be extended to multiple applications of the coproduct in the obvious way.

A *bialgebra* B is both an associative algebra with a unit and a coassociative coalgebra with a counit. Furthermore, the two structures are required to be compatible in the obvious way, i.e. $\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}$ and $\epsilon(ab) = \epsilon(a)\, \epsilon(a)$ and $\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ for all $a, b \in B$.

A *Hopf algebra* H is a bialgebra with a map $\mathrm{S} : H \to H$ with the property $(\mathrm{S}\, h_{(1)})h_{(2)} = h_{(1)}\, \mathrm{S}\, h_{(2)} = \epsilon(h)\mathbf{1}$ for all $h \in H$. S is called the *antipode*.

Definition 1.1.1. *A* quasitriangular structure *(also known as "universal R-matrix") on a Hopf algebra H is an invertible element $\mathcal{R} \in H \otimes H$ so that*

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \quad \tau \circ \Delta\, h = \mathcal{R}(\Delta\, h)\mathcal{R}^{-1}\ \forall h \in H,$$

where $\mathcal{R}$ with indices denotes the obvious extension to $H \otimes H \otimes H$ supplemented by the unit and τ denotes the map that exchanges tensor factors.

Definition 1.1.2. *A* coquasitriangular structure *on a Hopf algebra H is a convolution-invertible map $\mathcal{R} : H \otimes H \to \Bbbk$ so that*

$$\mathcal{R}(ab \otimes c) = \mathcal{R}(a \otimes c_{(1)})\mathcal{R}(b \otimes c_{(2)}), \quad \mathcal{R}(a \otimes bc) = \mathcal{R}(a_{(1)} \otimes c)\mathcal{R}(a_{(2)} \otimes b),$$
$$b_{(1)}a_{(1)}\mathcal{R}(a_{(2)} \otimes b_{(2)}) = \mathcal{R}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}$$

for all $a, b, c \in H$.

1.1.1 Quantum Groups

We comment here on our use of the term *quantum group* in this work.

Commutative Hopf algebras provide an equivalent description of group structures as in the following well-known example.

Example 1.1.3. *Let G be a compact Lie group. Denote the set of algebraic functions*[1] *on G by $\mathcal{C}(G)$. Then $\mathcal{C}(G)$ forms a commutative Hopf algebra as follows:*

$$\mathbf{1}(g) = 1, \quad (f \cdot h)(g) = f(g)h(g),$$
$$\epsilon\, f = f(e), \quad \Delta\, f(g, g') = f(gg'), \quad (\mathrm{S}\, f)(g) = f(g^{-1}),$$

[1] These are the functions that are obtained as matrix elements of finite dimensional representations.

where $f, h \in \mathcal{C}(G)$ while $g, g' \in G$ and e denotes the unit element in G. For the coproduct observe the implicit isomorphism $\mathcal{C}(G) \otimes \mathcal{C}(G) \cong \mathcal{C}(G \times G)$ which holds due to the Peter-Weyl theorem.

This example represents very much the point of view we take on how the concept of a (not necessarily commutative) Hopf algebra generalises the concept of a group. We use the term *quantum group* with this setting in mind: A Hopf algebra as a generalised function algebra on a group. Consequently, we are usually interested in coactions rather than actions (see Section 1.2.2) and coquasitriangular rather than quasitriangular structures.

Due to the self-duality of the axioms of a Hopf algebra there is also the dual point of view, where the Hopf algebra plays the role of the enveloping algebra of a Lie algebra or that of a group algebra. We occasionally employ this point of view as well, in particular in Chapters 3 and 4. An advantage of the "function algebra" over the "enveloping algebra" setting lies in the fact that the global structure of the "group" is not lost. This plays an essential role in Chapter 5.

Note that some authors use the term *quantum group* in a narrower sense, so as to only denote q-deformations of Lie groups and enveloping algebras.

1.2 Representation Theory

1.2.1 (Braided) Monoidal Categories

Intuitively speaking, a *monoidal category* is a category with a tensor product. In particular, the category of vector spaces is a monoidal category in this way. The formal definition is as follows:

Definition 1.2.1. *A* monoidal category *is a category $\mathcal{C}$ together with a functor $\odot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbf{1} \in \mathcal{C}$ satisfying the following conditions:*

(i) There is a natural equivalence $\Phi_{U,V,W} : U \odot (V \odot W) \to (U \odot V) \odot W$ satisfying

$$
\begin{tikzcd}
 & (U \odot V) \odot (W \odot X) \arrow[dr, "\Phi"] & \\
U \odot (V \odot (W \odot X)) \arrow[ur, "\Phi"] \arrow[d, "\mathrm{id} \odot \Phi"'] & & ((U \odot V) \odot W) \odot X \\
U \odot ((V \odot W) \odot X) \arrow[rr, "\Phi"'] & & (U \odot (V \odot W)) \odot X \arrow[u, "\Phi \odot \mathrm{id}"']
\end{tikzcd}
$$

(ii) There are natural equivalences $\rho_U : U \odot \mathbf{1} \to U$ and $\lambda_U : \mathbf{1} \odot U \to U$ satisfying

$$\begin{tikzcd} U \odot (\mathbf{1} \odot V) \arrow[rr, "\Phi"] \arrow[dr, "\mathrm{id} \odot \lambda"'] & & (U \odot \mathbf{1}) \odot V \arrow[dl, "\rho \odot \mathrm{id}"] \\ & U \odot V & \end{tikzcd}$$

Definition 1.2.2. *A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between monoidal categories is a* monoidal functor *if $\mathcal{F}(\mathbf{1}_\mathcal{C}) = \mathbf{1}_\mathcal{D}$ and there is a natural equivalence $c_{U,V} : \mathcal{F}(U) \odot_\mathcal{D} \mathcal{F}(V) \to \mathcal{F}(U \odot_\mathcal{C} V)$ satisfying the "associativity condition" (i)*

$$\begin{tikzcd} & \mathcal{F}(U \odot_\mathcal{C} V) \odot_\mathcal{D} \mathcal{F}(W) \arrow[dr, "c"] & \\ (\mathcal{F}(U) \odot_\mathcal{D} \mathcal{F}(V)) \odot_\mathcal{D} \mathcal{F}(W) \arrow[ur, "c \odot_\mathcal{D} \mathrm{id}"] \arrow[d, "\Phi_\mathcal{C}"'] & & \mathcal{F}((U \odot_\mathcal{C} V) \odot_\mathcal{C} W) \arrow[d, "\mathcal{F}(\Phi_\mathcal{C})"] \\ \mathcal{F}(U) \odot_\mathcal{D} (\mathcal{F}(V) \odot_\mathcal{D} \mathcal{F}(W)) \arrow[dr, "\mathrm{id} \odot_\mathcal{D} c"'] & & \mathcal{F}(U \odot_\mathcal{C} (V \odot_\mathcal{C} W)) \\ & \mathcal{F}(U) \odot_\mathcal{D} \mathcal{F}(V \odot_\mathcal{C} W) \arrow[ur, "c"'] & \end{tikzcd}$$

and the "unit conditions" (ii)

$$\begin{tikzcd} \mathcal{F}(\mathbf{1}_\mathcal{C}) \odot_\mathcal{D} \mathcal{F}(U) \arrow[r, "c"] \arrow[d, "\mathrm{id}"'] & \mathcal{F}(\mathbf{1}_\mathcal{C} \odot_\mathcal{C} U) \arrow[d, "\mathcal{F}(\lambda_\mathcal{C})"] \\ \mathbf{1}_\mathcal{D} \odot_\mathcal{D} \mathcal{F}(U) \arrow[r, "\lambda_\mathcal{D}"'] & \mathcal{F}(U) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(U) \odot_\mathcal{D} \mathcal{F}(\mathbf{1}_\mathcal{C}) \arrow[r, "c"] \arrow[d, "\mathrm{id}"'] & \mathcal{F}(U \odot_\mathcal{C} \mathbf{1}_\mathcal{C}) \arrow[d, "\mathcal{F}(\rho_\mathcal{C})"] \\ \mathcal{F}(U) \odot_\mathcal{D} \mathbf{1}_\mathcal{D} \arrow[r, "\rho_\mathcal{D}"'] & \mathcal{F}(U) \end{tikzcd}$$

A monoidal category is called *strict* if all the natural equivalences Φ, ρ, and λ are identities. This is the only case of interest to us in the following. In particular, it applies to the category of vector spaces with the ordinary tensor product $\otimes$.

An additional important structure for monoidal categories is a *braiding*. This is an invertible functor $\psi_{V,W} : V \odot W \to W \odot V$ for objects V, W obeying an associativity condition. A monoidal category equipped with such a braiding is also called a *braided monoidal category* (or "quasitensor category"). The definitions are as follows.

Definition 1.2.3. *A* braided monoidal category *is a monoidal category $\mathcal{C}$ together with a* *natural equivalence* $\psi_{U,V} : U \odot V \to V \odot U$ *satisfying (i)*

$$\begin{tikzcd}[column sep=tiny]
 & U \odot (V \odot W) \arrow[dl, "\mathrm{id} \odot \psi"'] \arrow[dr, "\Phi"] & \\
U \odot (W \odot V) \arrow[d, "\Phi"'] & & (U \odot V) \odot W \arrow[d, "\psi"] \\
(U \odot W) \odot V \arrow[dr, "\psi \odot \mathrm{id}"'] & & W \odot (U \odot V) \arrow[dl, "\Phi"] \\
 & (W \odot U) \odot V &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
 & (U \odot V) \odot W \arrow[dl, "\Phi^{-1}"'] \arrow[dr, "\psi \odot \mathrm{id}"] & \\
U \odot (V \odot W) \arrow[d, "\psi"'] & & (V \odot U) \odot W \arrow[d, "\Phi^{-1}"] \\
(V \odot W) \odot U \arrow[dr, "\Phi^{-1}"'] & & V \odot (U \odot W) \arrow[dl, "\mathrm{id} \otimes \psi"] \\
 & V \odot (W \odot U) &
\end{tikzcd}$$

and (ii)

$$\begin{tikzcd}
\mathbf{1} \odot U \arrow[rr, "\psi"] \arrow[dr, "\lambda"'] & & U \odot \mathbf{1} \arrow[dl, "\rho"] \\
 & U &
\end{tikzcd}
\qquad
\begin{tikzcd}
U \odot \mathbf{1} \arrow[rr, "\psi"] \arrow[dr, "\rho"'] & & \mathbf{1} \odot U \arrow[dl, "\lambda"] \\
 & U &
\end{tikzcd}$$

Definition 1.2.4. *A* braided monoidal functor *is a monoidal functor between braided monoidal categories* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ *satisfying*

$$\begin{tikzcd}
\mathcal{F}(U) \odot_{\mathcal{D}} \mathcal{F}(V) \arrow[r, "c"] \arrow[d, "\psi_{\mathcal{D}}"'] & \mathcal{F}(U \odot_{\mathcal{C}} V) \arrow[d, "\mathcal{F}(\psi_{\mathcal{C}})"] \\
\mathcal{F}(V) \odot_{\mathcal{D}} \mathcal{F}(U) \arrow[r, "c"'] & \mathcal{F}(V \odot_{\mathcal{C}} U)
\end{tikzcd}$$

Since we use braided monoidal categories extensively in the following, we introduce the customary diagrammatic notation to represent morphisms. A diagram is to be read from top to bottom. The top line and the bottom line represent tensor products of objects. The objects on the top line are connected with objects on the bottom line by strands which may cross. Over- and under-crossings are distinct and correspond to the braiding ψ and its inverse, see Figure 1.1. Strands that do not cross correspond to the identity map on that tensor factor.

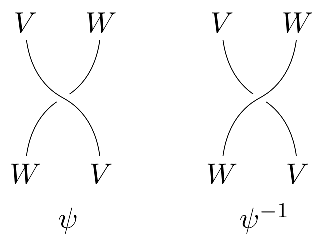


Figure 1.1: The braiding and its inverse in diagrammatic notation.

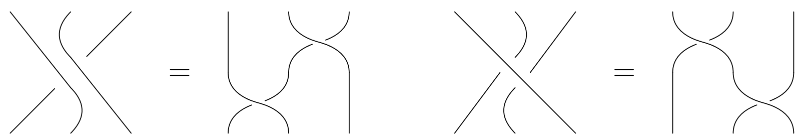


Figure 1.2: Compatibility of braiding and tensor product.

For example, we can express the hexagon identities of Definition 1.2.3.(i) by the diagrams of Figure 1.2 (Φ is implicit). Here, crossings with close parallel strands represent a braiding with a tensor product of the corresponding objects.

If $\psi = \psi^{-1}$ the braiding and the category are called *symmetric*. It means diagrammatically that over- and under-crossings are exchangeable.

The material of this section may be found in (the new edition of) Mac Lane's book on category theory [Mac98] or in some of the books on quantum groups [Maj95b, CP94].

1.2.2 Hopf Algebra Module Categories

Let A be an algebra. A *left action* of A on a vector space V is a linear map $\triangleright : A \otimes V \to V$ so that the following diagrams commute:

$$
\begin{tikzcd}
V \arrow[r, "\cong"] \arrow[d, "\mathrm{id}"'] & \Bbbk \otimes V \arrow[d, "\eta \otimes \mathrm{id}"] \\
V & A \otimes V \arrow[l, "\triangleright"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes V \arrow[r, "\cdot \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \triangleright"'] & A \otimes V \arrow[d, "\triangleright"] \\
A \otimes V \arrow[r, "\triangleright"'] & V
\end{tikzcd}
$$

A vector space with a left action of A is a *left A-module*. We denote the category of left A-modules by ${}_A\mathcal{M}$.

Now, let C be a coalgebra. A *left coaction* of C on a vector space V is a linear map $\beta : V \to C \otimes V$ so that the following diagrams commute:

$$
\begin{tikzcd}
V \arrow[r, "\beta"] \arrow[d, "\mathrm{id}"'] & C \otimes V \arrow[d, "\epsilon \otimes \mathrm{id}"] \\
V & \Bbbk \otimes V \arrow[l, "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\beta"] \arrow[d, "\beta"'] & C \otimes V \arrow[d, "\mathrm{id} \otimes \beta"] \\
C \otimes V \arrow[r, "\Delta \otimes \mathrm{id}"'] & C \otimes C \otimes V
\end{tikzcd}
$$

A vector space with a left coaction of C is a *left C-comodule*. We also use the notation $\beta(v) = v_{(1)} \otimes v_{(\underline{2})}$ for a left action, in analogy with the coproduct. Note that the index of the component living in the module is underlined. If C is a bialgebra, an element $v \in V$ is called *left-invariant* if $\beta(v) = \mathbf{1} \otimes v$. We denote the category of left C-comodules by ${}^{C}\mathcal{M}$.

We also have the corresponding right-sided notions. We often denote left coactions by β_L and right ones by β_R.

An important property of the categories of modules and comodules over a bialgebra is the fact that they are monoidal categories. More precisely, the tensor product of vector spaces provides a monoidal structure.

Let H be a bialgebra. Let V,W be left H-modules. Then $V \otimes W$ is a left H-module with the action

$$a \triangleright (v \otimes w) = (a_{(1)} \triangleright v) \otimes (a_{(2)} \triangleright w). \tag{1.1}$$

Dually, let V,W be left H-comodules. Then $V \otimes W$ is a left H-comodule with the coaction

$$\beta(v \otimes w) = v_{(1)} w_{(1)} \otimes (v_{(\underline{2})} \otimes w_{(\underline{2})}). \tag{1.2}$$

This defines the monoidal structure for the categories ${}_{H}\mathcal{M}$ and ${}^{H}\mathcal{M}$. The unit object in both categories is just the one-dimensional vector space isomorphic to $\Bbbk$ with trivial module ($a \triangleright v = \epsilon(a)v$) or comodule ($\beta(v) = \mathbf{1} \otimes v$) structure.

We can also consider more complicated types of modules. Over an algebra A, for example, we can combine left and right module structures. Let V be left A-module and a right A-module. The natural property to demand is that the module structures commute, i.e. that the diagram

$$\begin{tikzcd} A \otimes V \otimes A \arrow[r, "\triangleright\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\triangleleft"'] & V \otimes A \arrow[d, "\triangleleft"] \\ A \otimes V \arrow[r, "\triangleright"'] & V \end{tikzcd}$$

commutes. We say that V is an *A-bimodule* and denote the category of such objects by ${}_{A}\mathcal{M}_{A}$. We can also make this category monoidal by equipping it with the *tensor product over A* defined by the coequalizer diagram

$$\begin{tikzcd} V \otimes H \otimes W \arrow[r, shift left, "\mathrm{id}\otimes\triangleright"] \arrow[r, shift right, "\triangleleft\otimes\mathrm{id}"'] & V \otimes W \arrow[r] & V \odot_A W \end{tikzcd} \tag{1.3}$$

for objects V and W. The unit object is now A with the obvious bimodule structure. The action on the tensor product is given by

$$a \triangleright (v \odot_A w) = (a \triangleright v) \odot_A w \tag{1.4}$$

and correspondingly for the right action. If H is a bialgebra, we can alternatively endow the category ${}_H\mathcal{M}_H$ with a monoidal structure analogous to the one of ${}_H\mathcal{M}$ defined above. In this case the left action on the tensor product is given by (1.1) and the right action correspondingly. In order to distinguish between the two monoidal structures, we call the one with the tensor product and unit of $\mathcal{V}$ec (the category of vector spaces) *thin* and the one with the unit given by H *thick*. We reflect this in our notation by writing a bar under thick module categories with the tensor product (1.3).

We also have the dual notion of *bicomodules* over a coalgebra C. In the category ${}^C\mathcal{M}^C$ the natural monoidal structure is given by the *cotensor product over* C defined by the equalizer diagram

$$V \odot^C W \longrightarrow V \otimes W \underset{\beta_R \otimes \mathrm{id}}{\overset{\mathrm{id} \otimes \beta_L}{\rightrightarrows}} V \otimes C \otimes W \tag{1.5}$$

for objects V and W. Analogous to (1.4) the coaction on tensor products is

$$\beta_L(v \odot^C w) = v_{(1)} \otimes (v_{(2)} \odot^C w) \tag{1.6}$$

and accordingly for the right coaction. We denote this monoidal structure by a bar over the corresponding category. Again, for a bialgebra H we can alternatively equip ${}^H\mathcal{M}^H$ with the thin monoidal structure. The left coaction on a tensor product is then given by (1.2) and the right one accordingly.

Let us combine two of the notions defined so far to form a category which will be important in later considerations. Consider an algebra A in the thin monoidal category ${}^H\mathcal{M}^H$, i.e. an algebra so that the multiplication $A \otimes A \to A$ is an H-bicomodule map (notice that the tensor product here is formally the thin one in the category and not the one of $\mathcal{V}$ec). A is also called an H-bicomodule algebra. Now, consider the category of A-bimodules inside the category ${}^H\mathcal{M}^H$. Equipping it with the thick tensor product (1.3) we denote this monoidal category by ${}^H_A\underline{\mathcal{M}}^H_A$. One can easily check that the kernel of the projection $V \otimes W \to V \odot_A W$ (which is spanned by elements $(v \triangleleft a) \otimes w - v \otimes (a \triangleright w)$) is invariant under coactions of H so the tensor product $\odot_A$ indeed exists in ${}^H_A\underline{\mathcal{M}}^H_A$ and our definition is well. Note that we can also make this construction in ${}^H\mathcal{M}$ leading to the monoidal category ${}^H_A\underline{\mathcal{M}}_A$. We also have the left-right reversed notion of this and the dual ones with module and comodule structures interchanged.

Over a bialgebra H we have even more possibilities. In particular, we can combine module and comodule structures. We always demand the commutativity of the various structures and denote the categories in the obvious way. Such modules are also called *Hopf modules*. In particular, the category ${}_H\underline{\mathcal{M}}^H_H$ is a thick monoidal category with the tensor product over H (1.3) and the category ${}^H\overline{\mathcal{M}}^H_H$ is a thick monoidal category with the cotensor product over H (1.5) (correspondingly under left-right reversal). The category ${}^H_H\mathcal{M}^H_H$ can be equipped with both tensor products. Notice for example that ${}^H_H\underline{\mathcal{M}}^H_H$ is a special case of ${}^H_A\underline{\mathcal{M}}^H_A$ by setting $A = H$. Similar statements hold for other Hopf module categories.

There is another interesting type of module over a bialgebra H. Let V be a left H-module and H-comodule with the property

$$h_{(1)}v_{(1)} \otimes h_{(2)} \triangleright v_{(2)} = (h_{(1)} \triangleright v)_{(1)}h_{(2)} \otimes (h_{(1)} \triangleright v)_{(2)}.$$

We say that V is a *left crossed H-module* (also called "Yetter-Drinfeld module"). We denote the category of such objects by ${}^H_H\dot{\mathcal{M}}$. This category naturally has the thin monoidal structure with relations (1.1) and (1.2). We also have the corresponding right sided notion.

From this point on, let H be a Hopf algebra. It turns out that the classification of the different module structures is much simplified by the following theorem.

Theorem 1.2.5. *The following equivalences of categories hold.*

(i) $\mathcal{M}^H_H$ *and* $\mathcal{V}\mathrm{ec}$ *are equivalent,*

(ii) ${}_H\underline{\mathcal{M}}^H_H$ *and* ${}_H\mathcal{M}$ *are monoidal equivalent,*

(iii) ${}^H\overline{\mathcal{M}}^H_H$ *and* ${}^H\mathcal{M}$ *are monoidal equivalent,*

(iv) ${}^H_H\underline{\mathcal{M}}^H_H$*,* ${}^H_H\overline{\mathcal{M}}^H_H$ *and* ${}^H_H\dot{\mathcal{M}}$ *are (pre-)braided*[2] *monoidal equivalent.*

Proof. We only sketch the construction that establishes the equivalence. For a full proof see e.g. [Sch94].

In all cases the functor establishing the equivalence between the thin and thick categories (for the purpose of this discussion we also mean the pair $\mathcal{V}\mathrm{ec}$, $\mathcal{M}^H_H$ here) is given by the assignment $M \mapsto M \rtimes H$ for M an object in the thin category. $M \rtimes H$ is an object in the thick category built on the vector space $M \otimes H$. The right actions and coactions are given by the product and coproduct on H from the right. The left actions and/or coactions are given by (1.1) and (1.2). Conversely, for a thick module V we take its right-invariant subspace M (i.e. $M = \{m \in V | \beta_R(m) = m \otimes 1\}$) and find that $\triangleleft \circ (\mathrm{id} \otimes \mathrm{S}) \circ \beta_R$ projects from V to M. The projection of the left action and/or coaction gives M the structure required to be in the corresponding thin module category. On the other hand $M \rtimes H$ recovers V. For the tensor product one has $(V \otimes W) \rtimes H \cong (V \rtimes H) \odot (W \rtimes H)$ as required (here $\otimes$ denotes the thin and $\odot$ the relevant thick tensor product). $\square$

We also have the left-right reversed version of the theorem.

[2]See explanation below.

There is a thick module category that naturally has the structure of a braided category. This is ${}^H_H\underline{\mathcal{M}}^H_H$ (choosing the version with the tensor product (1.3)). The braiding is given by

$$\psi_{V,W}(v \odot_H w) = w \odot_H v \tag{1.7}$$

for $v \in V$ left-invariant and $w \in W$ right-invariant. This determines $\psi_{V,W}$ completely by the requirement that it is a bimodule map (to be a morphism in the category). To be precise, this is only a pre-braiding, i.e. ψ is not necessarily invertible. However, if the antipode is a bijection, the invertibility of ψ is guaranteed. The equivalent crossed module categories are also (pre-)braided and the equivalence 1.2.5.(iv) is an equivalence of (pre-)braided categories (see Definition 1.2.4). The (pre-)braiding in ${}^H_H\dot{\mathcal{M}}$ is given by

$$\psi_{V,W}(v \otimes w) = (v_{(1)} \triangleright w) \otimes v_{(2)}.$$

Correspondingly in $\dot{\mathcal{M}}^H_H$.

The category of (co)modules of H also acquires the structure of a braided category if H carries a (co)quasitriangular structure. Let $\mathcal{R} : H \otimes H \to \Bbbk$ denote a coquasitriangular structure (see Definition 1.1.2). The braiding on comodules V and W is then given by

$$\psi_{V,W}(v \otimes w) = \mathcal{R}(w_{(1)} \otimes v_{(1)})\, w_{(2)} \otimes v_{(2)}. \tag{1.8}$$

Similarly for right comodules and dually for quasitriangular structures and modules.

Bibliographical Notes

The treatment of Hopf modules appears e.g. in Sweedler's book [Swe69] and Abe's book [Abe80]. Crossed modules were introduced by Yetter [Yet90] and are also based on work by Drinfeld [Dri87] and Radford [Rad85].

Part (i) of Theorem 1.2.5 is essentially the structure theorem for bimodules of Sweedler [Swe69, Theorem 4.1.1]. The further aspects of Theorem 1.2.5 appear in the subsequent Hopf algebra literature. A complete formulation was given by Schauenburg [Sch94].

The attributes *thin* and *thick* are not standard and were introduced here for convenience.

1.3 Quantum Differentials

In this section we introduce the quantum analogue of differential forms.

1.3.1 First Order Differential Calculi

Definition 1.3.1. *Let A be an algebra. A* first order differential calculus Ω^1 *over A is an A-bimodule together with a linear map $\mathrm{d} : A \to \Omega^1$ obeying the Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + a\,\mathrm{d}b$ and $\Omega^1 = \mathrm{span}\{a\,\mathrm{d}b | a, b \in A\}$.*

This is the most general "quantum version" of the usual space of 1-forms over a manifold. A plays the role of the algebra of (say) smooth real valued functions on a differentiable manifold. Unfortunately, this definition appears to be far too general to be useful. To be more restrictive, we could require symmetries to be present. Classically, this would mean that a Lie group acts on the manifold, inducing an action on differential forms. This translates to a Hopf algebra H coacting on the algebra A, so that A becomes an H-comodule algebra. H plays the role of the algebra of functions on the Lie group. If A is a left H-comodule algebra we can demand that the first order differential calculus also lives in the category ${}^H\mathcal{M}$ of left H-comodules instead of $\mathcal{V}$ec. Similarly for right- and for bicomodules. This gives rise to the following definition.

Definition 1.3.2. *In the context of Definition 1.3.1,* Ω^1 *is called* left-/right-/bicovariant *if A is a left-/right-/bicomodule over H and the actions of A on Ω^1 as well as the map* d *are left/right/bicomodule maps.*

A case of special interest is the situation where A itself is a Hopf algebra. It is then natural to demand that a first order differential calculus be bicovariant under $H = A$ itself. This corresponds classically to the differential forms on a Lie group carrying actions by left and right translation. In the following we use "differential calculus" as a shorthand for "bicovariant first order differential calculus". The bimodule and bicomodule structure imply that such a differential calculus lives in ${}^A_A\underline{\mathcal{M}}^A_A$. Thus, by Theorem 1.2.5.(iv) there is a one-to-one correspondence to objects M in the category ${}^A_A\dot{\mathcal{M}}$ of crossed modules of A. Following Woronowicz [Wor89], the further structure of a differential calculus allows to identify the objects M as submodules of $A' := \ker \epsilon \subset A$, where A is a crossed module over itself via left multiplication and left adjoint coaction:

$$a \triangleright v = av, \qquad \mathrm{Ad_L}(v) = v_{(1)} \,\mathrm{S}\, v_{(3)} \otimes v_{(2)} \qquad \forall a \in A, v \in M.$$

More precisely, given a crossed submodule M, the corresponding calculus is given by $\Gamma = (A'/M) \otimes A$ with $\mathrm{d}a = \pi(\Delta\, a - \mathbf{1} \otimes a)$ (π the canonical projection). The right action and coaction on Γ are given by the right multiplication and coproduct on A, the left action and coaction by the tensor product ones with A'/M as a left crossed module.

Alternatively [Maj98], given in addition a Hopf algebra H dually paired with A (which we might think of as being of enveloping algebra type), we can express the coaction of A on itself as an action of H^{op} (the Hopf algebra with the opposite product) using the pairing:

$$h \triangleright v = \langle h, v_{(1)} \,\mathrm{S}\, v_{(3)} \rangle v_{(2)} \qquad \forall h \in H^{op}, v \in A.$$

Thereby we change from the category of (left) crossed A-modules to the equivalent category of left modules of the quantum double $A \bowtie H^{op}$.

In this picture the pairing between A and H descends to a pairing between $A/\Bbbk\mathbf{1}$ (which we may identify with A') and $H' := \ker\epsilon \subset H$. Further quotienting $A/\Bbbk\mathbf{1}$ by M (viewed in $A/\Bbbk\mathbf{1}$) leads to a pairing with the subspace $L \subseteq H'$ that annihilates M. L is called a *quantum tangent space* and is dual to the differential calculus Γ generated by M in the sense that $\Gamma \cong \mathrm{Lin}(L, A)$ via

$$A/(\Bbbk\mathbf{1}+M)\otimes A \to \mathrm{Lin}(L,A), \qquad v\otimes a \mapsto \langle\cdot, v\rangle a \tag{1.9}$$

if the pairing between $A/(\Bbbk\mathbf{1}+M)$ and L is non-degenerate.

The quantum tangent spaces are obtained directly by dualising the (left) action of the quantum double on A to a (right) action on H. Explicitly, this is the adjoint action and the coregular action

$$h \triangleright x = h_{(1)} x \,\mathrm{S}\, h_{(2)}, \qquad a \triangleright x = \langle x_{(1)}, a\rangle x_{(2)} \qquad \forall a \in A^{op},\ h, x \in H,$$

where we have converted the right action to a left action by going from $A \bowtie H^{op}$-modules to $H \bowtie A^{op}$-modules. Quantum tangent spaces are subspaces of H' invariant under the projection of this action to H' via $x \mapsto x - \epsilon(x)\mathbf{1}$. Alternatively, the left action of A^{op} can be converted to a left coaction of H which is the comultiplication (with subsequent projection onto $H \otimes H'$).

We can use the evaluation map (1.9) to define a "braided derivation" on elements of the quantum tangent space via

$$\partial_x : A \to A, \qquad \partial_x(a) = \mathrm{d}a(x) = \langle x, a_{(1)}\rangle a_{(2)} \qquad \forall x \in L, a \in A.$$

This obeys the braided derivation rule

$$\partial_x(ab) = (\partial_x a)b + a_{(2)}\partial_{a_{(1)}\triangleright x} b \qquad \forall x \in L, a \in A.$$

Given a right invariant basis $\{\eta_i\}_{i\in I}$ of Γ with a dual basis $\{\phi_i\}_{i\in I}$ of L we have

$$\mathrm{d}a = \sum_{i\in I} \eta_i \cdot \partial_i(a) \qquad \forall a \in A,$$

with $\partial_i := \partial_{\phi_i}$. (This can easily be seen to hold by evaluating against ϕ_i $\forall i$.)

1.3.2 Exterior Differential Algebras

We can further define the quantum version of a whole exterior algebra.

Definition 1.3.3. *Let A be an algebra. An* exterior differential algebra *over A is a graded A-bimodule algebra $\Omega = \oplus_{n=0}^{\infty}\Omega^n$ where $\Omega^0 = A$, together with a linear map $\mathrm{d} : \Omega \to \Omega$ of degree one. We require d to satisfy $\mathrm{d}^2 = 0$, the graded Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + (-1)^{|a|} a\, \mathrm{d}b$, and $\Omega^n = \mathrm{span}\{a_0\, \mathrm{d}a_1\, \mathrm{d}a_2 \ldots \mathrm{d}a_n | a_i \in A\}$.*

Notice that the span-condition implies that Ω is a quotient of the tensor algebra $\Omega^* = \oplus_{n=0}^{\infty}\tilde{\Omega}^n$ with $\tilde{\Omega}^n = \Omega^1 \odot_A \Omega^1 \odot_A \ldots \odot_A \Omega^1$ (n-fold).

In the same way as for a first order differential calculus we define the analogous covariant versions.

Definition 1.3.4. *In the context of Definition 1.3.3, Ω is called* left-/right-/bicovariant *if A is a left-/right-/bicomodule over H and the actions of A on Ω as well as the map* d *and the grading are left/right/bicomodule maps.*

We specialise again to the bicovariant case with $A = H$. It turns out that it is natural to consider exterior differential algebras with a super-Hopf algebra[3] structure in this case. This comes from the following Proposition.

Proposition 1.3.5 (cf. [Brz93]). *Let Γ be an H-bicovariant bimodule (that is $\Gamma \in {}^H_H\underline{\mathcal{M}}^H_H$). The tensor algebra $\Gamma^* = \oplus_{n=0}^{\infty}\Gamma^n$ with $\Gamma^0 = H$ and $\Gamma^n = \Gamma \odot_H \Gamma \odot_H \ldots \odot_H \Gamma$ (n-fold) is a super-Hopf algebra. The coproduct and antipode are*

$$\Delta = \beta_L + \beta_R, \quad \mathrm{S}\,\alpha = -(\mathrm{S}\,\alpha_{(1)}) \cdot \alpha_{(\underline{2})} \cdot (\mathrm{S}\,\alpha_{(3)})$$

on degree 1 and extended to Γ^* as a super-Hopf algebra.

Proof. The proof is by induction. First note that Δ as stated is a bimodule map since β_L, β_R are. We extend it by

$$\Delta(\alpha \odot_H \beta) = (-1)^{|\alpha_{(2)}||\beta_{(1)}|}(\alpha_{(1)} \odot_H \beta_{(1)}) \otimes (\alpha_{(2)} \odot_H \beta_{(2)})$$

which is well-defined since Δ on α, β is a bimodule map. Moreover, for the same reason Δ remains a bimodule map. Coassociativity on degree 1 follows from that of H and the bicomodule properties of Ω, and likewise extends to all degrees by induction. By construction, Δ is an algebra map with the super-tensor product. Hence we have a super-bialgebra.

Similarly, it is easy to see from β_L, β_R bimodule maps that $\mathrm{S}(h \cdot \alpha) = (\mathrm{S}\,\alpha) \cdot \mathrm{S}\,h$ and $\mathrm{S}(\alpha \cdot h) = (\mathrm{S}\,h) \cdot \mathrm{S}\,\alpha$ (S a skew-bimodule map). We extend S to higher products by $\mathrm{S}(\alpha \odot_H \beta) = (-1)^{|\alpha||\beta|}(\mathrm{S}\,\beta) \odot_H (\mathrm{S}\,\alpha)$ which is therefore well defined and remains a skew-bimodule map. That the antipode axiom is fulfilled then only has to be verified on degree 1, and extends by induction to all degrees. This is easily verified. □

This proposition is to be seen in connection with the previous remark that exterior differential algebras are quotients of the tensor algebra over the first order component. We

[3]The definition of a super-Hopf algebra is the same as for a Hopf algebra, except that it is $\mathbb{Z}_2$-graded as a vector space and that the compatiblity between algebra and coalgebra structure takes the modified form $\Delta(ab) = (-1)^{|a_{(2)}||b_{(1)}|} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}$.

can thus limit ourselves to quotients that inherit the super-Hopf algebra structure of Proposition 1.3.5. The $\mathbb{Z}_2$-grading is necessary to allow commutativity of the coproduct with d. These considerations give rise to the following definition.

Definition 1.3.6 ([Brz93]). *Let H be a Hopf algebra. A* bicovariant exterior differential Hopf algebra *over H is a graded super-Hopf algebra $\Omega = \oplus_{n=0}^{\infty}\Omega^n$ in ${}^H_H\underline{\mathcal{M}}^H_H$ where $\Omega^0 = H$ together with an H-bicomodule map $\mathrm{d} : \Omega \to \Omega$ of degree one. We require d to satisfy $\mathrm{d}^2 = 0$, the graded Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + (-1)^{|a|}a\,\mathrm{d}b$, and to commute with coproduct and antipode. We also require $\Omega^n = \mathrm{span}\{a_0\,\mathrm{d}a_1\,\mathrm{d}a_2 \ldots \mathrm{d}a_n | a_i \in A\}$.*

For Hopf algebras there is a standard way of extending a bicovariant first order differential calculus to a bicovariant exterior differential algebra. This is the Woronowicz construction. It starts with the tensor algebra and quotients by an ideal obtained as the kernel of certain "antisymmetriser" maps. Those antisymmetrisers are similar to the classical ones, but instead of being built from permutations they are built from braidings. (For a detailed account see [Wor89].) In the case of an ordinary Lie group the construction reduces to the ordinary construction of the exterior algebra out of the space of 1-forms.

Bibliographical Notes

For differential calculi on quantum groups see Woronowicz's seminal paper [Wor89]. A textbook treatment of some of the material can be found, e.g., in [KS97].

Chapter 2

Twisting Theory

Although there does not as yet seem to be a satisfactory concept of Hopf algebra cohomology, certain aspects of it are known. In particular, 1- and 2-cochains and corresponding boundary operators can be defined. These generalise the respective notions of both Lie algebra and group cohomology. However, Hopf algebra cohomology is much richer due to its "noncommutativity" and more symmetrical due to the self-duality of the Hopf algebra axioms.

Remarkably, a 2-cocycle (i.e. a closed 2-cochain) on a Hopf algebra gives rise to a deformation, called a *twist*. This is a purely "noncommutative effect" which disappears upon restriction to groups or Lie algebras. It turns out to be related to deformation quantisation as will be briefly reviewed in Section 2.3.

In this chapter we study the influence of the twist deformation on the representation theory. In the simplest case, if two Hopf algebras are related by a (coproduct) twist, their respective categories of modules are equivalent. This was found by Drinfeld who introduced the concept of twisting [Dri90]. We extend this result in Section 2.2 by showing that such an equivalence holds in fact for many types of representation categories (as introduced in Section 1.2.2).

We then proceed to apply the results to quantum differential geometry. Namely, we show in Section 2.4 that they give rise to a one-to-one correspondence between quantum differential calculi over quantum spaces that are related by a (product) twist. In particular, this means that a cocycle deformation quantisation (see Section 2.3) of an ordinary group naturally carries a deformation quantised differential calculus. This will be of interest in Chapter 4, where such a quantum group is studied as a toy model for Planck scale physics.

2.1 Hopf Algebras

We review in this section the theory of *twists* on Hopf algebras. This provides a way of obtaining new Hopf algebras from given ones using elements of a Hopf algebra cohomology theory. Twists were introduced by Drinfeld [Dri90] in the context of quasi-Hopf algebras. See also [Maj95b, CP94].

Let H denote a Hopf algebra.

Definition 2.1.1. *An element $\chi \in H \otimes H$ is a* counital 2-cocycle *if it has the following properties.*

(i) χ *is invertible, i.e. there exists* $\chi^{-1} \in H \otimes H$ *so that* $\chi\chi^{-1} = \chi^{-1}\chi = \mathbf{1} \otimes \mathbf{1}$.

(ii) $(\mathbf{1} \otimes \chi)(\mathrm{id} \otimes \Delta)\chi = (\chi \otimes \mathbf{1})(\Delta \otimes \mathrm{id})\chi$. *(cocycle condition)*

(iii) $(\mathrm{id} \otimes \epsilon)\chi = (\epsilon \otimes \mathrm{id})\chi = \mathbf{1}$. *(counitality)*

Proposition 2.1.2. *A counital 2-cocycle $\chi \in H \otimes H$ defines a* twisted *Hopf algebra H^χ with the same algebra structure and counit as H. The coproduct and antipode are given by*

$$\Delta^\chi h = \chi(\Delta h)\chi^{-1}, \quad \mathrm{S}^\chi h = U(\mathrm{S}\, h)U^{-1} \quad \textit{with} \quad U = \chi^{(1)}\, \mathrm{S}\, \chi^{(2)}.$$

If H has a quasitriangular structure $\mathcal{R} \in H \otimes H$, then H^χ has a quasitriangular structure $\mathcal{R}^\chi$ given by

$$\mathcal{R}^\chi = \chi_{21}\mathcal{R}\chi^{-1}.$$

Proof. Sketch (for the Hopf algebra structure): The associativity of the twisted coproduct follows from the cocycle condition 2.1.1.(ii) together with the corresponding condition for χ^{-1} using 2.1.1.(i). That the counit remains a counit follows from the unitality 2.1.1.(iii) with 2.1.1.(i). That the twisted coproduct remains an algebra map is obvious from its definition. It remains to check the antipode property for S^χ, which is done by explicit calculation. $\square$

For a complete proof see e.g. [Maj95b]. Note that the inverse operation to twisting with χ is twisting with χ^{-1}. In particular, χ^{-1} is a counital 2-cocycle with respect to H^χ.

By dualising this setting we obtain twists that modify the product structure instead of the coproduct structure.

Definition 2.1.3. *A linear map $\chi : H \otimes H \to \Bbbk$ is a* unital 2-cocycle *if it has the following properties.*

(i) χ *is convolution invertible, i.e. there exists* $\chi^{-1} : H \otimes H \to \Bbbk$ *so that*

$$\chi(a_{(1)} \otimes b_{(1)})\chi^{-1}(a_{(2)} \otimes b_{(2)}) = \chi^{-1}(a_{(1)} \otimes b_{(1)})\chi(a_{(2)} \otimes b_{(2)}) = \epsilon(a)\,\epsilon(b) \quad \forall a, b \in H.$$

(ii) $\chi(a_{(1)}\otimes b_{(1)})\chi(a_{(2)}b_{(2)}\otimes c) = \chi(b_{(1)}\otimes c_{(1)})\chi(a\otimes b_{(2)}c_{(2)}) \quad \forall a,b,c \in H.$ *(cocycle condition)*

(iii) $\chi(a \otimes 1) = \chi(1 \otimes a) = \epsilon(a) \quad \forall a \in H.$ *(unitality)*

Proposition 2.1.4. *A unital 2-cocycle $\chi : H \otimes H \to \Bbbk$ defines a* twisted *Hopf algebra H_χ with the same coalgebra structure and unit as H. The product and antipode are given by*

$$a \bullet b = \chi(a_{(1)} \otimes b_{(1)})\, a_{(2)}b_{(2)}\, \chi^{-1}(a_{(3)} \otimes b_{(3)}),$$
$$\mathrm{S}_\chi\, a = U(a_{(1)})\,\mathrm{S}\, a_{(2)}U^{-1}(a_{(3)}) \quad \textit{with} \quad U(a) = \chi(a_{(1)} \otimes \mathrm{S}\, a_{(2)}).$$

If H has a coquasitriangular structure $\mathcal{R} : H \otimes H \to \Bbbk$, then H_χ has a coquasitriangular structure $\mathcal{R}_\chi$ given by

$$\mathcal{R}_\chi(a \otimes b) = \chi(b_{(1)} \otimes a_{(1)})\, \mathcal{R}(a_{(2)} \otimes b_{(2)})\, \chi^{-1}(a_{(3)} \otimes b_{(3)}). \tag{2.1}$$

Proof. Sketch (for the Hopf algebra structure): The associativity of the twisted product follows from the cocycle condition 2.1.3.(ii) together with the corresponding condition for χ^{-1} using 2.1.3.(i). That the unit remains a unit follows from the unitality 2.1.3.(iii) with 2.1.3.(i). That the twisted product remains a coalgebra map is obvious from its definition. It remains to check the antipode property for S_χ, which is done by explicit calculation. □

For a complete proof see e.g. [Maj95b]. Again, χ^{-1} is a unital 2-cocycle with respect to H_χ and defines the inverse twist.

2.2 Module Categories

Drinfeld showed [Dri90] that a twist of a (quasitriangular) Hopf algebra extends to its category of modules. More precisely, it gives rise to an equivalence between the (braided) monoidal categories of modules of the original and the twisted Hopf algebra. (See Theorem 2.2.2 below, in dual formulation.)

We show that similar equivalences hold for other kinds of module categories of Hopf algebras as well. Since this case is more relevant in the following, we limit ourselves to the product twists. Each statement has an obvious dual version with coproduct twist. Theorems 2.2.5 and 2.2.6 are due to joint work with Shahn Majid [MO99].

To simplify the notation, we denote actions as multiplications. In particular, twisted actions are denoted with a $\bullet$. For clarity, we denote here (in contrast to Section 1.2.2) a thin tensor product by $\odot$ and its twisted counterpart by $\odot_\chi$.

Theorem 2.2.1. *Let H be a Hopf algebra, $\chi : H \otimes H \to \Bbbk$ a unital 2-cocycle. There is an isomorphism of monoidal categories $\mathcal{G}_\chi : {}^H\mathcal{M}^H \to {}^{H_\chi}\mathcal{M}^{H_\chi}$. $\mathcal{G}_\chi$ leaves the coactions unchanged. The monoidal structure is provided by the natural equivalence*

$$\begin{aligned} \sigma_\chi : \mathcal{G}_\chi(V) \odot_\chi \mathcal{G}_\chi(W) &\to \mathcal{G}_\chi(V \odot W) \\ v \odot_\chi w &\mapsto \chi(v_{(1)} \otimes w_{(1)})\, v_{(\underline{2})} \odot w_{(\underline{2})}\, \chi^{-1}(v_{(3)} \otimes w_{(3)}) \end{aligned}$$

for all $V, W \in {}^H\mathcal{M}^H$.

Proof. Since the twist does not change the coalgebra structure of H, H-bicomodules are H_χ-bicomodules and H-bicomodule morphisms are H_χ-bicomodule morphisms. Thus, $\mathcal{G}_\chi$ is a functor. Since the inverse operation to twisting by χ is twisting by χ^{-1} the invertibility is clear, and $\mathcal{G}_\chi$ is an isomorphism of categories. It remains to be shown that $\mathcal{G}_\chi$ is monoidal.

The first step is to see that σ_χ is a morphism in ${}^{H_\chi}\mathcal{M}^{H_\chi}$, i.e. that it commutes with left and right coactions. For the left coaction this is the commutativity of the diagram

$$\begin{tikzcd} \mathcal{G}_\chi(V) \odot_\chi \mathcal{G}_\chi(W) \arrow[r, "\beta_L"] \arrow[d, "\sigma_\chi"'] & H \otimes (\mathcal{G}_\chi(V) \odot_\chi \mathcal{G}_\chi(W)) \arrow[d, "\mathrm{id} \otimes \sigma_\chi"] \\ \mathcal{G}_\chi(V \odot W) \arrow[r, "\beta_L"'] & H \otimes \mathcal{G}_\chi(V \odot W) \end{tikzcd}$$

which we easily check as

$$\begin{aligned} &(\mathrm{id} \otimes \sigma_\chi) \circ \beta_L(v \odot_\chi w) \\ &= (\mathrm{id} \otimes \sigma_\chi)(v_{(1)} \bullet w_{(1)} \otimes (v_{(\underline{2})} \odot_\chi w_{(\underline{2})})) \\ &= v_{(1)} \bullet w_{(1)} \otimes \chi(v_{(2)} \otimes w_{(2)})(v_{(\underline{3})} \odot w_{(\underline{3})})\chi^{-1}(v_{(4)} \otimes w_{(4)}) \\ &= \chi(v_{(1)} \otimes w_{(1)})v_{(2)}w_{(2)} \otimes (v_{(\underline{3})} \odot w_{(\underline{3})})\chi^{-1}(v_{(4)} \otimes w_{(4)}) \\ &= \beta_L \circ \sigma_\chi(v \odot_\chi w). \end{aligned}$$

For the right coaction we leave out the proof since it is simply obtained by left-right reversal. $\mathcal{G}_\chi$ is the identity on morphisms, so the natural transformation property of σ_χ is its commutativity with morphisms. But this is clear since σ_χ is solely expressed in terms of coactions.

That the unit is preserved by $\mathcal{G}_\chi$ is clear. For the associativity condition (Definition 1.2.2) we observe that Φ is trivial in ${}_H\mathcal{M}_H$. One can then easily check that the commutativity of the diagram in Definition 1.2.2.(i) follows precisely from the cocycle condition for χ of Definition 2.1.3.(ii). Likewise follows the commutativity of the diagrams of Definition 1.2.2.(ii) from the unitality of χ (Definition 2.1.3.(iii)), since the coaction on the unit object just returns the unit object tensor the unit of H. This completes the proof. $\square$

Theorem 2.2.2 (Drinfeld). *Let H be a Hopf algebra, $\chi : H \otimes H \to \Bbbk$ a unital 2-cocycle. There is an isomorphism of monoidal categories $\mathcal{G}_\chi : {}^H\mathcal{M} \to {}^{H_\chi}\mathcal{M}$. $\mathcal{G}_\chi$ leaves the coaction unchanged. The monoidal structure is provided by the natural equivalence*

$$\begin{aligned}\sigma_\chi : \mathcal{G}_\chi(V) \odot_\chi \mathcal{G}_\chi(W) &\to \mathcal{G}_\chi(V \odot W)\\ v \odot_\chi w &\mapsto \chi(v_{(1)} \otimes w_{(1)})\, v_{(\underline{2})} \odot w_{(\underline{2})}\end{aligned}$$

for all $V, W \in {}^H\mathcal{M}$. If H is coquasitriangular, then $\mathcal{G}_\chi$ is an isomorphism of braided monoidal categories.

Proof. We observe that one can identify ${}^H\mathcal{M}$ as a full monoidal subcategory of ${}^H\mathcal{M}^H$ by equipping each left H-comodule with the trivial right H-comodule structure. This subcategory is preserved under the twisting in Theorem 2.2.1. Thus, we can simply restrict Theorem 2.2.1 to obtain the corresponding result for ${}^H\mathcal{M}$. The unitality of χ then yields the formula for σ_χ.

If H is coquasitriangular it remains to show that $\mathcal{G}_\chi$ is braided, i.e., that the diagram of Definition 1.2.4 commutes. Let $\mathcal{R} : H \otimes H \to \Bbbk$ denote the coquasitriangular structure. With (2.1) we find

$$\begin{aligned}&\sigma_\chi \circ \psi_\chi(v \odot_\chi w)\\ &= \mathcal{R}_\chi(w_{(1)} \otimes v_{(1)})\, \sigma_\chi(w_{(\underline{2})} \odot_\chi v_{(\underline{2})})\\ &= \mathcal{R}_\chi(w_{(1)} \otimes v_{(1)})\, \chi(w_{(2)} \otimes v_{(2)})\, w_{(\underline{3})} \odot v_{(\underline{3})}\\ &= \chi(v_{(1)} \otimes w_{(1)})\, \mathcal{R}(w_{(2)} \otimes v_{(2)})\, w_{(\underline{3})} \odot v_{(\underline{3})}\\ &= \chi(v_{(1)} \otimes w_{(1)})\, \psi(v_{(\underline{2})} \otimes w_{(\underline{2})})\\ &= \psi \circ \sigma_\chi(v \odot_\chi w).\end{aligned}$$

This completes the proof. □

Consider now an algebra A in ${}^H\mathcal{M}^H$. It is clear what happens with A under the twist ${}^H\mathcal{M}^H \to {}^{H_\chi}\mathcal{M}^{H_\chi}$. As a vector space, A remains the same, but the algebra map $A \odot A \to A$ becomes a map $\mathcal{G}_\chi(A \odot A) \to \mathcal{G}_\chi(A)$. Now, to recover a map $\mathcal{G}_\chi(A) \odot_\chi \mathcal{G}_\chi(A) \to \mathcal{G}_\chi(A)$ we need to compose with σ_χ. Thus, we obtain the twisted algebra A_χ in ${}^{H_\chi}\mathcal{M}^{H_\chi}$ with the twisted product $a \bullet b = \chi(a_{(1)} \otimes b_{(1)})\, a_{(\underline{2})} b_{(\underline{2})}\, \chi^{-1}(a_{(3)} \otimes b_{(3)})$. In fact, we can apply this procedure to the whole category ${}^H_A\underline{\mathcal{M}}^H_A$.

Theorem 2.2.3. *Let H be a Hopf algebra, $\chi : H \otimes H \to \Bbbk$ a unital 2-cocycle, A an H-bicomodule algebra. There is an isomorphism of monoidal categories $\underline{\mathcal{G}}_\chi : {}^H_A\underline{\mathcal{M}}^H_A \to {}^{H_\chi}_{A_\chi}\underline{\mathcal{M}}^{H_\chi}_{A_\chi}$. $\underline{\mathcal{G}}_\chi$ leaves the coactions unchanged and transforms the actions according to*

$$a \bullet v = \chi(a_{(1)} \otimes v_{(1)})\, a_{(2)} v_{(\underline{2})}\, \chi^{-1}(a_{(3)} \otimes v_{(3)}),$$

$$v \bullet a = \chi(v_{(1)} \otimes a_{(1)})\, v_{(\underline{2})} a_{(2)}\, \chi^{-1}(v_{(3)} \otimes a_{(3)}).$$

The monoidal structure is provided by the natural equivalence

$$\begin{aligned} \underline{\sigma}_\chi : \underline{\mathcal{G}}_\chi(V) \odot_{A_\chi} \underline{\mathcal{G}}_\chi(W) &\to \underline{\mathcal{G}}_\chi(V \odot_A W) \\ v \odot_{A_\chi} w &\mapsto \chi(v_{(1)} \otimes w_{(1)})\, v_{(\underline{2})} \odot_A w_{(\underline{2})}\, \chi^{-1}(v_{(3)} \otimes w_{(3)}) \end{aligned}$$

for all $V, W \in {}^H_A\underline{\mathcal{M}}^H_A$.

Proof. By Theorem 2.2.1 the transformation of the action maps is precisely obtained by composition with σ_χ. More generally, any diagram in ${}^H_A\underline{\mathcal{M}}^H_A$ is transformed into the corresponding diagram in ${}^{H_\chi}_{A_\chi}\underline{\mathcal{M}}^{H_\chi}_{A_\chi}$ by the application of σ_χ to tensor products. The consistence of this transformation is precisely guaranteed by the monoidality of the isomorphism. Thus, we clearly obtain an isomorphism of categories ${}^H_A\underline{\mathcal{M}}^H_A \to {}^{H_\chi}_{A_\chi}\underline{\mathcal{M}}^{H_\chi}_{A_\chi}$.

We proceed to show that $\underline{\mathcal{G}}_\chi$ is monoidal. First, we need to see that $\underline{\sigma}_\chi$ is a morphism. The proof that $\underline{\sigma}_\chi$ is an H-bicomodule map is exactly the same as in Theorem 2.2.1 (unaffected by the different tensor product). We check that $\underline{\sigma}_\chi$ is an A-bimodule map. For the left action this is the commutativity of the diagram

$$\begin{tikzcd} A \odot_\chi (\underline{\mathcal{G}}_\chi(V) \odot_{A_\chi} \underline{\mathcal{G}}_\chi(W)) \arrow[r, "\bullet"] \arrow[d, "\mathrm{id} \odot_\chi \underline{\sigma}_\chi"'] & \underline{\mathcal{G}}_\chi(V) \odot_{A_\chi} \underline{\mathcal{G}}_\chi(W) \arrow[d, "\underline{\sigma}_\chi"] \\ A \odot_\chi \underline{\mathcal{G}}_\chi(V \odot_A W) \arrow[r, "\bullet"'] & \underline{\mathcal{G}}_\chi(V \odot_A W) \end{tikzcd}$$

This is

$$\begin{aligned} &\underline{\sigma}_\chi(a \bullet (v \odot_{A_\chi} w)) \\ &= \underline{\sigma}_\chi(a \bullet v \odot_{A_\chi} w) \\ &= \chi(a_{(1)} \otimes v_{(1)})\, \underline{\sigma}_\chi(a_{(2)} v_{(\underline{2})} \odot_{A_\chi} w)\, \chi^{-1}(a_{(3)} \otimes v_{(3)}) \\ &= \chi(a_{(1)} \otimes v_{(1)})\, \chi(a_{(2)} v_{(2)} \otimes w_{(1)})\, a_{(3)} v_{(\underline{3})} \odot_A w_{(\underline{2})} \\ &\quad \chi^{-1}(a_{(4)} v_{(4)} \otimes w_{(3)})\, \chi^{-1}(a_{(5)} \otimes v_{(5)}) \\ &= \chi(a_{(1)} \otimes v_{(1)})\, \chi^{-1}(a_{(2)(1)} \otimes v_{(2)(1)}) \\ &\quad \chi(v_{(2)(2)} \otimes w_{(1)(1)})\, \chi(a_{(2)(2)} \otimes v_{(2)(3)} w_{(1)(2)})\, a_{(3)} v_{(\underline{3})} \odot_A w_{(\underline{2})} \\ &\quad \chi^{-1}(a_{(4)} v_{(4)} \otimes w_{(3)})\, \chi^{-1}(a_{(5)} \otimes v_{(5)}) \\ &= \chi(v_{(1)} \otimes w_{(1)})\, \chi(a_{(1)} \otimes v_{(2)} w_{(2)})\, a_{(2)} v_{(\underline{3})} \odot_A w_{(\underline{3})} \\ &\quad \chi^{-1}(a_{(3)} v_{(4)} \otimes w_{(4)})\, \chi^{-1}(a_{(4)} \otimes v_{(5)}) \\ &= \chi(v_{(1)} \otimes w_{(1)})\, \chi(a_{(1)} \otimes v_{(2)} w_{(2)})\, a_{(2)} v_{(\underline{3})} \odot_A w_{(\underline{3})} \\ &\quad \chi^{-1}(a_{(3)(1)} \otimes v_{(4)(1)} w_{(4)(1)})\, \chi^{-1}(v_{(4)(2)} \otimes w_{(4)(2)}) \end{aligned}$$

$$
\begin{aligned}
&\chi(a_{(3)(2)} \otimes v_{(4)(3)})\, \chi^{-1}(a_{(4)} \otimes v_{(5)}) \\
&= \chi(v_{(1)} \otimes w_{(1)})\, \chi(a_{(1)} \otimes v_{(2)}w_{(2)})\, a_{(2)}v_{(\underline{3})} \odot_A w_{(\underline{3})} \\
&\quad \chi^{-1}(a_{(3)} \otimes v_{(4)}w_{(4)})\, \chi^{-1}(v_{(5)} \otimes w_{(5)}) \\
&= \chi(v_{(1)} \otimes w_{(1)})\, \chi\left(a_{(1)} \otimes (v_{(\underline{2})} \odot_A w_{(\underline{2})})_{(1)}\right)\, a_{(2)}(v_{(\underline{2})} \odot_A w_{(\underline{2})})_{(\underline{2})} \\
&\quad \chi^{-1}\left(a_{(3)} \otimes (v_{(\underline{2})} \odot_A w_{(\underline{2})})_{(3)}\right)\, \chi^{-1}(v_{(3)} \otimes w_{(3)}) \\
&= a \bullet \left(\chi(v_{(1)} \otimes w_{(1)})\, v_{(\underline{2})} \odot_A w_{(\underline{2})}\, \chi^{-1}(v_{(3)} \otimes w_{(3)})\right) \\
&= a \bullet \underline{\sigma}_\chi(v \odot_{A_\chi} w).
\end{aligned}
$$

Similarly for the right action. The rest of the proof of the monoidality of $\underline{\mathcal{G}}_\chi$ is exactly the same as in Theorem 2.2.1 (and is not affected by having a different tensor product), so we omit it here. The only exceptions are the unit conditions 1.2.2.(ii). They follow using the fact that we can represent any element of $A \odot_A V$ as $\mathbf{1} \odot_A v$ for some $v \in V$, and correspondingly for $A_\chi \odot_{A_\chi} W$ (and the same left-right reversed). ☐

Theorem 2.2.4. *Let H be a Hopf algebra, $\chi : H \otimes H \to \Bbbk$ a unital 2-cocycle, A a left H-comodule algebra. There is an isomorphism of monoidal categories $\underline{\mathcal{G}}_\chi : {}^H_A\underline{\mathcal{M}}_A \to {}^{H_\chi}_{A_\chi}\underline{\mathcal{M}}_{A_\chi}$. $\underline{\mathcal{G}}_\chi$ leaves the coactions unchanged and transforms the actions according to*

$$
\begin{aligned}
a \bullet v &= \chi(a_{(1)} \otimes v_{(1)})\, a_{(2)}v_{(\underline{2})}, \\
v \bullet a &= \chi(v_{(1)} \otimes a_{(1)})\, v_{(\underline{2})}a_{(2)}.
\end{aligned}
$$

The monoidal structure is provided by the natural equivalence

$$
\begin{aligned}
\underline{\sigma}_\chi : \underline{\mathcal{G}}_\chi(V) \odot_{A_\chi} \underline{\mathcal{G}}_\chi(W) &\to \underline{\mathcal{G}}_\chi(V \odot_A W) \\
v \odot_{A_\chi} w &\mapsto \chi(v_{(1)} \otimes w_{(1)})\, v_{(\underline{2})} \odot_A w_{(\underline{2})}
\end{aligned}
$$

for all $V, W \in {}^H_A\underline{\mathcal{M}}_A$.

Proof. This follows from Theorem 2.2.3 in the same way as Theorem 2.2.2 follows from Theorem 2.2.1. ☐

Theorem 2.2.5. *Let H be a Hopf algebra, $\chi : H \otimes H \to \Bbbk$ a unital 2-cocycle. There is an isomorphism of (pre-)braided categories $\underline{\mathcal{G}}_\chi : {}^H_H\underline{\mathcal{M}}^H_H \to {}^{H_\chi}_{H_\chi}\underline{\mathcal{M}}^{H_\chi}_{H_\chi}$. $\underline{\mathcal{G}}_\chi$ leaves the coactions unchanged and transforms the actions according to*

$$
\begin{aligned}
h \bullet v &= \chi(h_{(1)} \otimes v_{(1)})\, h_{(2)}v_{(\underline{2})}\, \chi^{-1}(h_{(3)} \otimes v_{(3)}), \\
v \bullet h &= \chi(v_{(1)} \otimes h_{(1)})\, v_{(\underline{2})}h_{(2)}\, \chi^{-1}(v_{(3)} \otimes h_{(3)}).
\end{aligned}
$$

The monoidal structure is provided by the natural equivalence

$$\begin{aligned}\underline{\sigma}_\chi : \underline{\mathcal{G}}_\chi(V) \odot_{H_\chi} \underline{\mathcal{G}}_\chi(W) \to \underline{\mathcal{G}}_\chi(V \odot_H W)\\ v \odot_{H_\chi} w \mapsto \chi(v_{(1)} \otimes w_{(1)})\, v_{(\underline{2})} \odot_H w_{(\underline{2})}\, \chi^{-1}(v_{(3)} \otimes w_{(3)})\end{aligned}$$

for all $V, W \in {}^H_H\underline{\mathcal{M}}^H_H$.

Proof. As remarked in Section 1.2.2, ${}^H_H\underline{\mathcal{M}}^H_H$ as a monoidal category is a special case of ${}^H_A\underline{\mathcal{M}}^H_A$, so the monoidal isomorphy follows from Theorem 2.2.3.

It remains to show that $\underline{\mathcal{G}}_\chi$ is braided, i.e. that the diagram of Definition 1.2.4 commutes. The braiding (1.7) is determined by $\psi(hv \odot_H wg) = hw \odot_H vg$ with $h, g \in H, v \in V$ left-invariant and $w \in W$ right-invariant (similarly over H_χ). We check

$$\begin{aligned}
&\underline{\sigma}_\chi \circ \psi_\chi(h \bullet v \odot_{H_\chi} w \bullet g)\\
&= \underline{\sigma}_\chi(h \bullet w \odot_{H_\chi} v \bullet g)\\
&= \underline{\sigma}_\chi \left(\chi(h_{(1)} \otimes w_{(1)})\, h_{(2)} w_{(\underline{2})} \odot_{H_\chi} v_{(\underline{1})} g_{(1)}\, \chi^{-1}(v_{(2)} \otimes g_{(2)})\right)\\
&= \chi(h_{(1)} \otimes w_{(1)})\, \chi(h_{(2)} w_{(2)} \otimes g_{(1)})\, h_{(3)} w_{(\underline{3})} \odot_H v_{(\underline{1})} g_{(2)}\\
&\quad \chi^{-1}(h_{(4)} \otimes v_{(2)} g_{(3)})\, \chi^{-1}(v_{(3)} \otimes g_{(4)})\\
&= \psi(\chi(h_{(1)} \otimes w_{(1)})\, \chi(h_{(2)} w_{(2)} \otimes g_{(1)})\, h_{(3)} v_{(\underline{1})} \odot_H w_{(\underline{3})} g_{(2)}\\
&\quad \chi^{-1}(h_{(4)} \otimes v_{(2)} g_{(3)})\, \chi^{-1}(v_{(3)} \otimes g_{(4)}))\\
&= \psi(\chi(w_{(1)} \otimes g_{(1)})\, \chi(h_{(1)} \otimes w_{(2)} g_{(2)})\, h_{(2)} v_{(\underline{1})} \odot_H w_{(\underline{3})} g_{(3)}\\
&\quad \chi^{-1}(h_{(3)} \otimes v_{(2)} g_{(4)})\, \chi^{-1}(v_{(3)} \otimes g_{(5)}))\\
&= \psi(\chi(w_{(1)} \otimes g_{(1)})\, \chi(h_{(1)} \otimes w_{(2)} g_{(2)})\, h_{(2)} v_{(\underline{1})} \odot_H w_{(\underline{3})} g_{(3)}\\
&\quad \chi^{-1}(h_{(3)} v_{(2)} \otimes g_{(4)})\, \chi^{-1}(h_{(4)} \otimes v_{(3)}))\\
&= \psi \circ \underline{\sigma}_\chi(\chi(w_{(1)} \otimes g_{(1)})\, h_{(1)} v_{(\underline{1})} \odot_{H_\chi} w_{(\underline{2})} g_{(2)}\, \chi^{-1}(h_{(2)} \otimes v_{(2)}))\\
&= \psi \circ \underline{\sigma}_\chi(h \bullet v \odot_{H_\chi} w \bullet g).
\end{aligned}$$

This completes the proof. □

Theorem 2.2.6. *Let H be a Hopf algebra, $\chi : H \otimes H \to \Bbbk$ a unital 2-cocycle. There is an isomorphism of braided categories $\mathcal{F}_\chi : {}^H_H\dot{\mathcal{M}} \to {}^{H_\chi}_{H_\chi}\dot{\mathcal{M}}$ given by the identity on the underlying vector spaces and coactions, and transforming the action $\triangleright$ to*

$$h \triangleright_\chi v = \chi(h_{(1)} \otimes v_{(1)})(h_{(2)} \triangleright v_{(\underline{2})})_{(\underline{2})} \chi^{-1}((h_{(2)} \triangleright v_{(\underline{2})})_{(1)} \otimes h_{(3)}).$$

The monoidal structure is given by the natural transformation

$$\sigma_\chi : \mathcal{F}_\chi(V) \otimes \mathcal{F}_\chi(W) \to \mathcal{F}_\chi(V \otimes W), \quad v \otimes w \mapsto \chi(v_{(1)} \otimes w_{(1)}) v_{(\underline{2})} \otimes w_{(\underline{2})}.$$

Proof. We deduce this from Theorem 2.2.5 using the equivalence of braided monoidal categories ${}^H_H\underline{\mathcal{M}}^H_H \cong {}^H_H\dot{\mathcal{M}}$ from Theorem 1.2.5. As explained in the proof of 1.2.5, a bicovariant bimodule has the canonical form $V = M \rtimes H$ with M a crossed module. Conversely, M may be recovered as the space of right-invariant elements of V. Since the twisting in ${}^H_H\underline{\mathcal{M}}^H_H$ preserves the coactions, it preserves the decomposition $V = M \rtimes H$ and thus induces a twisting ${}^H_H\dot{\mathcal{M}} \to {}^{H_\chi}_{H_\chi}\dot{\mathcal{M}}$ by restriction to the right-invariant subspace M. The coaction of the crossed module remains unchanged, while the twisted action in ${}^{H^\chi}_{H^\chi}\dot{\mathcal{M}}$ is obtained from the action in ${}^{H_\chi}_{H_\chi}\underline{\mathcal{M}}^{H_\chi}_{H_\chi}$ by subsequent projection (see the proof of 1.2.5). Denoting the twisted actions by $\triangleright_\chi$ and $\bullet$ respectively, this is

$$
\begin{aligned}
h \triangleright_\chi v &= h_{(1)} \bullet v \bullet \mathrm{S}_\chi\, h_{(2)} \\
&= h_{(1)} \bullet v \bullet \mathrm{S}\, h_{(3)} U(h_{(2)})\, U^{-1}(h_{(4)}) \\
&= \chi(h_{(1)} \otimes v_{(1)})\, h_{(2)} \cdot v_{(\underline{2})} \bullet \mathrm{S}\, h_{(4)}\, U(h_{(3)}) U^{-1}(h_{(5)}) \\
&= \chi(h_{(1)} \otimes v_{(1)})\, \chi(h_{(2)} v_{(2)} \otimes \mathrm{S}\, h_{(8)})\, h_{(3)} \cdot v_{(\underline{3})} \cdot \mathrm{S}\, h_{(7)} \\
&\qquad \chi^{-1}(h_{(4)} \otimes \mathrm{S}\, h_{(6)})\, U(h_{(5)}) U^{-1}(h_{(9)}) \\
&= \chi(h_{(1)} \otimes v_{(1)})\, \chi(h_{(2)} v_{(2)} \otimes \mathrm{S}\, h_{(5)})\, h_{(3)} \cdot v_{(\underline{3})} \cdot \mathrm{S}\, h_{(4)}\, U^{-1}(h_{(6)}) \\
&= \chi(h_{(1)} \otimes v_{(1)})\, \chi(h_{(2)} v_{(2)} \otimes \mathrm{S}\, h_{(4)})\, h_{(3)} \triangleright v_{(\underline{3})}\, U^{-1}(h_{(5)}) \\
&= \chi(h_{(1)} \otimes v_{(1)})\, \chi((h_{(2)} \triangleright v_{(\underline{2})})_{(1)} h_{(3)} \otimes \mathrm{S}\, h_{(4)})\, (h_{(2)} \triangleright v_{(\underline{2})})_{(\underline{2})}\, U^{-1}(h_{(5)}) \\
&= \chi(h_{(1)} \otimes v_{(1)})\, (h_{(2)} \triangleright v_{(\underline{2})})_{(\underline{2})}\, \chi^{-1}((h_{(2)} \triangleright v_{(\underline{2})})_{(1)} \otimes h_{(3)}).
\end{aligned}
$$

We used the identity $\chi(ah_{(1)} \otimes \mathrm{S}\, h_{(2)})\, U^{-1}(h_{(3)}) = \chi^{-1}(a \otimes h)$ (which follows from Definition 2.1.3). For the monoidal structure, restricting $\underline{\sigma}_\chi$ given in Theorem 2.2.5 to the right-invariant subspace leads to the stated form. ☐

Note that we also have the obvious right comodule versions of Theorems 2.2.2 and 2.2.4 and the right crossed module version of Theorem 2.2.6.

2.3 Deformation Quantisation

We review (without proofs) a few basic results on the relation of (product) twists with deformation quantisation. They are ultimately due to Drinfeld and provided a motivation for him to introduce twists [Dri83a, Dri83b, Dri87].

Recall that a deformation quantisation of a manifold M with a Poisson bracket $\{\cdot,\cdot\}$ is an associative linear map $\bullet : \mathcal{C}(M) \otimes \mathcal{C}(M) \to \mathcal{C}(M)[[\hbar]]$ which satisfies $f \bullet g = fg + \mathcal{O}(\hbar)$ and $f \bullet g - g \bullet f = \hbar\{f, g\} + \mathcal{O}(\hbar^2)$. One usually also requires that the $\bullet$-product is defined for all orders in $\hbar$ by bidifferential operators. See e.g. [BFF$^+$78].

Given a Lie group, a twist of its Hopf algebra of functions provides a deformation quantisation by the twisted product of Proposition 2.1.4. Such deformations are called *strict*.

Proposition 2.3.1. *Let G be a Lie group. Denote by $H = \mathcal{C}(G)$ the (topological) Hopf algebra of functions on G. Then, a unital 2-cocycle $\chi : H \otimes H \to \mathbb{C}[[\hbar]]$ with $\chi(f \otimes h) = \epsilon(f)\,\epsilon(h) + \mathcal{O}(\hbar)$ defines a strict deformation quantisation of G. Furthermore, expanding $\chi = \sum_n \chi_n \hbar^n$, the Poisson bracket of which χ is the quantisation is given by $\{f, h\} = \chi_1(f_{(1)} \otimes h_{(1)}) f_{(2)} h_{(2)} - f_{(1)} h_{(1)} \chi_1(f_{(2)} \otimes h_{(2)})$. It makes G into a* Poisson-Lie group.

Given a Lie group G acting on a manifold M, $\mathcal{C}(M)$ is a comodule of $\mathcal{C}(G)$ and as such acquires a new product under a twist of $\mathcal{C}(G)$ due to Theorem 2.2.2. This is the following example.

Example 2.3.2. *Let H be a Hopf algebra, A a left H-comodule algebra and χ a unital 2-cocycle over H. Then, A_χ built on A with the new multiplication*

$$a \bullet b = \chi(a_{(1)} \otimes b_{(1)})\, a_{(\underline{2})} b_{(\underline{2})}$$

is an H_χ-comodule algebra.

In fact, this gives rise to a deformation quantisation as well.

Proposition 2.3.3. *Let G be a Lie group acting on a manifold M. Denote by $H = \mathcal{C}(G)$ the (topological) Hopf algebra of functions on G and by $A = \mathcal{C}(M)$ the H-comodule algebra of functions on M. Then, a unital 2-cocycle $\chi : H \otimes H \to \mathbb{C}[[\hbar]]$ so that $\chi(f \otimes h) = \epsilon(f)\,\epsilon(h) + \mathcal{O}(\hbar)$ defines a deformation quantisation of M.*

We refer to deformation quantisations arising as twists as *cocycle deformation quantisations.*

2.4 Quantum Differentials

In this section we show how the twisting theory applies to quantum differential forms.

The results of Section 2.4.2 are due to joint work with Shahn Majid [MO99].

2.4.1 Covariant Differentials

Let H be a Hopf algebra and A an H-comodule algebra. We have already seen that a twist of H induces a twist of A. In fact, this extends to quantum differentials.

Proposition 2.4.1. *Let H be a Hopf algebra, $\chi : H \otimes H \to \Bbbk$ a unital 2-cocycle, A an H-bicomodule algebra. Then, H-bicovariant exterior differential algebras over A are in one-to-one correspondence to H_χ-bicovariant exterior differential algebras over A_χ via the functor $\underline{\mathcal{G}}_\chi$ of Theorem 2.2.3. The corresponding statement holds for first order differential calculi.*

Proof. We use Theorem 2.2.3. Let Ω be an H-bicovariant exterior differential calculus over A. In particular Ω is a graded algebra object in ${}^H_A\underline{\mathcal{M}}^H_A$. Thus, $\underline{\mathcal{G}}_\chi(\Omega)$ is a graded algebra object in ${}^{H_\chi}_{A_\chi}\underline{\mathcal{M}}^{H_\chi}_{A_\chi}$ (note that this implies applying $\underline{\sigma}_\chi$ to obtain the new multiplication). $\Omega^0 = A$ is mapped to A_χ as required. We check that d is also an exterior derivative in the twisted setting.

$$\begin{aligned}
\mathrm{d}(\alpha \bullet \beta) &= \chi(\alpha_{(1)} \otimes \beta_{(1)})\, \mathrm{d}(\alpha_{(\underline{2})}\beta_{(\underline{2})})\, \chi^{-1}(\alpha_{(3)} \otimes \beta_{(3)}) \\
&= \chi(\alpha_{(1)} \otimes \beta_{(1)})\, ((\mathrm{d}\alpha_{(\underline{2})})\beta_{(\underline{2})} + (-1)^{|\alpha_{(\underline{2})}|}\alpha_{(\underline{2})}\mathrm{d}\beta_{(\underline{2})})\, \chi^{-1}(\alpha_{(3)} \otimes \beta_{(3)}) \\
&= \chi((\mathrm{d}\alpha)_{(1)} \otimes \beta_{(1)})\, (\mathrm{d}\alpha)_{(\underline{2})}\beta_{(\underline{2})}\, \chi^{-1}((\mathrm{d}\alpha)_{(3)} \otimes \beta_{(3)}) \\
&\quad + (-1)^{|\alpha|}\chi(\alpha_{(1)} \otimes (\mathrm{d}\beta)_{(1)})\, \alpha_{(\underline{2})}(\mathrm{d}\beta)_{(\underline{2})}\, \chi^{-1}(\alpha_{(3)} \otimes (\mathrm{d}\beta)_{(3)}) \\
&= (\mathrm{d}\alpha) \bullet \beta + (-1)^{|\alpha|}\alpha \bullet \mathrm{d}\beta.
\end{aligned}$$

It is also clear that the span-condition remains unchanged under the twist. Thus, $\underline{\mathcal{G}}_\chi(\Omega)$ with the same d is an H_χ-bicovariant exterior differential algebra over A_χ. Since $\underline{\mathcal{G}}_\chi$ is an isomorphism of categories we obtain a one-to-one correspondence.

For first order differential calculi we observe that they are special cases of exterior differential algebras with forms of degree higher than one vanishing and the product of 1-forms equal to zero. □

We have formulated the proposition for bicomodules here. However, as in Theorem 2.2.4 it specialises to left or to right H-comodules in the obvious way.

2.4.2 Bicovariant Differentials over Quantum Groups

Let us now specialise to the case $A = H$. We have immediately

Corollary 2.4.2. *Let H be a Hopf algebra, $\chi : H \otimes H \to \Bbbk$ a unital 2-cocycle. Then, H-bicovariant exterior differential algebras over H are in one-to-one correspondence to H_χ-bicovariant exterior differential algebras over H_χ via the functor $\underline{\mathcal{G}}_\chi$ of Theorem 2.2.5. The corresponding statement holds for first order differential calculi.*

In view of Proposition 1.3.5 and Definition 1.3.6 we can specialise to bicovariant exterior differential Hopf algebras. This leads to the following result.

Proposition 2.4.3. *Let H be a Hopf algebra, $\chi : H \otimes H \to \mathbb{k}$ a unital 2-cocycle. Then, bicovariant exterior differential Hopf algebras over H and H_χ are in one-to-one correspondence via the functor $\underline{\mathcal{G}}_\chi$ of Theorem 2.2.5.*

Proof. We only have to deal with the structure that is additional as compared to Corollary 2.4.2. Let Ω be a bicovariant exterior differential Hopf algebra. Observe that the twist can now be directly interpreted as a Hopf algebra twist of Ω with χ in the sense of Proposition 2.1.4 (by the trivial extension of χ to $\Omega \otimes \Omega \to \mathbb{k}$ and with the slight modification of a $\mathbb{Z}_2$-grading). This also gives us the twisted antipode.

Counit and coaction are invariant under the twist as is d. The only thing that remains to be checked is that the twisted antipode still commutes with d. This is

$$\begin{aligned}
\mathrm{S}^\chi \,\mathrm{d}\alpha &= U((\mathrm{d}\alpha)_{(1)})\,\mathrm{S}((\mathrm{d}\alpha)_{(2)})U^{-1}((\mathrm{d}\alpha)_{(3)}) \\
&= U(\alpha_{(1)})\,\mathrm{S}(\mathrm{d}\alpha_{(2)})U^{-1}(\alpha_{(3)}) \\
&= U(\alpha_{(1)})\mathrm{d}\,\mathrm{S}(\alpha_{(2)})U^{-1}(\alpha_{(3)}) \\
&= \mathrm{d}\,\mathrm{S}^\chi\,\alpha.
\end{aligned}$$

□

For the Woronowicz exterior algebra we obtain the following corollary.

Corollary 2.4.4. *The Woronowicz exterior algebra is stable under twisting, i.e. the Woronowicz construction on a twisted first order calculus is canonically isomorphic to the twisting of the Woronowicz construction on the original first order calculus.*

Proof. We use Corollary 2.4.2. Viewing the Woronowicz construction as a quotient of the tensor algebra over the first order calculus as described above, the isomorphism is given by $\underline{\sigma}_\chi$ of Theorem 2.2.5 extended to multiple tensor products. Thus, we have to ensure that $\underline{\sigma}_\chi$ is an intertwiner for the Woronowicz ideal by which we quotient. But this ideal is given as the kernel of a linear combination of identities and braidings (in ${}^H_H\underline{\mathcal{M}}^H_H$) and $\underline{\sigma}_\chi$ is an intertwiner for the braiding, so this is satisfied. □

Remark. Given a Lie group G, the standard exterior algebra on it is bicovariant in the sense discussed above. Thus, for any cocycle deformation quantisation of G (product twist of $\mathcal{C}(G)$), the corresponding exterior algebra is given by application of the functor $\underline{\mathcal{G}}_\chi$ of Theorem 2.2.5 to the standard exterior algebra according to Corollary 2.4.2.

Chapter 3

Differential Calculi on $\mathsf{U}_q(\mathfrak{b}_+)$

One of the fundamental ingredients in the theory of noncommutative or quantum differential geometry is the notion of a differential calculus as introduced in Section 1.3. Due to the removal of the commutativity constraint the uniqueness of a canonical calculus is lost. It is therefore desirable to classify the possible choices. The most important part is the space of one-forms or first order differential calculus to which we restrict our attention in the following. (We leave out the words "first order".)

The wealth of possible differential calculi can be somewhat constraint by imposing symmetries. This is particularly natural for differential calculi on quantum groups, where bicovariance can be imposed (Definition 1.3.2 with $A = H$). (We leave out the adjective "bicovariant" as well.) The study of such calculi was initiated by Woronowicz [Wor89].

Much attention has been devoted to the investigation of differential calculi on quantum groups $\mathcal{C}_q(G)$ that are q-deformations of function algebras for G a simple Lie group. Natural differential calculi on matrix quantum groups were obtained by Jurco [Jur91] and Carow-Watamura et al. [CSWW91]. A partial classification of calculi of the same dimension as the commutative ones was obtained by Schmüdgen and Schüler [SS95]. More recently, a classification theorem for factorisable cosemisimple quantum groups was obtained by Majid [Maj98], covering the general $\mathcal{C}_q(G)$ case. A similar result was obtained later by Baumann and Schmitt [BS98]. Also, Heckenberger and Schmüdgen [HS97] gave a complete classification on $\mathcal{C}_q(SL(N))$ and $\mathcal{C}_q(Sp(N))$.

In contrast, for G not simple or semisimple the differential calculi on $\mathcal{C}_q(G)$ are largely unknown. The smallest non-Abelian Lie group is the 2-dimensional matrix group

$$B_+ = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \right\}, \quad \alpha, \beta \in \mathbb{R}$$

which is the exponentiation of the Lie algebra $\mathfrak{b}_+$ generated by two elements X, H with the relation $[H, X] = X$. Although it is not simple, its q-deformation is known, since $\mathfrak{b}_+$ is the Borel subalgebra of $\mathfrak{sl}_2$. The quantum enveloping algebra $\mathsf{U}_q(\mathfrak{b}_+)$ is self-dual, i.e. is non-degenerately paired with itself [Dri87]. This has an interesting consequence: $\mathsf{U}_q(\mathfrak{b}_+)$ may be identified with (a certain algebraic model of) the q-deformed function algebra $\mathcal{C}_q(B_+)$. The differential calculi on this quantum group and on its "classical limits" $\mathcal{C}(B_+)$ and $\mathsf{U}(\mathfrak{b}_+)$ will be the main concern of this chapter. We pay hereby equal attention to the dual notion of quantum tangent space.

We have a further motivation to study differential calculi on $\mathcal{C}_q(B_+)$ in the commutative limit $q \to 1$. In Chapter 4 we study the "Planck scale Hopf algebra", which is a toy model for Planck scale physics. It turns out that it is a cocycle deformation quantisation of B_+ and thus that its differential calculi are in one-to-one correspondence to those of B_+ by the twisting theory of Chapter 2.

We start in Section 3.1 by reviewing the definition of $\mathsf{U}_q(\mathfrak{b}_+)$ and its classical limits.

In Section 3.2 we obtain the complete classification of differential calculi on $\mathcal{C}_q(B_+)$. It turns out that (finite dimensional) differential calculi are characterised by finite subsets $I \subset \mathbb{N}$. These sets determine the decomposition into coirreducible (i.e. not admitting quotients) differential calculi characterised by single integers. For the coirreducible calculi the explicit formulas for the commutation relations and braided derivations are given.

In Section 3.3 we give the complete classification for the classical function algebra $\mathcal{C}(B_+)$. It is essentially the same as in the q-deformed setting, and we stress this by giving an almost one-to-one correspondence of differential calculi to those obtained in the previous section. In contrast, however, the decomposition and coirreducibility properties do not hold at all. (One may even say that they are maximally violated). We give the explicit formulas for those calculi corresponding to coirreducible ones.

More interesting perhaps is the "dual" classical limit. That is, we view $\mathsf{U}(\mathfrak{b}_+)$ as a quantum function algebra with quantum enveloping algebra $\mathcal{C}(B_+)$. This is investigated in Section 3.4. It turns out that in this setting we have considerably more freedom in choosing a differential calculus since the bicovariance condition becomes much weaker. This shows that this dual classical limit is in a sense "unnatural" as compared to the ordinary classical limit of Section 3.3. However, we can still establish a correspondence of certain differential calculi to those of Section 3.2. The decomposition properties are conserved while the coirreducibility properties are not. We give the formulas for the calculi corresponding to coirreducible ones.

Another interesting aspect of viewing $\mathsf{U}(\mathfrak{b}_+)$ as a quantum function algebra is the connection to quantum deformed models of space-time and its symmetries. In particular, the κ-deformed Minkowski space coming from the κ-deformed Poincaré algebra [LNR92, MR94] is just a simple generalisation of $\mathsf{U}(\mathfrak{b}_+)$. We use this in Section 3.5 to equip it with a natural

4-dimensional differential calculus. Then we show (in a formal context) that integration is given by the usual Lebesgue integral on $\mathbb{R}^n$ after normal ordering. This is obtained in an intrinsic context different from the standard κ-Poincaré approach.

We provide a required formula for the adjoint coaction on $\mathsf{U}_q(\mathfrak{b}_+)$ in Appendix 3.A.

Conventions

Throughout this chapter, $\Bbbk$ denotes a field of characteristic 0 and $\Bbbk(q)$ denotes the field of rational functions in one parameter q over $\Bbbk$. $\Bbbk(q)$ is our ground field in the q-deformed setting, while $\Bbbk$ is the ground field in the "classical" settings. Within Section 3.2 one could equally well view $\Bbbk$ as the ground field with $q \in \Bbbk^*$ not a root of unity. This point of view is problematic, however, when obtaining "classical limits" as in Sections 3.3 and 3.4.

The positive integers are denoted by $\mathbb{N}$ while the non-negative integers are denoted by $\mathbb{N}_0$. We define q-integers, q-factorials and q-binomials as follows:

$$[n]_q := \sum_{i=0}^{n-1} q^i, \qquad [n]_q! := [1]_q[2]_q \cdots [n]_q, \qquad \begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{[n]_q!}{[m]_q![n-m]_q!}.$$

For a function of several variables (among them x) over $\Bbbk$ we define

$$(T_{a,x}f)(x) := f(x+a),$$
$$(\nabla_{a,x}f)(x) := \frac{f(x+a)-f(x)}{a}$$

with $a \in \Bbbk$, and similarly over $\Bbbk(q)$

$$(Q_{m,x}f)(x) := f(q^m x),$$
$$(\partial_{q,x}f)(x) := \frac{f(x)-f(qx)}{x(1-q)}$$

with $m \in \mathbb{Z}$.

We frequently use the notion of a polynomial in an extended sense. Namely, if we have an algebra with an element g and its inverse g^{-1} (as in $\mathsf{U}_q(\mathfrak{b}_+)$) we mean by a polynomial in g, g^{-1} a finite power series in g with exponents in $\mathbb{Z}$. The length of such a polynomial is defined as the difference between highest and lowest degree.

3.1 $\mathsf{U}_q(\mathfrak{b}_+)$ and its Classical Limits

We recall that, in the framework of quantum groups, the duality between enveloping algebra $\mathsf{U}(\mathfrak{g})$ of the Lie algebra and algebra of functions $\mathcal{C}(G)$ on the Lie group carries over to q-deformations. In the case of $\mathfrak{b}_+$, the q-deformed enveloping algebra $\mathsf{U}_q(\mathfrak{b}_+)$ defined over $\Bbbk(q)$

as

$$\begin{gathered}
\mathsf{U}_q(\mathfrak{b}_+) = \Bbbk(q)\langle X, g, g^{-1}\rangle \qquad \text{with relations} \\
g g^{-1} = 1, \qquad X g = q g X, \\
\Delta X = X \otimes 1 + g \otimes X, \qquad \Delta g = g \otimes g, \\
\epsilon(X) = 0, \qquad \epsilon(g) = 1, \qquad \mathrm{S}\, X = -g^{-1} X, \qquad \mathrm{S}\, g = g^{-1}
\end{gathered}$$

is self-dual. Consequently, it may alternatively be viewed as the quantum algebra $\mathcal{C}_q(B_+)$ of functions on the Lie group B_+ associated with $\mathfrak{b}_+$. It has two classical limits, the enveloping algebra $\mathsf{U}(\mathfrak{b}_+)$ and the function algebra $\mathcal{C}(B_+)$. The transition to the classical enveloping algebra is achieved by replacing q by e^{-t} and g by e^{tH} in a formal power series setting in t, introducing a new generator H. Now, all expressions are written in the form $\sum_j a_j t^j$ and only the lowest order in t is kept. The transition to the classical function algebra on the other hand is achieved by setting $q = 1$. This may be depicted as follows:

$$\begin{tikzcd}[column sep=small]
& \mathsf{U}_q(\mathfrak{b}_+) \cong \mathcal{C}_q(B_+) \arrow[dl, "{\begin{array}{l} q = e^{-t} \\ g = e^{tH} \end{array}\Big|_{t\to 0}}"'] \arrow[dr, "q=1"] & \\
\mathsf{U}(\mathfrak{b}_+) & < \cdots \text{dual} \cdots > & \mathcal{C}(B_+)
\end{tikzcd}$$

The self-duality of $\mathsf{U}_q(\mathfrak{b}_+)$ is expressed as a pairing $\mathsf{U}_q(\mathfrak{b}_+) \times \mathsf{U}_q(\mathfrak{b}_+) \to \Bbbk$ with itself:

$$\langle X^n g^m, X^r g^s\rangle = \delta_{n,r} [n]_q!\, q^{-n(n-1)/2} q^{-ms} \qquad \forall n, r \in \mathbb{N}_0\ m, s \in \mathbb{Z}.$$

In the classical limit this becomes the pairing $\mathsf{U}(\mathfrak{b}_+) \times \mathcal{C}(B_+) \to \Bbbk$

$$\langle X^n H^m, X^r g^s\rangle = \delta_{n,r} n!\, s^m \qquad \forall n, m, r \in \mathbb{N}_0\ s \in \mathbb{Z}. \tag{3.1}$$

3.2 Classification on $\mathcal{C}_q(B_+)$ and $\mathsf{U}_q(\mathfrak{b}_+)$

In this section we completely classify differential calculi on $\mathcal{C}_q(B_+)$ and, dually, quantum tangent spaces on $\mathsf{U}_q(\mathfrak{b}_+)$. We start by classifying the relevant crossed modules and then proceed to a detailed description of the calculi.

Lemma 3.2.1. *(a) Left crossed $\mathcal{C}_q(B_+)$-submodules $M \subseteq \mathcal{C}_q(B_+)$ by left multiplication and left adjoint coaction, are in one-to-one correspondence to pairs (P, I) where $P \in \Bbbk(q)[g]$ is a polynomial with $P(0) = 1$ and $I \subset \mathbb{N}$ is finite.* $\operatorname{codim} M < \infty$ *iff $P = 1$. In particular,* $\operatorname{codim} M = \sum_{n\in I} n$ *if $P = 1$.*

(b) The finite codimensional maximal M correspond to the pairs $(1, \{n\})$ with n the codimension. The infinite codimensional maximal M are characterised by $(P, \emptyset)$ with P irreducible and $P(g) \neq 1 - q^{-k}g$ for any $k \in \mathbb{N}_0$.

(c) Crossed submodules M of finite codimension are intersections of maximal ones. In particular, $M = \bigcap_{n \in I} M^n$, with M^n corresponding to $(1, \{n\})$.

Proof. (a) Let $M \subseteq \mathcal{C}_q(B_+)$ be a crossed $\mathcal{C}_q(B_+)$-submodule by left multiplication and left adjoint coaction and let $\sum_n X^n P_n(g) \in M$, where P_n are polynomials in g, g^{-1} (every element of $\mathcal{C}_q(B_+)$ can be expressed in this form). From the formula for the coaction ((3.2), see Appendix 3.A) we observe that for all n and for all $t \leq n$ the element

$$X^t P_n(g) \prod_{s=1}^{n-t} (1 - q^{s-n} g)$$

lies in M. In particular, this is true for $t = n$, meaning that elements of constant degree in X lie separately in M. It is therefore sufficient to consider such elements.

Let now $X^n P(g) \in M$. By left multiplication $X^n P(g)$ generates any element of the form $X^k P(g) Q(g)$, where $k \geq n$ and Q is any polynomial in g, g^{-1}. (Note that $Q(q^k g) X^k = X^k Q(g)$.) We see that M contains the following elements:

$$\begin{array}{ll}
\vdots & \\
X^{n+2} & P(g) \\
X^{n+1} & P(g) \\
X^n & P(g) \\
X^{n-1} & P(g)(1 - q^{1-n} g) \\
X^{n-2} & P(g)(1 - q^{1-n} g)(1 - q^{2-n} g) \\
\vdots & \\
X & P(g)(1 - q^{1-n} g)(1 - q^{2-n} g) \ldots (1 - q^{-1} g) \\
 & P(g)(1 - q^{1-n} g)(1 - q^{2-n} g) \ldots (1 - q^{-1} g)(1 - g)
\end{array}$$

Moreover, if M is generated by $X^n P(g)$ as a module then these elements generate M as a vector space by left multiplication with polynomials in g, g^{-1}. (Observe that the application of the coaction to any of the elements shown does not generate elements of new type.)

Now, let M be a given crossed submodule. We pick, among the elements in M of the form $X^n P(g)$ with P of minimal length, one with lowest degree in X. Then certainly the elements listed above are in M. Furthermore, for any element of the form $X^k Q(g)$, Q must contain P as a factor and for $k < n$, Q must contain $P(g)(1 - q^{1-n} g)$ as a factor. We continue by picking the smallest n_2, so that $X^{n_2} P(g)(1 - q^{1-n} g) \in M$. Certainly $n_2 < n$. Again, for any element of $X^l Q(g)$ in M with $l < n_2$, we have that $P(g)(1-q^{1-n}g)(1-q^{1-n_2}g)$ divides Q(g). We proceed by induction, until we arrive at degree zero in X.

We obtain the following elements generating M as a vector space by left multiplication with polynomials in g, g^{-1} (rename $n_1 := n$):

$$\begin{array}{ll}
\vdots & \\
X^{n_1+1} & P(g) \\
X^{n_1} & P(g) \\
X^{n_1-1} & P(g)(1-q^{1-n_1}g) \\
\vdots & \\
X^{n_2} & P(g)(1-q^{1-n_1}g) \\
X^{n_2-1} & P(g)(1-q^{1-n_1}g)(1-q^{1-n_2}) \\
\vdots & \\
X^{n_3} & P(g)(1-q^{1-n_1}g)(1-q^{1-n_2}g) \\
X^{n_3-1} & P(g)(1-q^{1-n_1}g)(1-q^{1-n_2}g)(1-q^{1-n_3}) \\
\vdots & \\
 & P(g)(1-q^{1-n_1}g)(1-q^{1-n_2}g)(1-q^{1-n_3}g)\dots(1-q^{1-n_m}g)
\end{array}$$

We see that the integers $n_1, \dots, n_m$ uniquely determine the shape of this picture. The polynomial $P(g)$ on the other hand can be shifted (by g and g^{-1}) or renormalised. To determine M uniquely we shift and normalise P in such a way that it contains no negative powers and has unit constant coefficient. P can then be viewed as a polynomial $\in \Bbbk(q)[g]$.

We see that the codimension of M is the sum of the lengths of the polynomials in g over all degrees in X in the above picture. Finite codimension corresponds to $P = 1$. In this case the codimension is the sum $n_1 + \ldots + n_m$.

(b) We observe that polynomials of the form $1-q^j g$ have no common divisors for distinct j. Therefore, finite codimensional crossed submodules are maximal if and only if there is just one integer ($m = 1$). Thus, the maximal left crossed submodule of codimension k is generated by X^k and $1-q^{1-k}g$. For an infinite codimensional crossed submodule we certainly need $m = 0$. Then, the maximality corresponds to irreducibility of P.

(c) This is again due to the distinctness of factors $1 - q^j g$. $\square$

Corollary 3.2.2. *(a) Left crossed $\mathcal{C}_q(B_+)$-submodules $M \subseteq \ker \epsilon \subset \mathcal{C}_q(B_+)$ are in one-to-one correspondence to pairs (P, I) as in Lemma 3.2.1 with the additional constraint that $(1-g)$ divides $P(g)$ or $1 \in I$.* $\operatorname{codim} M < \infty$ *iff* $P = 1$. *In particular,* $\operatorname{codim} M = (\sum_{n\in I} n) - 1$ *if* $P = 1$.

(b) The finite codimensional maximal M correspond to the pairs $(1,\{1,n\})$ with $n \geq 2$ the codimension. The infinite codimensional maximal M correspond to pairs $(P,\{1\})$ with P irreducible and $P(g) \neq 1-q^{-k}g$ for any $k \in \mathbb{N}_0$.

(c) Crossed submodules M of finite codimension are intersections of maximal ones. In particular, $M = \bigcap_{n\in I} M^n$, with M^n corresponding to $(1,\{1,n\})$.

Proof. First observe that $\sum_n X^n P_n(g) \in \ker\epsilon$ if and only if $(1-g)$ divides $P_0(g)$. This is to say that $\ker\epsilon$ is the crossed submodule corresponding to the pair $(1,\{1\})$ in Lemma 3.2.1. We obtain the classification from the one of Lemma 3.2.1 by intersecting everything with this crossed submodule. In particular, this reduces the codimension by one in the finite codimensional case. $\square$

Lemma 3.2.3. *(a) Left crossed $\mathsf{U}_q(\mathfrak{b}_+)$-submodules $L \subseteq \mathsf{U}_q(\mathfrak{b}_+)$ via the left adjoint action and left regular coaction are in one-to-one correspondence with the set $\{\mathbb{N}_0 \to \{1,2,3\}\}\times\{\mathbb{N} \to \{1,2\}\}$. Finite dimensional L are in one-to-one correspondence with finite sets $I \subset \mathbb{N}$ and $\dim L = \sum_{n\in I} n$.*

(b) Finite dimensional irreducible L correspond to $\{n\}$ with n the dimension.

(c) Finite dimensional L are direct sums of irreducible ones. In particular, $L = \oplus_{n\in I} L^n$ with L^n corresponding to $\{n\}$.

Proof. (a) The action takes the explicit form

$$g \triangleright X^n g^k = q^{-n} X^n g^k, \qquad X \triangleright X^n g^k = X^{n+1} g^k (1 - q^{-(n+k)}),$$

while the coproduct is

$$\Delta(X^n g^k) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q q^{-r(n-r)} X^{n-r} g^{k+r} \otimes X^r g^k,$$

which we view as a left coaction here. Let now $L \subseteq \mathsf{U}_q(\mathfrak{b}_+)$ be a crossed $\mathsf{U}_q(\mathfrak{b}_+)$-submodule via this action and coaction. For $\sum_n X^n P_n(g) \in L$ invariance under the action by g clearly means that $X^n P_n(g) \in L\ \forall n$. Then, from invariance under the coaction we can conclude that if $X^n \sum_j a_j g^j \in L$ we must have $X^n g^j \in L\ \forall j$. That is, elements of the form $X^n g^j$ lie separately in L and it is sufficient to consider such elements. From the coaction we learn that if $X^n g^j \in L$ we have $X^m g^j \in L\ \forall m \leq n$. The action by X leads to $X^n g^j \in L \Rightarrow X^{n+1} g^j \in L$ except if $n + j = 0$. The classification is given by the possible choices we have for each power in g. For every positive integer j we can choose whether or not to include the span of $\{X^n g^j | \forall n\}$ in L and for every non-positive integer we can choose to include either the span of $\{X^n g^j | \forall n\}$ or just $\{X^n g^j | \forall n \leq -j\}$ or neither. That is, for positive integers ($\mathbb{N}$) we have two choices while for non-positive (identified with $\mathbb{N}_0$) ones we have three choices.

Clearly, the finite dimensional L are those where we choose only to include finitely many powers of g and also only finitely many powers of X. The latter is only possible for the non-positive powers of g. By identifying positive integers n with powers $1-n$ of g, we obtain a classification by finite subsets of $\mathbb{N}$.

(b) Irreducibility clearly corresponds to just including one power of g in the finite dimensional case.

(c) The decomposition property is obvious from the discussion. □

Corollary 3.2.4. *(a) Left crossed $\mathsf{U}_q(\mathfrak{b}_+)$-submodules $L \subseteq \ker\epsilon \subset \mathsf{U}_q(\mathfrak{b}_+)$ via the left adjoint action and left regular coaction (with subsequent projection to $\ker\epsilon$ via $x \mapsto x - \epsilon(x)1$) are in one-to-one correspondence to the set $\{\mathbb{N} \to \{1,2,3\}\} \times \{\mathbb{N}_0 \to \{1,2\}\}$. Finite dimensional L are in one-to-one correspondence to finite sets $I \subset \mathbb{N}\setminus\{1\}$ and $\dim L = \sum_{n\in I} n$.*

(b) Finite dimensional irreducible L correspond to $\{n\}$ with $n \geq 2$ the dimension.

(c) Finite dimensional L are direct sums of irreducible ones. In particular, $L = \oplus_{n\in I} L^n$ with L^n corresponding to $\{n\}$.

Proof. Only a small modification of Lemma 3.2.3 is necessary. Elements of the form $P(g)$ are replaced by elements of the form $P(g) - P(1)$. Monomials with non-vanishing degree in X are unchanged. The choices for elements of degree 0 in g are reduced to either including the span of $\{X^k | \forall k > 0\}$ in the crossed submodule or not. In particular, the crossed submodule characterised by $\{1\}$ in Lemma 3.2.3 is projected out. □

Differential calculi in the original sense of Woronowicz are classified by Corollary 3.2.2 while from the quantum tangent space point of view the classification is given by Corollary 3.2.4. In the finite dimensional case the duality is strict in the sense of a one-to-one correspondence. The infinite dimensional case on the other hand depends strongly on the algebraic models we use for the function or enveloping algebras. It is therefore not surprising that in the present purely algebraic context the classifications are quite different in this case. We will restrict ourselves to the finite dimensional case in the following description of the differential calculi.

Theorem 3.2.5. *(a) Finite dimensional differential calculi Γ on $\mathcal{C}_q(B_+)$ and corresponding quantum tangent spaces L on $\mathsf{U}_q(\mathfrak{b}_+)$ are in one-to-one correspondence to finite sets $I \subset \mathbb{N}\setminus\{1\}$. In particular, $\dim\Gamma = \dim L = \sum_{n\in I} n$.*

(b) Coirreducible Γ and irreducible L correspond to $\{n\}$ with $n \geq 2$ the dimension. Such a Γ has a right invariant basis $\eta_0, \ldots, \eta_{n-1}$ so that the relations

$$\mathrm{d}X = \eta_1 + (q^{n-1} - 1)\eta_0 X, \qquad \mathrm{d}g = (q^{n-1} - 1)\eta_0 g,$$

$$[a, \eta_0] = \mathrm{d}a \quad \forall a \in \mathcal{C}_q(B_+),$$

$$[g, \eta_i]_{q^{n-1-i}} = 0 \quad \forall i, \qquad [X, \eta_i]_{q^{n-1-i}} = \begin{cases} \eta_{i+1} & \text{if } i < n-1 \\ 0 & \text{if } i = n-1 \end{cases}$$

hold, where $[a,b]_p := ab - pba$. By choosing the dual basis on the corresponding irreducible L we obtain the braided derivations

$$\partial_i\colon f := : Q_{n-1-i,g} Q_{n-1-i,X} \frac{1}{[i]_q!} (\partial_{q,X})^i f : \qquad \forall i \geq 1,$$
$$\partial_0\colon f := : Q_{n-1,g} Q_{n-1,X} f - f :$$

for $f \in \Bbbk(q)[X, g, g^{-1}]$ with normal ordering $\Bbbk(q)[X, g, g^{-1}] \to \mathcal{C}_q(B_+)$ given by $g^n X^m \mapsto g^n X^m$.

(c) Finite dimensional Γ and L decompose into direct sums of coirreducible respectively irreducible ones. In particular, $\Gamma = \oplus_{n \in I} \Gamma^n$ and $L = \oplus_{n \in I} L^n$ with Γ^n and L^n corresponding to $\{n\}$.

Proof. (a) We observe that the classifications of Lemma 3.2.1 and Lemma 3.2.3 or Corollary 3.2.2 and Corollary 3.2.4 are dual to each other in the finite (co)dimensional case. More precisely, for $I \subset \mathbb{N}$ finite the crossed submodule M corresponding to $(1, I)$ in Lemma 3.2.1 is the annihilator of the crossed submodule L corresponding to I in Lemma 3.2.3 and vice versa. $\mathcal{C}_q(B_+)/M$ and L are dual spaces with the induced pairing. For $I \subset \mathbb{N} \setminus \{1\}$ finite this descends to M corresponding to $(1, I \cup \{1\})$ in Corollary 3.2.2 and L corresponding to I in Corollary 3.2.4. For the dimension of Γ observe $\dim \Gamma = \dim \ker \epsilon / M = \operatorname{codim} M$.

(b) Coirreducibility (having no proper quotient) of Γ clearly corresponds to maximality of M. The statement then follows from parts (b) of Corollaries 3.2.2 and 3.2.4. The formulas are obtained by choosing the basis $\eta_0, \ldots, \eta_{n-1}$ of $\ker \epsilon / M$ as the equivalence classes of

$$(g-1)/(q^{n-1}-1), X, \ldots, X^{n-1}.$$

The dual basis of L is then given by

$$g^{1-n} - 1, X g^{1-n}, \ldots, q^{k(k-1)} \frac{1}{[k]_q!} X^k g^{1-n}, \ldots, q^{(n-1)(n-2)} \frac{1}{[n-1]_q!} X^{n-1} g^{1-n}.$$

(c) The statement follows from Corollaries 3.2.2 and 3.2.4 parts (c) with the observation

$$\ker \epsilon / M = \ker \epsilon / \bigcap_{n \in I} M^n = \oplus_{n \in I} \ker \epsilon / M^n.$$

□

Corollary 3.2.6. *There is precisely one differential calculus on $\mathcal{C}_q(B_+)$ which is natural in the sense that it has dimension 2. It is coirreducible and obeys the relations*

$$[g, \mathrm{d}X] = 0, \qquad [g, \mathrm{d}g]_q = 0, \qquad [X, \mathrm{d}X]_q = 0, \qquad [X, \mathrm{d}g]_q = (q-1)(\mathrm{d}X)g$$

with $[a,b]_q := ab - qba$. *In particular, we have*

$$\mathrm{d}\colon f := \mathrm{d}g\colon \partial_{q,g} f : + \mathrm{d}X\colon \partial_{q,X} f : \qquad \forall f \in \Bbbk(q)[X, g, g^{-1}].$$

Proof. This is a special case of Theorem 3.2.5. The formulas follow from (b) with $n = 2$. □

3.3 Classification in the Classical Limit

In this section we give the complete classification of differential calculi and quantum tangent spaces in the classical case of $\mathcal{C}(B_+)$ along the lines of the previous section. We pay particular attention to the relation to the q-deformed setting.

The classical limit $\mathcal{C}(B_+)$ of the quantum group $\mathcal{C}_q(B_+)$ is simply obtained by substituting the parameter q with 1. The classification of left crossed submodules in part (a) of Lemma 3.2.1 remains unchanged, as one may check by going through the proof. In particular, we get a correspondence of crossed modules in the q-deformed setting with crossed modules in the classical setting as a map of pairs $(P, I) \mapsto (P, I)$ that converts polynomials $\Bbbk(q)[g]$ to polynomials $\Bbbk[g]$ (if defined) and leaves sets I unchanged. This is one-to-one in the finite dimensional case. However, we did use the distinctness of powers of q in part (b) and (c) of Lemma 3.2.1 and have to account for changing this. The only place where we used it, was in observing that factors $1 - q^j g$ have no common divisors for distinct j. This was crucial to conclude the maximality (b) of certain finite codimensional crossed submodules and the intersection property (c). Now, all those factors become $1 - g$.

Corollary 3.3.1. *(a) Left crossed $\mathcal{C}(B_+)$-submodules $M \subseteq \mathcal{C}(B_+)$ by left multiplication and left adjoint coaction are in one-to-one correspondence to pairs (P, I), where $P \in \Bbbk[g]$ is a polynomial with $P(0) = 1$ and $I \subset \mathbb{N}$ is finite.* $\operatorname{codim} M < \infty$ *iff $P = 1$. In particular,* $\operatorname{codim} M = \sum_{n \in I} n$ *if $P = 1$.*

(b) The infinite codimensional maximal M are characterised by $(P, \emptyset)$ with P irreducible and $P(g) \neq 1 - g$ for any $k \in \mathbb{N}_0$.

In the restriction to $\ker \epsilon \subset \mathcal{C}(B_+)$ corresponding to Corollary 3.2.2 we observe another difference to the q-deformed setting. Since the condition for a crossed submodule to lie in $\ker \epsilon$ is exactly to have factors $1 - g$ in the X-free monomials this condition may now be satisfied more easily. If the characterising polynomial does not contain this factor it is now sufficient to have just any non-empty characterising integer set I and it need not contain 1. Consequently, the map $(P, I) \mapsto (P, I)$ does not reach all crossed submodules now.

Corollary 3.3.2. *(a) Left crossed $\mathcal{C}(B_+)$-submodules $M \subseteq \ker \epsilon \subset \mathcal{C}(B_+)$ are in one-to-one correspondence to pairs (P, I) as in Corollary 3.3.1 with the additional constraint* $(1 - g)$

divides $P(g)$ *or* I *non-empty.* $\operatorname{codim} M < \infty$ *iff* $P = 1$*. In particular,* $\operatorname{codim} M = (\sum_{n\in I} n)-1$ *if* $P = 1$*.*

(b) The infinite codimensional maximal M *correspond to pairs* $(P, \{1\})$ *with* P *irreducible and* $P(g) \neq 1 - g$*.*

Let us now turn to quantum tangent spaces on $\mathsf{U}(\mathfrak{b}_+)$. Here, the process to go from the q-deformed setting to the classical one is not quite so straightforward.

Lemma 3.3.3. *Proper left crossed* $\mathsf{U}(\mathfrak{b}_+)$*-submodules* $L \subset \mathsf{U}(\mathfrak{b}_+)$ *via the left adjoint action and left regular coaction are in one-to-one correspondence to pairs* (l, I) *with* $l \in \mathbb{N}_0$ *and* $I \subset \mathbb{N}$ *finite.* $\dim L < \infty$ *iff* $l = 0$*. In particular,* $\dim L = \sum_{n\in I} n$ *if* $l = 0$*.*

Proof. The left adjoint action takes the form

$$X \triangleright X^n H^m = X^{n+1}(H^m - (H+1)^m), \qquad H \triangleright X^n H^m = n X^n H^m,$$

while the coaction is

$$\Delta(X^n H^m) = \sum_{i=1}^{n}\sum_{j=1}^{m} \binom{n}{i}\binom{m}{j} X^i H^j \otimes X^{n-1} H^{m-j}.$$

Let L be a crossed submodule invariant under the action and coaction. The (repeated) action of H separates elements by degree in X. It is therefore sufficient to consider elements of the form $X^n P(H)$, where P is a polynomial. By acting with X on an element $X^n P(H)$ we obtain $X^{n+1}(P(H) - P(H+1))$. Subsequently applying the coaction and projecting on the left hand side of the tensor product onto X (in the basis $X^i H^j$ of $\mathsf{U}(\mathfrak{b}_+)$) leads to the element $X^n(P(H) - P(H+1))$. Now the degree of $P(H) - P(H+1)$ is exactly the degree of $P(H)$ minus 1. Thus, we have polynomials $X^n P_i(H)$ of any degree $i = \deg(P_i) \leq \deg(P)$ in L by induction. In particular, $X^n H^m \in L$ for all $m \leq \deg(P)$. It is thus sufficient to consider elements of the form $X^n H^m$. Given such an element, the coaction generates all elements of the form $X^i H^j$ with $i \leq n, j \leq m$.

For given n, the characterising datum is the maximal m so that $X^n H^m \in L$. Due to the coaction this cannot decrease with decreasing n and due to the action of X this can decrease at most by 1 when increasing n by 1. This leads to the classification given. For $l \in \mathbb{N}_0$ and $I \subset \mathbb{N}$ finite, the corresponding crossed submodule is generated by

$$\begin{gathered} X^{n_m-1}H^{l+m-1}, X^{n_m+n_{m-1}-1}H^{l+m-2}, \ldots, X^{(\sum_i n_i)-1}H^l \\ \text{and} \qquad X^{(\sum_i n_i)+k}H^{l-1} \quad \forall k \geq 0 \quad \text{if} \quad l > 0 \end{gathered}$$

as a crossed module. $\square$

For the transition from the q-deformed (Lemma 3.2.3) to the classical case we observe that the space spanned by $g^{s_1}, \ldots, g^{s_m}$ with m different integers $s_i \in \mathbb{Z}$ maps to the space spanned by $1, H, \ldots, H^{m-1}$ in the prescription of the classical limit (as described in Section 3.1). That is, the classical crossed submodule characterised by an integer l and a finite set $I \subset \mathbb{N}$ comes from a crossed submodule characterised by this same I and additionally l other integers $j \in \mathbb{Z}$ for which $X^k g^{1-j}$ is included. In particular, we have a one-to-one correspondence in the finite dimensional case.

To formulate the analogue of Corollary 3.2.4 for the classical case is essentially straightforward now. However, as for $\mathcal{C}(B_+)$, we obtain more crossed submodules than those from the q-deformed setting. This is due to the degeneracy introduced by forgetting the powers of g and just retaining the number of different powers.

Corollary 3.3.4. *(a) Proper left crossed $\mathsf{U}(\mathfrak{b}_+)$-submodules $L \subset \ker\epsilon \subset \mathsf{U}(\mathfrak{b}_+)$ via the left adjoint action and left regular coaction (with subsequent projection to $\ker\epsilon$ via $x \mapsto x - \epsilon(x)1$) are in one-to-one correspondence to pairs (l, I) with $l \in \mathbb{N}_0$ and $I \subset \mathbb{N}$ finite where $l \neq 0$ or $I \neq \emptyset$. $\dim L < \infty$ iff $l = 0$. In particular, $\dim L = (\sum_{n\in I} n) - 1$ if $l = 0$.*

As in the q-deformed setting, we give a description of the finite dimensional differential calculi where we have a strict duality to quantum tangent spaces.

Proposition 3.3.5. *(a) Finite dimensional differential calculi Γ on $\mathcal{C}(B_+)$ and finite dimensional quantum tangent spaces L on $\mathsf{U}(\mathfrak{b}_+)$ are in one-to-one correspondence to non-empty finite sets $I \subset \mathbb{N}$. In particular, $\dim\Gamma = \dim L = (\sum_{n\in I} n) - 1$.*

The Γ with $1 \in \mathbb{N}$ are in one-to-one correspondence to the finite dimensional calculi and quantum tangent spaces of the q-deformed setting (Theorem 3.2.5(a)).

(b) The differential calculus Γ of dimension $n \geq 2$ corresponding to the coirreducible one of $\mathcal{C}_q(B_+)$ (Theorem 3.2.5(b)) has a right invariant basis $\eta_0, \ldots, \eta_{n-1}$ so that

$$\mathrm{d}X = \eta_1 + \eta_0 X, \qquad \mathrm{d}g = \eta_0 g,$$

$$[g, \eta_i] = 0 \;\forall i, \qquad [X, \eta_i] = \begin{cases} 0 & \text{if } i = 0 \text{ or } i = n-1 \\ \eta_{i+1} & \text{if } 0 < i < n-1 \end{cases}$$

hold. The braided derivations obtained from the dual basis of the corresponding L are given by

$$\partial_i f = \frac{1}{i!}\left(\frac{\partial}{\partial X}\right)^i f \qquad \forall i \geq 1,$$
$$\partial_0 f = \left(X\frac{\partial}{X} + g\frac{\partial}{g}\right) f$$

for $f \in \mathcal{C}(B_+)$.

(c) The differential calculus of dimension $n-1$ *corresponding to the one in (b) with* 1 *removed from the characterising set is the same as the one above, except that we set* $\eta_0 = 0$ *and* $\partial_0 = 0$.

Proof. (a) We observe that the classifications of Corollary 3.3.1 and Lemma 3.3.3 or Corollary 3.3.2 and Corollary 3.3.4 are dual to each other in the finite (co)dimensional case. More precisely, for $I \subset \mathbb{N}$ finite, the crossed submodule M corresponding to $(1, I)$ in Corollary 3.3.1 is the annihilator of the crossed submodule L corresponding to $(0, I)$ in Lemma 3.3.3 and vice versa. $\mathcal{C}(B_+)/M$ and L are dual spaces with the induced pairing. For non-empty I this descends to M corresponding to $(1, I)$ in Corollary 3.3.2 and L corresponding to $(0, I)$ in Corollary 3.3.4. For the dimension of Γ note $\dim \Gamma = \dim \ker \epsilon / M = \operatorname{codim} M$.

(b) For $I = \{1, n\}$ we choose in $\ker \epsilon \subset \mathcal{C}(B_+)$ the basis $\eta_0, \ldots, \eta_{n-1}$ as the equivalence classes of $g-1, X, \ldots, X^{n-1}$. The dual basis in L is then $H, X, \ldots, \frac{1}{k!}X^k, \ldots, \frac{1}{(n-1)!}X^{n-1}$. This leads to the formulas given.

(c) For $I = \{n\}$ we get the same as in (b) except that η_0 and ∂_0 disappear. $\square$

The classical commutative calculus is the special case of (b) with $n = 2$. It is the only calculus of dimension 2 with $\mathrm{d}g \neq 0$. Note that it is not coirreducible.

3.4 The Dual Classical Limit

We proceed in this section to the more interesting point of view where we consider the classical algebras, but with their roles interchanged. That is, we view $\mathsf{U}(\mathfrak{b}_+)$ as the "function algebra" and $\mathcal{C}(B_+)$ as the "enveloping algebra". Due to the self-duality of $\mathsf{U}_q(\mathfrak{b}_+)$, we can again view the differential calculi and quantum tangent spaces as classical limits of the q-deformed setting investigated in Section 3.2.

In this dual setting the bicovariance constraint for differential calculi becomes much weaker. In particular, the adjoint action on a classical function algebra is trivial due to commutativity, and the adjoint coaction on a classical enveloping algebra is trivial due to cocommutativity. In effect, the correspondence with the q-deformed setting is much weaker than in the ordinary case of Section 3.3. There are many more differential calculi and quantum tangent spaces than in the q-deformed setting.

We will not attempt to classify all of them in the following but essentially content ourselves with those objects coming from the q-deformed setting.

Lemma 3.4.1. *Left* $\mathcal{C}(B_+)$*-subcomodules* $\subseteq \mathcal{C}(B_+)$ *via the left regular coaction are* $\mathbb{Z}$*-graded subspaces of* $\mathcal{C}(B_+)$ *with* $|X^n g^m| = n + m$, *stable under formal derivation in* X.

By choosing any ordering in $\mathcal{C}_q(B_+)$, left crossed submodules via left regular action and adjoint coaction are in one-to-one correspondence to certain subcomodules of $\mathcal{C}(B_+)$ by setting $q = 1$. Direct sums correspond to direct sums.

This descends to $\ker \epsilon \subset \mathcal{C}(B_+)$ *by the projection $x \mapsto x - \epsilon(x)1$.*

Proof. The coproduct on $\mathcal{C}(B_+)$ is

$$\Delta(X^n g^k) = \sum_{r=0}^{n} \binom{n}{r} X^{n-r} g^{k+r} \otimes X^r g^k,$$

which we view as a left coaction. Projecting on the left hand side of the tensor product onto g^l in a basis $X^n g^k$, we observe that coacting on an element $\sum_{n,k} a_{n,k} X^n g^k$ we obtain elements $\sum_n a_{n,l-n} X^n g^{l-n}$ for all l. That is, elements of the form $\sum_n b_n X^n g^{l-n}$ lie separately in a subcomodule, and it is sufficient to consider such elements. Writing the coaction on such an element as

$$\sum_t \frac{1}{t!} X^t g^{l-t} \otimes \sum_n b_n \frac{n!}{(n-t)!} X^{n-t} g^{l-n},$$

we see that the coaction generates all formal derivatives in X of this element. This gives us the classification: $\mathcal{C}(B_+)$-subcomodules $\subseteq \mathcal{C}(B_+)$ under the left regular coaction are $\mathbb{Z}$-graded subspaces with $|X^n g^m| = n + m$, stable under formal derivation in X given by $X^n g^m \mapsto nX^{n-1} g^m$.

The correspondence with the $\mathcal{C}_q(B_+)$ case follows from the trivial observation that the coproduct of $\mathcal{C}(B_+)$ is the same as that of $\mathcal{C}_q(B_+)$ with $q = 1$.

The restriction to $\ker \epsilon$ is straightforward. □

Lemma 3.4.2. *The process of obtaining the classical limit $\mathsf{U}(\mathfrak{b}_+)$ from $\mathsf{U}_q(\mathfrak{b}_+)$ is well defined for subspaces and sends crossed $\mathsf{U}_q(\mathfrak{b}_+)$-submodules $\subset \mathsf{U}_q(\mathfrak{b}_+)$ by regular action and adjoint coaction to $\mathsf{U}(\mathfrak{b}_+)$-submodules $\subset \mathsf{U}(\mathfrak{b}_+)$ by regular action. This map is injective in the finite codimensional case. Intersections and codimensions are preserved in this case.*

This descends to $\ker \epsilon$.

Proof. To obtain the classical limit of a left ideal it is sufficient to apply the limiting process (as described in Section 3.1) to the module generators. (We can forget the additional comodule structure.) On the one hand, any element generated by left multiplication with polynomials in g corresponds to some element generated by left multiplication with a polynomial in H, that is, there will be no more generators in the classical setting. On the other hand, left multiplication by a polynomial in H comes from left multiplication by the same polynomial in $g - 1$, that is, there will be no fewer generators.

The maximal left crossed $\mathsf{U}_q(\mathfrak{b}_+)$-submodule $\subseteq \mathsf{U}_q(\mathfrak{b}_+)$ by left multiplication and adjoint coaction of codimension n ($n \geq 1$) is generated as a left ideal by $\{1 - q^{1-n}g, X^n\}$ (see Lemma 3.2.1). Applying the limiting process to this leads to the left ideal of $\mathsf{U}(\mathfrak{b}_+)$ (which is not maximal for $n \neq 1$) generated by $\{H + n - 1, X^n\}$ having also codimension n.

More generally, the picture given for arbitrary finite codimensional left crossed modules of $\mathsf{U}_q(\mathfrak{b}_+)$ in terms of generators with respect to polynomials in g, g^{-1} in Lemma 3.2.1 carries over by replacing factors $1 - q^{1-n}g$ with factors $H + n - 1$ leading to generators with respect to polynomials in H. In particular, intersections go to intersections since the distinctness of the factors for different n is conserved.

The restriction to $\ker \epsilon$ is straightforward. $\square$

We are now in a position to give a detailed description of the differential calculi induced from the q-deformed setting by the limiting process.

Proposition 3.4.3. *(a) Certain finite dimensional differential calculi Γ on $\mathsf{U}(\mathfrak{b}_+)$ and quantum tangent spaces L on $\mathcal{C}(B_+)$ are in one-to-one correspondence to finite dimensional differential calculi on $\mathsf{U}_q(\mathfrak{b}_+)$ and quantum tangent spaces on $\mathcal{C}_q(B_+)$. Intersections correspond to intersections.*

(b) In particular, Γ and L corresponding to coirreducible differential calculi on $\mathsf{U}_q(\mathfrak{b}_+)$ and irreducible quantum tangent spaces on $\mathcal{C}_q(B_+)$ via the limiting process are given as follows: Γ has a right invariant basis $\eta_0, \dots, \eta_{n-1}$ so that

$$\mathrm{d}X = \eta_1, \qquad \mathrm{d}H = (1-n)\eta_0,$$

$$[H, \eta_i] = (1 - n + i)\eta_i \quad \forall i, \qquad [X, \eta_i] = \begin{cases} \eta_{i+1} & \text{if } \; i < n-1 \\ 0 & \text{if } \; i = n-1 \end{cases}$$

holds. The braided derivations corresponding to the dual basis of L are given by

$$\partial_i\colon f := \,:T_{1-n+i,H} \frac{1}{i!}\left(\frac{\partial}{\partial X}\right)^i f: \qquad \forall i \geq 1,$$
$$\partial_0\colon f := \,:T_{1-n,H} f - f:$$

for $f \in \Bbbk[X, H]$ with the normal ordering $\Bbbk[X, H] \to \mathsf{U}(\mathfrak{b}_+)$ via $H^n X^m \mapsto H^n X^m$.

Proof. (a) The strict duality between $\mathcal{C}(B_+)$-subcomodules $L \subseteq \ker \epsilon$ given by Lemma 3.4.1 and Corollary 3.2.4 and $\mathsf{U}(\mathfrak{b}_+)$-modules $\mathsf{U}(\mathfrak{b}_+)/(\Bbbk 1 + M)$ with M given by Lemma 3.4.2 and Corollary 3.2.2 can be checked explicitly. It is essentially due to mutual annihilation of factors $H + k$ in $\mathsf{U}(\mathfrak{b}_+)$ with elements g^k in $\mathcal{C}(B_+)$.

(b) L is generated by $\{g^{1-n}-1, Xg^{1-n}, \ldots, X^{n-1}g^{1-n}\}$ and M is generated by $\{H(H+n-1), X(H+n-1), X^n\}$. The formulas are obtained by denoting with $\eta_0, \ldots, \eta_{n-1}$ the equivalence classes of $H/(1-n), X, \ldots, X^{n-1}$ in $\mathsf{U}(\mathfrak{b}_+)/(\Bbbk 1+M)$. The dual basis of L is then

$$g^{1-n}-1, Xg^{1-n}, \ldots, \frac{1}{(n-1)!}X^{n-1}g^{1-n}.$$

□

In contrast to the q-deformed setting and to the usual classical setting the many freedoms in choosing a calculus leave us with many 2-dimensional calculi. It is not obvious which one we should consider to be the "natural" one. Let us first look at the 2-dimensional calculus coming from the q-deformed setting as described in (b). The relations become

$$[\mathrm{d}H, a] = \mathrm{d}a, \qquad [\mathrm{d}X, a] = 0 \qquad \forall a \in \mathsf{U}(\mathfrak{b}_+),$$
$$\mathrm{d}\colon f := \mathrm{d}H\colon \nabla_{1,H} f : + \,\mathrm{d}X\colon \frac{\partial}{\partial X} f :$$

for $f \in \Bbbk[X, H]$.

We might want to consider calculi which are closer to the classical theory in the sense that derivatives are not finite differences but usual derivatives. Let us therefore demand

$$\mathrm{d}P(H) = \mathrm{d}H \frac{\partial}{\partial H} P(H) \qquad \text{and} \qquad \mathrm{d}P(X) = \mathrm{d}X \frac{\partial}{\partial X} P(X)$$

for polynomials P, and $\mathrm{d}X \neq 0$ and $\mathrm{d}H \neq 0$.

Proposition 3.4.4. *There is precisely one differential calculus of dimension* 2 *meeting these conditions. It obeys the relations*

$$[a, \mathrm{d}H] = 0, \qquad [X, \mathrm{d}X] = 0, \qquad [H, \mathrm{d}X] = \mathrm{d}X,$$
$$\mathrm{d}\colon f := \mathrm{d}H\colon \frac{\partial}{\partial H} f : + \,\mathrm{d}X\colon \frac{\partial}{\partial X} f :,$$

where the normal ordering $\Bbbk[X, H] \to \mathsf{U}(\mathfrak{b}_+)$ *is given by* $X^n H^m \mapsto X^n H^m$.

Proof. Let M be the left ideal corresponding to the calculus. It is easy to see that for a primitive element a the classical derivation condition corresponds to $a^2 \in M$ and $a \notin M$. In our case $X^2, H^2 \in M$. If we take the ideal generated from these two elements we obtain an ideal of $\ker \epsilon$ of codimension 3. Now, without loss of generality, it is sufficient to add a generator of the form $\alpha H + \beta X + \gamma XH$. α and β must then be zero in order not to generate X or H in M. That is, M is generated by H^2, XH, X^2. The relations stated follow. □

3.5 Remarks on κ-Minkowski Space and Integration

There is a straightforward generalisation of $\mathsf{U}(\mathfrak{b}_+)$. Let us define the Lie algebra $\mathfrak{b}_{n+}$ as generated by $x_0, \ldots, x_{n-1}$ with relations

$$[x_0, x_i] = x_i, \qquad [x_i, x_j] = 0 \qquad \forall i, j \geq 1.$$

Its enveloping algebra $\mathsf{U}(\mathfrak{b}_{n+})$ is nothing but (rescaled) κ-Minkowski space as introduced in [MR94]. In this section we make some remarks about its intrinsic geometry.

We have a surjective Lie algebra homomorphism $\mathfrak{b}_{n+} \to \mathfrak{b}_+$ given by $x_0 \mapsto H$ and $x_i \mapsto X$. This is an isomorphism for $n = 2$. The surjective Lie algebra homomorphism extends to a surjective homomorphism of enveloping algebras $\mathsf{U}(\mathfrak{b}_{n+}) \to \mathsf{U}(\mathfrak{b}_+)$ in the obvious way. This gives rise to an injective map from the set of submodules of $\mathsf{U}(\mathfrak{b}_+)$ to the set of submodules of $\mathsf{U}(\mathfrak{b}_{n+})$ by taking the pre-image. In particular this induces an injective map from the set of differential calculi on $\mathsf{U}(\mathfrak{b}_+)$ to the set of differential calculi on $\mathsf{U}(\mathfrak{b}_{n+})$ which are invariant under permutations of the x_i, $i \geq 1$.

Corollary 3.5.1. *There is a natural n-dimensional differential calculus on $\mathsf{U}(\mathfrak{b}_{n+})$ induced from the one considered in Proposition 3.4.4. It obeys the relations*

$$[a, \mathrm{d}x_0] = 0 \quad \forall a \in \mathsf{U}(\mathfrak{b}_{n+}), \qquad [x_i, \mathrm{d}x_j] = 0, \quad [x_0, \mathrm{d}x_i] = \mathrm{d}x_i \qquad \forall i, j \geq 1,$$
$$\mathrm{d}\!: f : = \sum_{\mu=0}^{n-1} \mathrm{d}x_\mu \!: \frac{\partial}{\partial x_\mu} f :,$$

where the normal ordering is given by

$$\Bbbk[x_0, \ldots, x_{n-1}] \to \mathsf{U}(\mathfrak{b}_{n+}) \quad \textit{via} \quad x_{n-1}^{m_{n-1}} \cdots x_0^{m_0} \mapsto x_{n-1}^{m_{n-1}} \cdots x_0^{m_0}.$$

Proof. The calculus is obtained from the ideal generated by

$$x_0^2, x_i x_j, x_i x_0 \qquad \forall i, j \geq 1$$

being the pre-image of X^2, XH, X^2 in $\mathsf{U}(\mathfrak{b}_+)$. $\square$

Let us try to push the analogy with the commutative case further and take a look at the notion of integration. The natural way to encode the condition of translation invariance from the classical context in the quantum group context is given by the condition

$$\left(\int \otimes \mathrm{id}\right) \circ \Delta\, a = 1 \int a \qquad \forall a \in A,$$

which defines a right integral on a Hopf algebra A [Swe69]. (Correspondingly, we have the notion of a left integral.) Let us formulate a slightly weaker version of this equation in the context of a Hopf algebra H dually paired with A. We write

$$\int (h - \epsilon(h)) \triangleright a = 0 \qquad \forall h \in H, a \in A,$$

where the action of H on A is the coregular action $h \triangleright a = a_{(1)} \langle a_{(2)}, h \rangle$ given by the pairing.

In the present context we set $A = \mathsf{U}(\mathfrak{b}_{n+})$ and $H = \mathcal{C}(B_{n+})$. We define the latter as a generalisation of $\mathcal{C}(B_+)$ with commuting generators $g, p_1, \ldots, p_{n-1}$ and coproducts

$$\Delta\, p_i = p_i \otimes 1 + g \otimes p_i, \qquad \Delta\, g = g \otimes g.$$

This can be identified (upon rescaling) as the momentum sector of the full κ-Poincaré algebra (with $g = e^{p_0}$). The pairing is the natural extension of (3.1):

$$\langle x_{n-1}^{m_{n-1}} \cdots x_1^{m_1} x_0^k, p_{n-1}^{r_{n-1}} \cdots p_1^{r_1} g^s \rangle = \delta_{m_{n-1},r_{n-1}} \cdots \delta_{m_1,r_1} m_{n-1}! \cdots m_1! s^k.$$

The resulting coregular action is conveniently expressed as (see also [MR94])

$$p_i \triangleright : f : = : \frac{\partial}{\partial x_i} f :, \qquad g \triangleright : f : = : T_{1,x_0} f :$$

with $f \in \Bbbk[x_0, \ldots, x_{n-1}]$. Due to cocommutativity, the notions of left and right integral coincide. The invariance conditions for integration become

$$\int : \frac{\partial}{\partial x_i} f : = 0 \quad \forall i \in \{1, \ldots, n-1\} \qquad \text{and} \qquad \int : \nabla_{1,x_0} f : = 0.$$

The condition on the left is familiar and states the invariance under infinitesimal translations in the x_i. The condition on the right states the invariance under integer translations in x_0. However, we should remember that we use a certain algebraic model of $\mathcal{C}(B_{n+})$. We might add, for example, a generator p_0 to $\mathcal{C}(B_{n+})$ that is dual to x_0 and behaves as the "logarithm" of g, i.e. acts as an infinitesimal translation in x_0. We then have the condition of infinitesimal translation invariance

$$\int : \frac{\partial}{\partial x_\mu} f : = 0$$

for all $\mu \in \{0, 1, \ldots, n-1\}$.

In the present purely algebraic context these conditions do not make much sense. In fact they would force the integral to be zero on the whole algebra. This is not surprising, since we are dealing only with polynomial functions which would not be integrable in the classical case either. In contrast, if we had for example the algebra of smooth functions in two real variables, the conditions just characterise the usual Lebesgue integral (up to normalisation).

Let us assume $\Bbbk = \mathbb{R}$ and suppose that we have extended the normal ordering vector space isomorphism $\mathbb{R}[x_0, \ldots, x_{n-1}] \cong \mathsf{U}(\mathfrak{b}_{n+})$ to a vector space isomorphism of some sufficiently large class of functions on $\mathbb{R}^n$ with a suitable completion $\hat{\mathsf{U}}(\mathfrak{b}_{n+})$ in a functional analytic framework (embedding $\mathsf{U}(\mathfrak{b}_{n+})$ in some operator algebra on a Hilbert space). It is then natural to define the integration on $\hat{\mathsf{U}}(\mathfrak{b}_{n+})$ by

$$\int : f : = \int_{\mathbb{R}^n} f \; dx_0 \cdots dx_{n-1},$$

where the right hand side is just the usual Lebesgue integral in n real variables $x_0, \ldots, x_{n-1}$. This integral is unique (up to normalisation) in satisfying the covariance condition since, as we have seen, these correspond just to the usual translation invariance in the classical case via normal ordering, for which the Lebesgue integral is the unique solution. It is also the $q \to 1$ limit of the translation invariant integral on $\mathsf{U}_q(\mathfrak{b}_+)$ obtained in [Maj90b].

We see that the natural differential calculus in Corollary 3.5.1 is compatible with this integration in that the appearing braided derivations are exactly the actions of the translation generators p_μ. However, we should stress that this calculus is not covariant under the full κ-Poincaré algebra, since it was shown in [GKM96] that in $n = 4$ there is no such calculus of dimension 4. Our results therefore indicate a new intrinsic approach at κ-Minkowski space that allows a bicovariant differential calculus of dimension 4 and a unique translation invariant integral by normal ordering and Lebesgue integration.

3.A Appendix: The Adjoint Coaction on $\mathsf{U}_q(\mathfrak{b}_+)$

The coproduct on X^n is

$$\Delta(X^n) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q g^r X^{n-r} \otimes X^r,$$

$$(\mathrm{id} \otimes \Delta)\, \Delta(X^n) = \sum_{r=0}^{n} \sum_{i=0}^{r} \begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} r \\ i \end{bmatrix}_q g^r X^{n-r} \otimes g^i X^{r-i} \otimes X^i.$$

From this we get

$$\begin{aligned}
\mathrm{Ad}_{\mathrm{L}}(X^n) &= \sum_{r=0}^{n}\sum_{s=0}^{r}\begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} r \\ s \end{bmatrix}_q g^r X^{n-r}\,\mathrm{S}\,X^s \otimes g^s X^{r-s} \\
&= \sum_{r=0}^{n}\sum_{s=0}^{r}\begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} r \\ s \end{bmatrix}_q g^r X^{n-r}(-g^{-1}X)^s \otimes g^s X^{r-s} \\
&= \sum_{t=0}^{n}\sum_{s=0}^{n-t}\begin{bmatrix} n \\ t+s \end{bmatrix}_q \begin{bmatrix} t+s \\ s \end{bmatrix}_q g^{t+s} X^{n-t-s}(-g^{-1}X)^s \otimes g^s X^{t} \\
&= \sum_{t=0}^{n}\sum_{s=0}^{n-t}\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} n-t \\ s \end{bmatrix}_q g^{t+s} X^{n-t-s}(-g^{-1}X)^s \otimes g^s X^{t} \\
&= \sum_{t=0}^{n}\begin{bmatrix} n \\ t \end{bmatrix}_q g^t X^{n-t} \otimes X^t \sum_{s=0}^{n-t}\begin{bmatrix} n-t \\ s \end{bmatrix}_q q^{s(s+1)/2}(-q^{-n}g)^s \\
&= \sum_{t=0}^{n}\begin{bmatrix} n \\ t \end{bmatrix}_q g^t X^{n-t} \otimes X^t \prod_{u=1}^{n-t}(1-q^{u-n}g),
\end{aligned}$$

where we have used

$$\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_q q^{i(i+1)/2}x^i = \prod_{j=1}^{n}(1+q^j x),$$

which can be easily checked by induction. Using the property

$$\mathrm{Ad}_{\mathrm{L}}(ag^n) = \mathrm{Ad}_{\mathrm{L}}(a)(1\otimes g^n) \qquad \forall n \in \mathbb{Z},$$

we obtain for any polynomial P in g, g^{-1}:

$$\mathrm{Ad}_{\mathrm{L}}(X^n P(g)) = \sum_{t=0}^{n}\begin{bmatrix} n \\ t \end{bmatrix}_q g^t X^{n-t} \otimes X^t P(g) \prod_{u=1}^{n-t}(1-q^{u-n}g). \tag{3.2}$$

Chapter 4

Quantum Geometry of the Planck Scale Hopf Algebra

The *Planck scale Hopf algebra* introduced by Majid [Maj88] is an interesting toy model for Planck scale physics. It describes a quantum mechanical particle in one dimension in a curved phase space. The definition is in terms of the algebra of observables which has a quantum group structure as the bicrossproduct $\mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p]$. Explicitly, it is given by generators x, p with the following Hopf algebra structure:

$$
\begin{gathered}
[x,p] = \mathrm{i}\hbar(1 - e^{-\frac{x}{\mathsf{G}}}),\\
\Delta\, p = p \otimes e^{-\frac{x}{\mathsf{G}}} + 1 \otimes p, \qquad \Delta\, x = x \otimes 1 + 1 \otimes x,\\
\mathrm{S}\, p = -pe^{\frac{x}{\mathsf{G}}}, \quad \epsilon\, p = 0, \qquad \mathrm{S}\, x = -x \quad , \epsilon\, x = 0.
\end{gathered}
$$

The two parameters $\hbar$ and G play respectively the role of Planck's constant and the "gravitational" curvature scale.

As it turns out that the Planck scale Hopf algebra is a cocycle deformation quantisation of the Lie group B_+ (considered in Chapter 3), the methods of Chapter 2 enable us to study its quantum differential geometry. This is the object of the present chapter.

In Section 4.1 we start by constructing the Planck scale Hopf algebra as a twist of the algebra of functions $\mathcal{C}(B_+)$ on the Lie group B_+. Thus, B_+ is the classical phase space underlying the quantum system. However, covariance under the non-Abelian group structure of B_+ means that this phase space has a kind of curvature, the scale of which is determined by the parameter G.

Next, we apply the methods of Chapter 2 in Section 4.2 to deformation quantise the differential calculi of B_+ obtained in Chapter 3 yielding all differential calculi on $\mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p]$. We further obtain the exterior algebra on $\mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p]$ corresponding to the standard one on B_+. It turns out that the curvature is essential for the quantum geometry: It becomes singular in the limit of flat space quantum mechanics.

In Section 4.3 we explore elements of a "quantum Poisson geometry" on $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$. While quantum mechanics normally "forgets" the geometric picture attached with the classical phase space, quantum geometry is able to carry over geometrical notions such as differential forms and vector fields. We investigate the Poisson structure in this context. This leads to a non-standard suggestion for the quantum equations of motion motivated by the geometry.

In Section 4.4 we develop a Fourier transformation that relates $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$ to its dual $\mathbb{C}[\bar{p}]\blacktriangleright\!\!\triangleleft_{\frac{1}{\hbar},\frac{\mathsf{G}}{\hbar}}\mathbb{C}[\bar{x}]$. Remarkably, $\mathbb{C}[\bar{p}]\blacktriangleright\!\!\triangleleft_{\frac{1}{\hbar},\frac{\mathsf{G}}{\hbar}}\mathbb{C}[\bar{x}]$ is essentially isomorphic to $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$, but with Planck's constant $\hbar$ and the curvature scale G "inverted". Thus, we obtain a kind of T-duality. In the classical limit as well as the flat space limit the transformation degenerates to a transformation between the two quite different spaces $\mathcal{C}(B_+)$ and $\mathsf{U}(\mathfrak{b}_+)$.

This chapter is based on joint work with Shahn Majid [MO99].

4.1 The Cocycle Twist

For the purposes of the present section we work algebraically with $g = e^{-\frac{x}{\mathsf{G}}}$ and g^{-1} instead of x. Then, the explicit formulae become

$$[p, g] = \mathrm{i}A(1-g)g,$$
$$\Delta\, p = p\otimes g + 1\otimes p, \qquad \Delta\, g = g\otimes g,$$
$$\mathrm{S}\, p = -pg^{-1}, \quad \epsilon\, p = 0, \qquad \mathrm{S}\, g = g^{-1}, \quad \epsilon\, g = 1,$$

where $A := \frac{\hbar}{\mathsf{G}}$. Also, as $\hbar \to 0$ (corresponding to $A \to 0$), we obtain $\mathcal{C}(B_+)$ (see Chapter 3). In terms of the coordinate functions g and p the group multiplication may be written as

$$\begin{pmatrix} g & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} g' & 0 \\ p' & 1 \end{pmatrix} = \begin{pmatrix} gg' & 0 \\ pg' + p' & 1 \end{pmatrix}.$$

Our starting points are the known facts that the Hopf algebra $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$ is of a self-dual form and at the same time a twisting (of the coproduct) of $\mathsf{U}(\mathfrak{b}_+)$ [Maj88, Maj95b]. Combining these observations, one may expect that it is also a product twist of $\mathcal{C}(B_+)$ by a cocycle. This turns out to be the case.

Proposition 4.1.1. *χ defined by*

$$\chi = (\epsilon\otimes\epsilon)\circ\exp\left(\mathrm{i}A\frac{\partial}{\partial p}\otimes\frac{\partial}{\partial g^{-1}}\right), \qquad \chi^{-1} = (\epsilon\otimes\epsilon)\circ\exp\left(\mathrm{i}A\frac{\partial}{\partial p}\otimes\frac{\partial}{\partial g}\right)$$

is a unital 2-cocycle on $\mathcal{C}(B_+)$, and $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p] = \mathcal{C}(B_+)_\chi$.

Proof. In order to show that χ is a unital 2-cocycle we have to show its invertibility, the cocycle condition, and the unitality (see Definition 2.1.3). It will be useful to have the explicit expressions of χ and χ^{-1} on a basis $\{p^n g^r | n \in \mathbb{N}_0, r \in \mathbb{Z}\}$ of $\mathcal{C}(B_+)$:

$$\chi(p^n g^r \otimes p^m g^s) = \delta_{m,0}(\mathrm{i}A)^n \prod_{k=0}^{n-1}(-s-k), \qquad \chi^{-1}(p^n g^r \otimes p^m g^s) = \delta_{m,0}(\mathrm{i}A)^n \prod_{k=0}^{n-1}(s-k).$$

For the invertibility we require

$$\chi(a_{(1)} \otimes b_{(1)})\, \chi^{-1}(a_{(2)} \otimes b_{(2)}) = \epsilon(a)\, \epsilon(b), \qquad \chi^{-1}(a_{(1)} \otimes b_{(1)})\, \chi(a_{(2)} \otimes b_{(2)}) = \epsilon(a)\, \epsilon(b).$$

To see this we take $a = p^n g^r$ and $b = p^m g^s$, and the first expression becomes

$$\begin{aligned}
&\sum_{k,l} \binom{n}{k}\binom{m}{l} \chi(p^k g^r \otimes p^l g^s)\, \chi^{-1}(p^{n-k} g^{k+r} \otimes p^{m-l} g^{l+s}) \\
&= \sum_{k,l} \binom{n}{k}\binom{m}{l} \delta_{l,0}\, \delta_{m,l} (\mathrm{i}A)^n \prod_{i=0}^{k-1}(-s-i) \prod_{j=0}^{n-k}(l+s-j) \\
&= \delta_{m,0}(\mathrm{i}A)^n \sum_k \binom{n}{k} \prod_{i=0}^{k-1}(-s-i) \prod_{j=0}^{n-k}(s-j) \\
&= \delta_{m,0}\, \delta_{n,0} = \epsilon(p^n g^r)\, \epsilon(p^m g^s).
\end{aligned}$$

We have used

$$\sum_{k=0}^{n} \binom{n}{k} \prod_{i=0}^{k-1}(-s-i) \prod_{j=0}^{n-k}(s-j) = \delta_{n,0}, \quad \forall s \in \mathbb{Z}\ \forall n \in \mathbb{N}_0.$$

While this is obvious for $s = 0$ it follows easily by induction for $s \neq 0$. Note that the exchange of χ and χ^{-1} in the above calculation is equivalent to replacing s by $-s$. Thus follows the second equation as well. Next, for the cocycle condition we take $p^n g^r$, $p^m g^s$, $p^l g^t$ for a, b, c in Definition 2.1.3.(ii). The right hand side evaluates to

$$\begin{aligned}
&\sum_{j,k} \binom{m}{j}\binom{l}{k} \chi(p^j g^s \otimes p^k g^t)\, \chi(p^n g^r \otimes p^{m-j+l-k} g^{j+s+k+t}) \\
&= \sum_{j,k} \binom{m}{j}\binom{l}{k} \delta_{k,0} \delta_{m-j+l-k,0} (\mathrm{i}A)^{j+n} \prod_{i=0}^{j-1}(-t-i) \prod_{h=0}^{n-1}(-j-s-k-t-h) \\
&= \sum_{j} \binom{m}{j} \delta_{m-j+l,0} (\mathrm{i}A)^{j+n} \prod_{i=0}^{j-1}(-t-i) \prod_{h=0}^{n-1}(-j-s-t-h) \\
&= \delta_{l,0} (\mathrm{i}A)^{n+m} \prod_{i=0}^{m-1}(-t-i) \prod_{h=0}^{n-1}(-m-s-t-h).
\end{aligned}$$

The left hand side is

$$\begin{aligned}
&\sum_{j,k}\binom{n}{j}\binom{m}{k}\chi(p^j g^r\otimes p^k g^s)\,\chi(p^{n-j+m-k}g^{j+r+k+s}\otimes p^l g^t)\\
&=\sum_{j,k}\binom{n}{j}\binom{m}{k}\delta_{k,0}\delta_{l,0}(\mathrm{i}A)^{n+m-k}\prod_{i=0}^{j-1}(-s-i)\prod_{h=0}^{n-j+m-k-1}(-t-h)\\
&=\delta_{l,0}(\mathrm{i}A)^{n+m}\sum_{j}\binom{n}{j}\prod_{i=0}^{j-1}(-s-i)\prod_{h=0}^{n-j+m-1}(-t-h)\\
&=\delta_{l,0}(\mathrm{i}A)^{n+m}\prod_{i=0}^{m-1}(-t-i)\prod_{h=0}^{n-1}(-m-s-t-h).
\end{aligned}$$

The last equality can be checked by induction in n. Finally, the unitality (Definition 2.1.3.(iii)) follows easily from the explicit formula for χ. It remains to check that the twist of $\mathcal{C}(B_+)$ defined by χ is indeed $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$. For that, it is sufficient to check the commutator between p and g. For clarity, we distinguish the twisted product here from the untwisted one by denoting the former with a $\bullet$.

$$\begin{aligned}
g\bullet g &= \chi(g\otimes g)\,gg\,\chi^{-1}(g\otimes g)=gg,\\
p\bullet g &= \chi(p\otimes g)\,gg\,\chi^{-1}(g\otimes g)+\chi(1\otimes g)\,pg\,\chi^{-1}(g\otimes g)+\chi(1\otimes g)\,g\,\chi^{-1}(p\otimes g)\\
&= -\mathrm{i}Agg+pg+\mathrm{i}Ag,\\
g\bullet p &= \chi(g\otimes p)\,gg\,\chi^{-1}(g\otimes g)+\chi(g\otimes 1)\,gp\,\chi^{-1}(g\otimes g)+\chi(g\otimes 1)\,g\,\chi^{-1}(g\otimes p)=gp.
\end{aligned}$$

In particular, we obtain

$$p\bullet g-g\bullet p=\mathrm{i}A(1-g)g=\mathrm{i}A(1-g)\bullet g,$$

which is the correct relation in $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$. $\square$

4.2 Differential Calculi

Let us now turn to the differential calculi on $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$. We have given a complete classification of the first order differential calculi of the untwisted Hopf algebra $\mathcal{C}(B_+)$ in Chapter 3. However, we use a different basis of $\mathcal{C}(B_+)$ here so that it will be convenient to restate the result.

Proposition 4.2.1 (3.3.5). *(a) Finite dimensional differential calculi Ω^1 on $\mathcal{C}(B_+)$ are in one-to-one correspondence to non-empty finite sets $I\subset\mathbb{N}$ and have dimension $(\sum_{n\in I}n)-1$.*

(b) The differential calculus of dimension $n\geq 2$ corresponding to $I=\{n,1\}$ has a right invariant basis $\eta_0,\dots\eta_{n-1}$ so that

$$\mathrm{d}g=g\eta_0,\qquad \mathrm{d}p=g\eta_1,$$

$$[g,\eta_k]=0,\qquad [p,\eta_k]=\begin{cases}0 & \textit{if } k=0 \textit{ or } k=n-1\\ g\eta_{k+1} & \textit{if } 0<k<n-1\end{cases}$$

$$\beta_L(\eta_k)=\begin{cases}g^{-k}\otimes\eta_k & \textit{if } k\neq 1\\ g^{-1}\otimes\eta_1+g^{-1}p\otimes\eta_0 & \textit{if } k=1\end{cases}$$

(c) The differential calculus of dimension $n-1\geq 1$ corresponding to $I=\{n\}$ is the same as (b) except that $\eta_0=0$.

Proof. We refer to the proof of Proposition 3.3.5. However, due to the different choice of basis of $\mathcal{C}(B_+)$ there, the coproduct appeared in a different form. In the conventions of this chapter the crossed submodule $M\subset\ker\epsilon$ corresponding to $I=\{n,1\}$ is generated by $(g-1)(g-1),p(g-1),\ldots,p^n$ as a crossed module. Denoting the equivalence classes of $g-1,p,\ldots,p^{n-1}$ in $\ker\epsilon/M$ by $\eta_0,\ldots,\eta_{n-1}$, we obtain the derivative and commutation relations as stated. For the left coaction note that the left adjoint action on $\mathcal{C}(B_+)$ takes the form

$$\mathrm{Ad_L}(f(g)p^k)=\sum_{t=0}^{k}\binom{k}{t}g^{-k}p^{k-t}\otimes f(g)(g-1)^{k-t}p^t,$$

similarly to (3.2) but at $q=1$ and in our present basis. β is then obtained by composition with the projection to $\ker\epsilon/M$. $\square$

We can now apply our twisting theory of Chapter 2 to solve the classification problem for calculi on the Planck scale Hopf algebra,

Proposition 4.2.2. *(a) Finite dimensional differential calculi Ω^1 on $\mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p]$ are in one-to-one correspondence to non-empty finite subsets $I\subset\mathbb{N}$ with dimensions as in Proposition 4.2.1.*

(b) The differential calculus of dimension $n\geq 2$ corresponding to $I=\{n,1\}$ has a right invariant basis $\eta_0,\ldots\eta_{n-1}$ so that

$$\mathrm{d}g=g\eta_0,\qquad \mathrm{d}p=g\eta_1,$$

$$[g,\eta_k]=\begin{cases}0 & \textit{if } k\neq 1\\ \mathrm{i}Ag\eta_0 & \textit{if } k=1\end{cases}\qquad [p,\eta_k]=\mathrm{i}Akg\eta_k+\begin{cases}0 & \textit{if } k=0 \textit{ or } k=n-1\\ g\eta_{k+1} & \textit{if } 0<k<n-1\end{cases}$$

and β is of the same form as in Proposition 4.2.1.

(c) The differential calculus of dimension $n-1\geq 1$ corresponding to $I=\{n\}$ is the same as (b) except that $\eta_0=0$.

Proof. We apply Corollary 2.4.2 to Proposition 4.2.1. Part (a) remains unchanged. For part (b) we calculate the twisted actions in terms of the untwisted ones (using a $\bullet$ to denote the twisted ones).

$$\begin{aligned}
g \bullet \eta_k &= \chi(g \otimes g^{-k})\, g\eta_k + \delta_{k,1}\chi(g \otimes g^{-1}p)\, g\eta_0 = g\eta_k,\\
\eta_k \bullet g &= \chi(g^{-k} \otimes g)\, \eta_k g + \delta_{k,1}\chi(g^{-1}p \otimes g)\, \eta_0 g = \eta_k g - \mathrm{i}A\delta_{k,1}\eta_0 g,\\
p \bullet \eta_k &= \chi(p \otimes g^{-k})\, g\eta_k + \chi(1 \otimes g^{-k})\, p\eta_k + \delta_{k,1}(\chi(p \otimes g^{-1}p)\, g\eta_0 + \chi(1 \otimes g^{-1}p)\, p\eta_0)\\
&= \mathrm{i}Akg\eta_k + p\eta_k,\\
\eta_k \bullet p &= \chi(g^{-k} \otimes p)\, \eta_k g + \chi(g^{-k} \otimes 1)\, \eta_k p + \delta_{k,1}(\chi(g^{-1}p \otimes p)\, \eta_0 g + \chi(g^{-1}p \otimes 1)\, \eta_0 p) = \eta_k p.
\end{aligned}$$

This gives the new commutators and the expressions for the differentials. For the coaction we observe that $g^{-1} \bullet p = g^{-1}p$ so that its form does not change. Part (c) remains unchanged. $\square$

For the remainder of the section we concentrate on the calculus $\{2, 1\}$ which is the quantisation of the standard classical calculus on B_+. We can use the twisting theory to quantise in fact the entire exterior algebra in this case. We use Proposition 2.4.3 to take the whole super-Hopf algebra structure alongside.

Proposition 4.2.3. *The exterior algebra Ω of $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$ corresponding via twisting to the classical one of $\mathcal{C}(B_+)$ has the following properties. The first order calculus has a basis $\{\xi, \eta\}$ of right-invariant 1-forms with*

$$\mathrm{d}g = g\xi, \qquad \mathrm{d}p = g\eta, \qquad [a, \xi] = 0, \qquad [a, \eta] = \mathrm{i}A\mathrm{d}a, \qquad \forall a \in \mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p],$$
$$\beta_L(\xi) = 1 \otimes \xi, \qquad \beta_L(\eta) = g^{-1} \otimes \eta + g^{-1}p \otimes \xi.$$

The 2-forms have relations

$$\xi \wedge \xi = 0, \qquad \eta \wedge \xi = -\xi \wedge \eta, \qquad \eta \wedge \eta = \mathrm{i}A\xi \wedge \eta,$$
$$\mathrm{d}\xi = 0, \qquad \mathrm{d}\eta = \eta \wedge \xi.$$

As a super-Hopf algebra Ω has the structure

$$\Delta\,\xi = \xi \otimes 1 + 1 \otimes \xi, \quad \Delta\,\eta = g^{-1} \otimes \eta + g^{-1}p \otimes \xi - \eta \otimes 1,$$
$$\epsilon(\xi) = \epsilon(\eta) = 0, \qquad \mathrm{S}\,\xi = -\xi, \quad \mathrm{S}\,\eta = -g\eta + p\xi.$$

Proof. For the first order calculus we define $\xi := \eta_0$ and $\eta := \eta_1$. The commutation relations in the $n = 2$ case (b) of Proposition 4.2.2 become as stated. Next, the classical space of 2-forms on $\mathcal{C}(B_+)$ is spanned by $\xi \wedge \eta = -\eta \wedge \xi$. Denoting the wedge product on $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$ by $\wedge_\bullet$ we have

$$\eta \wedge_\bullet \eta = \chi(g^{-1} \otimes g^{-1})\, \eta \wedge \eta + \chi(g^{-1}p \otimes g^{-1})\, \xi \wedge \eta$$

$$\begin{aligned}&+\chi(g^{-1}\otimes g^{-1}p)\,\eta\wedge\xi+\chi(g^{-1}p\otimes g^{-1}p)\,\xi\wedge\xi\\ &=\mathrm{i}A\xi\wedge\eta.\end{aligned}$$

The other wedge products involving ξ and η are identical to the classical ones due to the bi-invariance of ξ. This leads to the relations stated. Finally, for the differentials of the 1-forms observe (the twisted and untwisted wedge products are the same here)

$$\mathrm{d}\xi=\mathrm{d}(g^{-1}\mathrm{d}g)=\mathrm{d}g^{-1}\wedge\mathrm{d}g=-g^{-1}\xi\wedge g\xi=0,$$
$$\mathrm{d}\eta=\mathrm{d}(g^{-1}\mathrm{d}p)=\mathrm{d}g^{-1}\wedge\mathrm{d}p=-g^{-1}\xi\wedge g\eta=\eta\wedge\xi.$$

The coproduct and antipode are readily obtained using Proposition 1.3.5. The exterior algebra here coincides with the Woronowicz prolongation of the first order part. □

In terms of generators x and p, the exterior algebra is generated by $\mathrm{d}x=\xi,\mathrm{d}p=g^{-1}\eta$ with the relations

$$a\mathrm{d}x=(\mathrm{d}x)a,\quad a\mathrm{d}p=(\mathrm{d}p)a+\frac{\mathrm{i}\hbar}{\mathsf{G}}\mathrm{d}a,$$
$$\mathrm{d}x\wedge\mathrm{d}x=0,\quad \mathrm{d}x\wedge\mathrm{d}p=-\mathrm{d}p\wedge\mathrm{d}x,\quad \mathrm{d}p\wedge\mathrm{d}p=0.$$

From this we see explicitly that in the classical limit $\hbar\to 0$ we obtain the usual exterior algebra on B_+. By contrast, the other limit $\mathsf{G}\to 0$ is highly singular with these generators, so that the exterior algebra is not even defined in this case. In other words, the presence of "gravity" in the form of G restores the geometrical picture not visible in flat space quantum mechanics.

We also know from Section 1.3 that associated to a first order calculus is a quantum tangent space L dual to the space V of right-invariant 1-forms. The right-invariant derivatives are generated by elements of L and obey a braided Leibniz rule.

Proposition 4.2.4. *Let $\{\xi^*,\eta^*\}$ be the basis of L dual to the basis $\{\xi,\eta\}$ above. Then,*

$$\partial_\xi(:f(g,p):)=:g\frac{\partial}{\partial g}f(g,p+\mathrm{i}A)+g(f(g,p+\mathrm{i}A)-f(g,p)):,$$
$$\partial_\eta(:f(g,p):)=:g\frac{f(g,p+\mathrm{i}A)-f(g,p)}{\mathrm{i}A}:.$$

Proof. We observe that $\mathrm{d}g^n=(\mathrm{d}g)ng^{n-1}$ and $\mathrm{d}p^n=(\mathrm{d}p)\frac{(p+\mathrm{i}A)^n-p^n}{\mathrm{i}A}$ (this can be easily checked by induction), so that

$$\begin{aligned}\mathrm{d}(g^np^m)&=(\mathrm{d}g^n)p^m+g^n\mathrm{d}p^m\\ &=(\mathrm{d}g^n)p^m+g^n(\mathrm{d}p)\frac{(p+\mathrm{i}A)^m-p^m}{\mathrm{i}A}\end{aligned}$$

$$
\begin{aligned}
&= (\mathrm{d}g^n)p^m + (\mathrm{d}p)g^n \frac{(p+\mathrm{i}A)^m - p^m}{\mathrm{i}A} + \mathrm{i}A(\mathrm{d}g^n)\frac{(p+\mathrm{i}A)^m - p^m}{\mathrm{i}A} \\
&= (\mathrm{d}g^n)(p+\mathrm{i}A)^m + (\mathrm{d}p)g^n \frac{(p+\mathrm{i}A)^m - p^m}{\mathrm{i}A} \\
&= \xi\, n g^n (p+\mathrm{i}A)^m + (\eta\, g + \mathrm{i}A\,\xi\, g)g^n \frac{(p+\mathrm{i}A)^m - p^m}{\mathrm{i}A} \\
&= \xi\, \left(n g^n (p+\mathrm{i}A)^m + g^{n+1}\left((p+\mathrm{i}A)^m - p^m\right)\right) + \eta\, g^{n+1}\frac{(p+\mathrm{i}A)^m - p^m}{\mathrm{i}A},
\end{aligned}
$$

which we compare with the property $\mathrm{d}f = \xi\,\partial_\xi(f) + \eta\,\partial_\eta(f)$ of the partial derivatives. $\square$

In terms of coordinates x, p we can similarly write the action of the basis of L dual to $\{\mathrm{d}x, \eta\}$ as

$$\partial_x(:f(x,p):) =: \frac{\partial}{\partial x} f(x, p + \frac{\mathrm{i}\hbar}{\mathsf{G}}) - \frac{e^{-\frac{x}{\mathsf{G}}}}{\mathsf{G}}(f(x, p + \frac{\mathrm{i}\hbar}{\mathsf{G}}) - f(x,p)) :, \tag{4.1}$$

$$\partial_\eta(:f(x,p):) = \frac{\mathsf{G}}{\mathrm{i}\hbar} : e^{-\frac{x}{\mathsf{G}}}(f(x, p + \frac{\mathrm{i}\hbar}{\mathsf{G}}) - f(x,p)) : \tag{4.2}$$

(here ∂_x denotes the action of the basis element dual to $\mathrm{d}x$ by a slight abuse of notation).

Finally, for completeness we note that all these formulae are for right-invariant differential forms. There is an equally good theory based on $L, V \in \dot{\mathcal{M}}^H_H$ and left-invariant partial derivatives. We take a left-invariant basis of the 1-forms to be $\{\xi = g^{-1}\mathrm{d}g, \bar\eta = \mathrm{d}p - pg^{-1}\mathrm{d}g\}$. The relations of the the calculus become

$$
\begin{gathered}
[a, \xi] = 0, \qquad [a, \bar\eta] = \mathrm{i}A\mathrm{d}a, \qquad \forall a \in \mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p], \\
\beta_R(\xi) = \xi \otimes 1, \qquad \beta_R(\bar\eta) = \bar\eta \otimes g - \xi \otimes p, \\
\xi \wedge \xi = 0, \qquad \bar\eta \wedge \xi = -\xi \wedge \bar\eta, \qquad \bar\eta \wedge \bar\eta = \mathrm{i}A\bar\eta \wedge \xi.
\end{gathered}
$$

Moreover, the differential in Ω is generated by (graded) commutation with the element $\theta = -\frac{1}{2}(\eta + \bar\eta)$ as

$$[\theta, \alpha] = \mathrm{i}A\mathrm{d}\alpha, \qquad \forall \alpha \in \Omega. \tag{4.3}$$

This is a step towards a Connes spectral triple description of this calculus. The generator $\frac{1}{\mathrm{i}A}\theta$ is singular in the limit $A \to 0$ ($\hbar \to 0$) so that the presence of $\hbar$ allows for nicer properties of the differential calculus than visible classically. This is a typical feature of q-deformation known for calculi on standard quantum groups.

We likewise bar the braided derivations in the left-invariant version of the theory to avoid confusion with the above right-invariant ones. The left-invariant derivations corresponding to $\{\mathrm{d}x, \bar\eta\}$ in the sense $\mathrm{d}f = (\bar\partial_x f)\mathrm{d}x + (\bar\partial_\eta f)\bar\eta$ are

$$\bar\partial_x(:f(x,p):) =: \frac{\partial}{\partial x} f(x,p) + \frac{p}{\mathrm{i}\hbar}\left(f(x, p - \frac{\mathrm{i}\hbar}{\mathsf{G}}) - f(x,p)\right) :, \tag{4.4}$$

$$\bar\partial_\eta(:f(x,p):) = -\frac{\mathsf{G}}{\hbar} : (f(x, p - \frac{\hbar}{\mathsf{G}}) - f(x,p)) : . \tag{4.5}$$

4.3 Quantum Poisson Bracket

We present in this section some elements of "quantum Poisson geometry". We recall first the classical situation. According to Proposition 2.3.1, any twisting of the Hopf algebra of functions on a Lie group G by a cocycle χ admitting a reasonable expansion in a parameter $\hbar$ defines a deformation quantisation. Furthermore, the underlying Poisson bracket makes G into a Poisson-Lie group. Note that the Poisson bracket, as with all Poisson-Lie groups, cannot be symplectic since it must vanish at least at the group identity.

In the present case, we obtain a Poisson bracket making B_+ into a Poisson-Lie group.

Proposition 4.3.1. *The Poisson bracket on $\mathcal{C}(B_+)$, for which the cocycle χ of Proposition 4.1.1 provides the quantisation, is*

$$\{a,b\} = (e^{-\frac{x}{\mathsf{G}}} - 1)\left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial p} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial p}\right).$$

Proof. Expanding χ of Proposition 4.1.1 in $\hbar$ and expressing everything in terms of the coordinates x, p yields

$$\begin{aligned} a \bullet b &= ab + \mathrm{i}\hbar\,\epsilon\left(\frac{\partial}{\partial p}a_{(1)}\right)\epsilon\left(e^{-\frac{x}{\mathsf{G}}}\frac{\partial}{\partial x}b_{(1)}\right)a_{(2)}b_{(2)} \\ &\quad - \mathrm{i}\hbar a_{(1)}b_{(1)}\,\epsilon\left(\frac{\partial}{\partial p}a_{(2)}\right)\epsilon\left(e^{\frac{x}{\mathsf{G}}}\frac{\partial}{\partial x}b_{(2)}\right) + \mathcal{O}\left(\hbar^2\right) \\ &= ab + \mathrm{i}\hbar\left(e^{-\frac{x}{\mathsf{G}}}\frac{\partial a}{\partial p}\frac{\partial b}{\partial x} - \frac{\partial a}{\partial p}\frac{\partial b}{\partial x}\right) + \mathcal{O}\left(\hbar^2\right) \\ &= ab + \mathrm{i}\hbar\left(e^{-\frac{x}{\mathsf{G}}} - 1\right)\left(\frac{\partial a}{\partial p}\frac{\partial b}{\partial x}\right) + \mathcal{O}\left(\hbar^2\right), \\ a \bullet b - b \bullet a &= \frac{\hbar}{\mathrm{i}}\left(e^{-\frac{x}{\mathsf{G}}} - 1\right)\left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial p} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial p}\right) + \mathcal{O}\left(\hbar^2\right). \end{aligned}$$

□

The idea is now to take the classical Poisson bracket and apply the twisting to it in a generalised quantum geometric setting, yielding a *quantum Poisson bracket*. We start by developing an appropriate setting. We work over a general field $\Bbbk$ for this.

Since $L, V \in {}^H_H\dot{\mathcal{M}}$, we can take their arbitrary tensor powers to define tensor fields of arbitrary mixed rank using the same correspondence with bicovariant bimodules. Thus $\Omega^{-1} = L \otimes H$ and $\Omega^{-1} \otimes_H \Omega^{-1} = L \otimes L \otimes H$ etc. We have a super-Hopf algebra $T_{-1}(\Omega^{-1})$ and a theory of twisting of quantum vector fields using the same theory of Chapter 2. Also, since morphisms in ${}^H_H\dot{\mathcal{M}}$ induce morphisms between bicovariant bimodules, the evaluation map $\langle\ ,\ \rangle : L \otimes V \to \Bbbk$ induces the pairing between vector fields and 1-forms. Thus

$$\Omega^{-1} \otimes_H \Omega^1 \to H, \quad \langle x \otimes h, v \otimes g\rangle = \langle x, h_{(1)} \triangleright v\rangle h_{(2)} g, \tag{4.6}$$

$$\Omega^{-1}\otimes_H\Omega^{-1}\otimes_H\Omega^1\to\Omega^{-1},\quad \langle x\otimes y\otimes h, v\otimes g\rangle = x\otimes\langle y, h_{(1)}\triangleright v\rangle h_{(2)}g, \tag{4.7}$$

$$\begin{gathered}\Omega^{-1}\otimes_H\Omega^{-1}\otimes_H\Omega^1\otimes_H\Omega^1\to H,\\ \langle x\otimes y\otimes h, v\otimes w\otimes g\rangle = \langle y, h_{(1)}\triangleright v\rangle\langle x, h_{(2)}\triangleright w\rangle h_{(3)}g,\end{gathered} \tag{4.8}$$

etc. The pairing $L\otimes L\otimes V\otimes V\to \mathbb{k}$ in (4.8) is the natural one in a braided category, namely to evaluate the inner $L\otimes V$ first and then the outer. The resulting pairing is also the same as applying (4.7) to the first factor of $\Omega^1\otimes_H\Omega^1$ to obtain an element of $\Omega^{-1}\otimes_H\Omega^1$ and then applying (4.6).

This is not the only way to formulate vector fields (for example a more left-right symmetric way is to consider $L\in\dot{\mathcal{M}}_H^H$ and $\Omega^{-1}=H\otimes L$, extending the pairing by $\langle h\otimes x, v\otimes g\rangle = h\langle x,v\rangle g$), but it is the one natural in the context of the Woronowicz exterior algebra (which can be viewed as based on a fixed identification of bicovariant bimodules with ${}^H_H\dot{\mathcal{M}}$ (say)). Taking now Ω^n defined by quotients of $V^{\otimes n}$ in the exterior algebra in this approach, the natural definition of antisymmetric vector fields is as corresponding to the appropriate subspace of $L^{\otimes n}$ dual to this quotient. In particular, the Poisson bivector field should be an element

$$\Pi\in\Omega^{-2}=\{x\otimes y-\psi_{L,L}(x\otimes y)|\ x,y\in L\}\otimes H$$

since $V\otimes V$ is quotiented by $\ker(\mathrm{id}-\psi_{V,V})$ in degree 2.

In general, we also need to impose a "Jacobi identity" on Π, which can be done as follows, at least in the nice case where the quantum Poisson bracket is non-degenerate: We can consider Π by the above as a map $\Omega^1\to\Omega^{-1}$ and demand that it is invertible, and that the inverse corresponds to evaluation against some $\omega\in\Omega^2$ which we can demand to be closed. Alternatively, one may attempt to develop a theory of "quantum-Lie algebras" and use the "quantum-Lie bracket" on L, thereby avoiding the invertibility assumption. This will not be attempted here, however; for our present purposes we note that in 2 dimensions with the classical differential calculus the Jacobi identity is redundant (similarly, every 2-form is closed). For our particular exterior algebras the dimensions are the classical ones (so that every 2-form is closed) and one may similarly consider any antisymmetric bivector field as a Poisson structure. We now give the explicit form of the quantum Poisson bracket corresponding to the above classical Poisson bracket.

Proposition 4.3.2. *For the Planck scale Hopf algebra with the standard quantum differential calculus as above, we consider* Π *of the form*

$$\Pi=(\eta^*\otimes\xi^*-\psi_{L,L}(\eta^*\otimes\xi^*))\otimes\pi(g)$$

for an arbitrary function $\pi(g)$*. Then the corresponding quantum Poisson bracket is*

$$\{a,b\}=\pi(g)\left(a_\xi b_\eta-a_\eta b_\xi+\mathrm{i}A(a_\eta b_\eta+(a_\xi)_\eta b_\eta-(a_\eta)_\xi b_\eta)+(\mathrm{i}A)^2(a_\eta)_\eta b_\eta\right)$$

where $a_\xi := \partial_\xi a$, etc. In particular, $\pi(g) = \frac{1}{\mathsf{G}}(g^{-1}-1)$ gives the twisting of the classical Poisson structure in Proposition 4.3.1.

Proof. We first of all use $\mathrm{d}a = \xi\partial_\xi a + \eta\partial_\eta a$ and the relations of the exterior algebra to obtain

$$\mathrm{d}a \wedge \mathrm{d}b = \xi \wedge \eta f, \quad f = a_\xi b_\eta - a_\eta b_\xi + \mathrm{i}A(a_\eta b_\eta + (a_\xi)_\eta b_\eta - (a_\eta)_\xi b_\eta) + (\mathrm{i}A)^2 (a_\eta)_\eta b_\eta.$$

Now the pairing can be computed as

$$\begin{aligned}\{a,b\} &= \langle \Pi, \mathrm{d}a\wedge \mathrm{d}b\rangle = \langle (\eta^*\otimes\xi^* - \psi(\eta^*\otimes\xi^*))\pi(g), \xi\wedge\eta f\rangle \\ &= \langle \eta^*\otimes\xi^* - \psi(\eta^*\otimes\xi^*), \xi\otimes\pi(g)_{(1)}\triangleright\eta\rangle\pi(g)_{(2)}f \\ &= \langle \eta^*\otimes\xi^*, \xi\otimes\pi(g)_{(1)}\triangleright\eta - \psi_{V,V}(\xi\otimes\pi(g)_{(1)}\triangleright\eta)\rangle\pi(g)_{(2)}f \\ &= \langle \eta^*\otimes\xi^*, \xi\otimes\pi(g)_{(1)}\triangleright\eta - \pi(g)_{(1)}\triangleright\eta\otimes\xi\rangle\pi(g)_{(2)}f = \langle\eta^*\otimes\xi^*,\xi\otimes\eta\rangle\pi(g)f,\end{aligned}$$

where we used functoriality of the braiding under the evaluation morphism to deduce

$$\langle \psi_{L,L}(\eta^*\otimes\xi^*), v\otimes w)\rangle = \langle \eta^*\otimes\xi^*, \psi_{V,V}(v\otimes w)\rangle$$

for any $v, w \in V$, and then $\psi_{V,V}(\xi\otimes w) = w_{(1)}\triangleright\xi\otimes w_{(\underline{2})} = w\otimes\xi$ since ξ is an invariant element of the crossed module. In the last line we used $g\triangleright\eta = \eta + \mathrm{i}A\xi$ to see that, although η is not invariant, the evaluation $\langle\eta^*, g^n\triangleright\eta\rangle = \langle\eta^*,\eta\rangle$ behaves as if it is. In terms of functions $a(g,p)$, $b(g,p)$ we obtain

$$\begin{aligned}&\{: a(g,p) :, : b(g,p) :\} \\ &= \pi(g) : \left(g(g-2)a(g,p) + g(\frac{\partial}{\partial g} - 2g + 3)a(g,p+\mathrm{i}A) + g(g-1)a(g,p+2\mathrm{i}A)\right) : \\ &\quad \bullet : g\frac{b(g,p+\mathrm{i}A) - b(g,p)}{\mathrm{i}A} : \\ &\quad - \pi(g) : g\frac{a(g,p+\mathrm{i}A)-a(g,p)}{\mathrm{i}A} : \bullet : \left(g\frac{\partial}{\partial g} b(g,p+\mathrm{i}A) + g(b(g,p+\mathrm{i}A) - b(g,p))\right) : .\end{aligned}$$

The classical limit $A \to 0$ is

$$\{a(g,p), b(g,p)\} = \pi(g)g^2\left(\frac{\partial a}{\partial g}\frac{\partial b}{\partial p} - \frac{\partial a}{\partial p}\frac{\partial b}{\partial g}\right).$$

Thus, to get the correct Poisson structure, we need $\pi(g) = \frac{1}{\mathsf{G}}(g^{-1}-1)$ (note that $-\mathsf{G}\frac{\partial}{\partial x} = g\frac{\partial}{\partial g}$). □

Also, if $: h :\in \mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p]$ is a choice of Hamiltonian then

$$\dot{x} = \{x, : h :\} = \frac{\mathsf{G}}{\mathrm{i}\hbar} : (e^{-\frac{x}{\mathsf{G}}} - 1)(h(x, p + \frac{\mathrm{i}\hbar}{\mathsf{G}}) - h(x,p)) :, \tag{4.9}$$

$$\dot{p} = \{p, : h :\} =: -(e^{-\frac{x}{\mathsf{G}}} - 1)\frac{\partial}{\partial x} h(x, p + \frac{\mathrm{i}\hbar}{\mathsf{G}})) : \tag{4.10}$$

are the corresponding quantum Hamilton equations of motion. For a simple concrete example, choosing the Hamiltonian $h(x,p) = \frac{p^2}{2m} + V(x)$ for a free particle of mass m in a potential $V(x)$, we obtain

$$\dot{x} = \frac{1}{2m}(e^{-\frac{x}{\mathsf{G}}} - 1)(2p - \frac{\mathrm{i}\hbar}{\mathsf{G}}), \quad \dot{p} = (e^{-\frac{x}{\mathsf{G}}} - 1)\frac{\partial}{\partial x}V(x). \tag{4.11}$$

Standard quantum mechanics (i.e. using the commutator with h) leads by contrast to

$$\dot{x} = \frac{\mathrm{i}}{\hbar}[x,h] = \frac{1}{2m}(e^{-\frac{x}{\mathsf{G}}} - 1)(2p - \frac{\mathrm{i}\hbar}{\mathsf{G}}e^{-\frac{x}{\mathsf{G}}}), \quad \dot{p} = \frac{\mathrm{i}}{\hbar}[p,h] = (e^{-\frac{x}{\mathsf{G}}} - 1)\frac{\partial}{\partial x}V(x).$$

Thus the quantum Hamiltonian equations of motion reduce to the classical ones when $\hbar \to 0$ as they should, but differ from the conventional quantum mechanical equations of motion at small momenta with Compton wavelength $\gg \mathsf{G}$. (We recall that G is the background curvature scale.) On the other hand, the quantum Hamiltonian equations retain a full (quantum) geometrical interpretation which is lost in conventional quantum mechanics. This suggests a geometrical modification of conventional quantum mechanics.

4.4 Fourier Theory

In this section we make some remarks about the noncommutative Fourier theory which is known to exist on any Hopf algebra equipped with a suitable translation-invariant integral and a suitable coevaluation element. We recall first the general formulation, which works basically when the Hopf algebra H is finite-dimensional, and in conventions suitable for our particular example. Thus, we require $\int : H \to \Bbbk$ such that $(\int h_{(1)})h_{(2)} = (\int h)1$ for all $h \in H$ (a right-integral) and $\int^* : H^* \to \Bbbk$ such that $\phi_{(1)} \int^* \phi_{(2)} = 1 \int \phi$ for all $\phi \in H^*$ (a left-integral), and we let $\sum e_a \otimes f^a \in H \otimes H^*$ denote the canonical coevaluation element (here $\{e_a\}$ is a basis of H and $\{f^a\}$ a dual basis). Then the Fourier transform in these conventions is

$$\mathcal{T}(h) = \left(\int e_a h\right) f^a, \quad \mathcal{T}^*(\phi) = e_a \int^* f^a \phi \tag{4.12}$$

and obeys

$$\begin{gathered} \mathcal{T}\mathcal{T}^*(\phi) = \mathrm{S}^{-1}\,\phi \int e_a \int^* f^a, \\ \mathcal{T}(h_{(1)}\langle\phi, h_{(2)}\rangle) = \mathcal{T}(h)\,\mathrm{S}^{-1}\,\phi, \quad \mathcal{T}^*(\langle\phi_{(1)}, h\rangle\phi_{(2)}) = \mathrm{S}\,h\mathcal{T}^*(\phi). \end{gathered} \tag{4.13}$$

These elementary facts are easily proven once one notes that $(\int g_{(1)}h)g_{(2)} = (\int gh_{(1)})Sh_{(2)}$ for all $h, g \in H$ and a similar identity on H^*. See also [KM94] for more discussion (and the extension to braided groups).

In our case the Planck scale Hopf algebra $\mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p]$ is not finite-dimensional and there is, moreover, no purely algebraic integral. For a full treatment one needs to introduce a Hopf-von Neumann algebra setting along the lines in [Maj91] and work with the integral as a weight, or one has to work with a C^*-algebra setting extended to include unbounded operators. Both of these are nontrivial and beyond our scope here. However, the bicrossproduct form of the Hopf algebra allows one to identify elements as normal ordered versions of ordinary functions $f(x,p)$ and thereby to reduce integration to ordinary integration of ordinary functions, for any class of functions and any topological setting to which the normal ordering extends. Therefore in this section we will initially work formally with x,p as generators (unlike the algebraic setting in the preceding sections) and proceed to consider formal power series in them; however, what we arrive at in this way is a well-defined deformed Fourier theory on functions on $\mathbb{R}^2$ of suitably rapid decay, *motivated* by the Hopf algebra $\mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p]$ and consistent with any operator algebra setting to which normal ordering extends. This is what we shall outline in this section.

First of all, the bicrossproduct form of the Hopf algebra implies that

$$\int :f(x,p):=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}p\,f(x,p)$$

is a left-integral on $\mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p]$. This is also evident from the explicit form of the right-invariant derivatives (4.1) and (4.2), from which we see that the integrals of $\partial_x :f:$ and $\partial_\eta :f:$ vanish for suitably decaying f. On the other hand, the right-integral desired in our preferred conventions for the Fourier theory can be similarly obtained using the left-invariant partial differentials (4.4) and (4.5). One finds

$$\int :f(x,p):=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}p\,e^{\frac{x}{\mathsf{G}}}f(x,p), \tag{4.14}$$

which is the right-integral that we shall use. (Although apparently more complicated, the resulting Fourier theory turns out to be more computable in these conventions.)

Next, we recall from [Maj88, Maj95b] that the Planck scale Hopf algebra is essentially self-dual. More precisely, if we let $\bar{x},\bar{p}$ be dual to the p,x generators in the sense $\langle\bar{x},x^np^m\rangle=\mathrm{i}\,\delta_{n,0}\delta_{m,1}$ and $\langle\bar{p},x^np^m\rangle=\mathrm{i}\,\delta_{n,1}\delta_{m,0}$, we have an algebraic model of the dual of $\mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p]$ as

$$\mathbb{C}[\bar{p}]{\triangleright\!\!\blacktriangleleft}_{\frac{1}{\hbar},\frac{\mathsf{G}}{\hbar}}\mathbb{C}[\bar{x}]\subseteq(\mathbb{C}[x]{\blacktriangleright\!\!\triangleleft}_{\hbar,\mathsf{G}}\mathbb{C}[p])^*,$$

where

$$[\bar{p},\bar{x}]=\frac{\mathrm{i}}{\hbar}(1-e^{-\bar{x}\frac{\hbar}{\mathsf{G}}}),\quad \Delta\bar{x}=\bar{x}\otimes1+1\otimes\bar{x},\quad \Delta\bar{p}=\bar{p}\otimes1+e^{-\bar{x}\frac{\hbar}{\mathsf{G}}}\otimes\bar{p}.$$

This has the same form as $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$, but with different parameter values and with the opposite product and opposite coproduct. On this Hopf algebra we define normal ordering as putting all the $\bar{x}$ to the right and the corresponding left-integral is

$$\int^* : f(\bar{p},\bar{x}) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\bar{p}\,\mathrm{d}\bar{x}\, e^{\bar{x}\frac{\hbar}{\mathsf{G}}} f(\bar{p},\bar{x}). \tag{4.15}$$

Also from the bicrossproduct form, the canonical element is [Maj95b]

$$\sum_{n,m} \frac{1}{n!m!\,\mathrm{i}^{n+m}} x^n p^m \otimes \bar{p}^n \bar{x}^m. \tag{4.16}$$

Finally, we will need explicitly the actions [Maj95b]

$$p \triangleright f(x) = \mathrm{i}\hbar(e^{-\frac{x}{\mathsf{G}}}-1)\frac{\partial}{\partial x} f, \quad f(\bar{x}) \triangleleft \bar{p} = \frac{\mathrm{i}}{\hbar}(e^{-\frac{\hbar}{\mathsf{G}}\bar{x}}-1)\frac{\partial}{\partial \bar{x}} f \tag{4.17}$$

in the bicrossproduct construction and its dual.

Proposition 4.4.1. *The quantum Fourier transform on the Planck scale Hopf algebra is*

$$\mathcal{T}(: f(x,p) :) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}p\, e^{-\mathrm{i}(\bar{p}+\frac{\mathrm{i}}{\mathsf{G}})\cdot x} e^{-\mathrm{i}\bar{x}\cdot(p+p\triangleright)} f(x,p)$$

and its dual is

$$\mathcal{T}^*(: f(\bar{p},\bar{x}) :) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\bar{p}\,\mathrm{d}\bar{x}\, e^{-\mathrm{i}\bar{p}\cdot x} e^{-\mathrm{i}\bar{x}\cdot p} f(\triangleleft\bar{p}+\bar{p},\bar{x}) e^{\frac{\hbar}{\mathsf{G}}\bar{x}},$$

where $p\triangleright$ acts only on the functions in x to the right in the integral ($\triangleleft\bar{p}$ acts only on functions in $\bar{x}$ to the left).

Proof. We use the reordering equality

$$: f(p) :: h(x) :=: e^{p\triangleright\cdot\frac{\partial}{\partial p}} h(x) f(p) :=: f(p+p\triangleright)h(x) :$$

in $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$ for functions f,h ($p\triangleright$ only acts on functions of x). This follows from the relation $[p,f(x)] = p\triangleright f(x)$ for functions $f(x)$, which is the semidirect product form of the algebra in the bicrossproduct. Hence,

$$\begin{aligned}
\mathcal{T}(: f(x,p) :) &= \sum_{n,m} \frac{1}{n!m!\,\mathrm{i}^{n+m}} \bar{p}^n \bar{x}^m \int x^n p^m : f(x,p) : \\
&= \sum_{n,m} \frac{1}{n!m!\,\mathrm{i}^{n+m}} \bar{p}^n \bar{x}^m \int : x^n (p+p\triangleright)^m f(x,p) : \\
&= \sum_{n,m} \frac{1}{n!m!\,\mathrm{i}^{n+m}} \bar{p}^n \bar{x}^m \int \mathrm{d}x\,\mathrm{d}p\, e^{\frac{x}{\mathsf{G}}} x^n (p+p\triangleright)^m f(x,p) \\
&= \int \mathrm{d}x\,\mathrm{d}p\, e^{-\mathrm{i}(\bar{p}+\frac{\mathrm{i}}{\mathsf{G}})\cdot x} e^{-\mathrm{i}\bar{x}\cdot(p+p\triangleright)} f(x,p),
\end{aligned}$$

where $p\triangleright$ only acts in the powers of x to its right. In $\mathbb{C}[\bar{p}]\blacktriangleright\!\!\triangleleft_{\frac{1}{\hbar},\frac{\mathsf{G}}{\hbar}}\mathbb{C}[\bar{x}]$ we have similarly

$$:f(\bar{x})::h(\bar{p}):=:f(\bar{x})e^{\triangleleft\bar{p}\cdot\frac{\partial}{\partial\bar{p}}}h(\bar{p}):=:f(\bar{x})h(\triangleleft\bar{p}+p):.$$

Hence,

$$\begin{aligned}\mathcal{T}^*(:f(\bar{p},\bar{x}):) &= \sum_{n,m}\frac{1}{n!m!\,\mathrm{i}^{n+m}}x^np^m\int^* \bar{p}^n\bar{x}^m :f(\bar{p},\bar{x}): \\ &= \sum_{n,m}\frac{1}{n!m!\,\mathrm{i}^{n+m}}x^np^m\int^* :\bar{p}^n\bar{x}^m f(\triangleleft\bar{p}+\bar{p},\bar{x}): \\ &= \sum_{n,m}\frac{1}{n!m!\,\mathrm{i}^{n+m}}x^np^m\int \mathrm{d}\bar{p}\,\mathrm{d}\bar{x}\,\bar{p}^n\bar{x}^m f(\triangleleft\bar{p}+\bar{p},\bar{x})e^{\frac{\hbar}{\mathsf{G}}\bar{x}} \\ &= \int \mathrm{d}\bar{p}\,\mathrm{d}\bar{x}\,e^{-\mathrm{i}\bar{p}\cdot x}e^{-\mathrm{i}\bar{x}\cdot p}f(\triangleleft\bar{p}+\bar{p},\bar{x})e^{\frac{\hbar}{\mathsf{G}}\bar{x}}.\end{aligned}$$

□

From the properties of the Fourier transform, we see in particular that it turns the (left-invariant) derivatives $\bar{\partial}_x$ and $\bar{\partial}_\eta$ in (4.4)–(4.5) into multiplication by the corresponding element of the dual. Also, these derivatives become right-handed derivatives $\partial_{\bar{x}}$ and $\partial_{\bar{\eta}}$ on $\mathbb{C}[\bar{p}]\blacktriangleright\!\!\triangleleft_{\frac{1}{\hbar},\frac{\mathsf{G}}{\hbar}}\mathbb{C}[\bar{x}]$ by identifying it with the opposite algebra and coalgebra to $\mathbb{C}[x]\triangleright\!\!\blacktriangleleft_{\frac{1}{\hbar},\frac{\mathsf{G}}{\hbar}}\mathbb{C}[p]$ and making the corresponding notational and parameter changes.

Proposition 4.4.2.

$$\mathcal{T}(\bar{\partial}_x a)=\mathcal{T}(a)\mathrm{i}\bar{p}e^{\frac{\hbar}{\mathsf{G}}\bar{x}},\qquad \mathcal{T}(\bar{\partial}_\eta a)=\mathcal{T}(a)\frac{\mathrm{i}\mathsf{G}}{\hbar}(e^{\frac{\hbar}{\mathsf{G}}\bar{x}}-1),$$
$$\mathcal{T}^*(\partial_{\bar{x}}\phi)=\mathrm{i}pe^{\frac{x}{\mathsf{G}}}\mathcal{T}^*(\phi),\qquad \mathcal{T}^*(\partial_{\bar{\eta}}\phi)=\mathrm{i}\mathsf{G}(e^{\frac{x}{\mathsf{G}}}-1)\mathcal{T}^*(\phi),$$
$$\mathcal{T}\mathcal{T}^*(\phi)=(2\pi)^2\,\mathrm{S}^{-1}\,\phi.$$

Proof. This is a short computation to identify the partial derivatives as $\bar{\partial}_x(a)=a_{(1)}\langle -\mathrm{i}\bar{p},a_{(2)}\rangle$ and $\bar{\partial}_\eta(a)=a_{(1)}\langle\frac{\mathrm{i}\mathsf{G}}{\hbar}(e^{-\frac{\hbar}{\mathsf{G}}\bar{x}}-1),a_{(2)}\rangle$, i.e. to identify the corresponding elements of L. Similarly, $\partial_{\bar{x}}$ corresponds to $-\mathrm{i}p$ and $\partial_{\bar{\eta}}$ corresponds to $\mathrm{i}\mathsf{G}(e^{-\frac{x}{\mathsf{G}}}-1)$ via the right coregular action. One can then verify the analogue of (4.13) directly in our setting for functions of suitably rapid decay. □

Note that when we take the limit $\hbar\to 0$ the Hopf algebra $\mathbb{C}[\bar{p}]\blacktriangleright\!\!\triangleleft_{\frac{1}{\hbar},\frac{\mathsf{G}}{\hbar}}\mathbb{C}[\bar{x}]$ becomes $\mathsf{U}(\mathfrak{b}_+)$ or κ-Minkowski space [MR94] with the relations

$$[\bar{p},\bar{x}]=\frac{\mathrm{i}}{\mathsf{G}}\bar{x}$$

(i.e. $\kappa=\frac{\mathsf{G}}{\mathrm{i}}$) regarded as a noncommutative space. Thus,

Corollary 4.4.3. *In the classical limit $\hbar \to 0$ the Fourier transform becomes*

$$\mathcal{T} : \mathcal{C}(B_+) \to \mathsf{U}(\mathfrak{b}_+), \qquad \mathcal{T}(: f(x,p) :) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}p\, e^{-\mathrm{i}(\bar{p}+\frac{1}{\kappa})\cdot x} e^{-\mathrm{i}\bar{x}\cdot p} f(x,p),$$

$$\mathcal{T}^* : \mathsf{U}(\mathfrak{b}_+) \to \mathcal{C}(B_+), \qquad \mathcal{T}^*(: f(\bar{p},\bar{x}) :) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\bar{p}\,\mathrm{d}\bar{x}\, e^{-\mathrm{i}\bar{p}\cdot x} e^{-\mathrm{i}\bar{x}\cdot p} f(\triangleleft\bar{p}+\bar{p},\bar{x})$$

with $f(\bar{x}) \triangleleft \bar{p} = -\frac{\bar{x}}{\kappa}\frac{\partial}{\partial \bar{x}} f$. *Moreover,*

$$\mathcal{T}(\bar{\partial}_x a) = \mathcal{T}(a)\mathrm{i}\bar{p}, \qquad \mathcal{T}(\bar{\partial}_\eta a) = \mathcal{T}(a)\mathrm{i}\bar{x},$$

$$\bar{\partial}_x(: f(x,p) :) =: \frac{\partial}{\partial x} f(x,p) + \frac{\mathrm{i}p}{\kappa}\frac{\partial}{\partial p} f(x,p) :, \qquad \bar{\partial}_\eta(: f(x,p) :) =: \frac{\partial}{\partial p} f(x,p) : .$$

The intertwiner properties of $\mathcal{T}^*$ in this limit are read off from Proposition 4.4.2 while required right-derivatives simplify to

$$\partial_{\bar{x}}(: f(\bar{p},\bar{x}) :) =: \frac{\partial}{\partial \bar{x}} f(\bar{p},\bar{x}) :, \qquad \partial_{\bar{\eta}}(: f(\bar{p},\bar{x}) :) =: -\kappa(f(\bar{p}-\frac{1}{\kappa},\bar{x}) - f(\bar{p},\bar{x})) : . \tag{4.18}$$

We also have a dual limit $\hbar, \mathsf{G} \to \infty$ with $\frac{\mathsf{G}}{\mathrm{i}\hbar} = \kappa$ constant, where $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p]$ becomes $\mathsf{U}(\mathfrak{b}_-)$ (with the opposite Lie algebra to $\mathfrak{b}_+$) and $\mathbb{C}[\bar{p}]\blacktriangleright\!\!\triangleleft_{\frac{1}{\hbar},\frac{\mathsf{G}}{\hbar}}\mathbb{C}[\bar{x}]$ becomes $\mathcal{C}(B_-)$. We regard the former as another version of κ-Minkowski space (with opposite commutation relations).

Corollary 4.4.4. *In the limit $\hbar, \mathsf{G} \to \infty$ with $\frac{\mathsf{G}}{\mathrm{i}\hbar} = \kappa$ the Fourier transform becomes*

$$\mathcal{T} : \mathsf{U}(\mathfrak{b}_-) \to \mathcal{C}(B_-), \qquad \mathcal{T}(: f(x,p) :) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}p\, e^{-\mathrm{i}\bar{p}\cdot x} e^{-\mathrm{i}\bar{x}\cdot(p+p\triangleright)} f(x,p),$$

$$\mathcal{T}^* : \mathcal{C}(B_-) \to \mathsf{U}(\mathfrak{b}_-), \qquad \mathcal{T}^*(: f(\bar{p},\bar{x}) :) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\bar{p}\,\mathrm{d}\bar{x}\, e^{-\mathrm{i}\bar{p}\cdot(x+\frac{1}{\kappa})} e^{-\mathrm{i}\bar{x}\cdot p} f(\bar{p},\bar{x})$$

with $p \triangleright f(x) = -\frac{1}{\kappa}\frac{\partial}{\partial x} f$. *Moreover,*

$$\mathcal{T}^*(\partial_{\bar{x}}\phi) = \mathrm{i}p\mathcal{T}^*(\phi), \qquad \mathcal{T}^*(\partial_{\bar{\eta}}\phi) = \mathrm{i}x\mathcal{T}^*(\phi),$$

$$\partial_{\bar{x}}(: f(\bar{x},\bar{p}) :) =: \frac{\partial}{\partial \bar{x}} f(\bar{x},\bar{p}) + \frac{\mathrm{i}p}{\kappa}\frac{\partial}{\partial \bar{p}} f(\bar{x},\bar{p}) :, \qquad \partial_{\bar{\eta}}(: f(\bar{x},\bar{p}) :) =: \frac{\partial}{\partial \bar{p}} f(\bar{x},\bar{p}) : .$$

In this limit the intertwiner properties of $\mathcal{T}$ do not simplify (we refer to Proposition 4.4.2), but the corresponding derivatives become

$$\bar{\partial}_x(: f(x,p) :) =: \frac{\partial}{\partial x} f(x,p) :, \qquad \bar{\partial}_\eta(: f(x,p) :) =: -\kappa(f(x,p-\frac{1}{\kappa}) - f(x,p)) : . \tag{4.19}$$

Therefore we obtain in fact two versions of Fourier theory on κ-Minkowski space as two limits of Fourier theory on the Planck scale Hopf algebra. This Hopf algebra, being of self-dual form, has the power to become both a classical but curved phase space (the classical limit) and its dual (the second limit), in addition to the flat space quantum mechanics limit.

There are many further possible developments of the geometry and Fourier theory on the noncommutative phase space in this toy model of Planck scale physics, among them quantum field theory (second quantisation) in a first-order formalism. There is also a physical interpretation of the self-duality as an observable-state duality [Maj88, Maj95a] which should be related to the noncommutative geometric picture above. Finally, we note that there are higher dimensional models of the bicrossproduct form [Maj90a, Maj91] which could be investigated from a similar point of view. These are some directions for further work.

Chapter 5

Spin and Statistics

While spin in quantum physics arises from the geometry of space-time, statistics is connected to the geometry of configuration space. Half-integer spin and Bose-Fermi statistics arise in 3 or higher dimensions, while in 2 dimensions more general fractional spin and anyonic statistics are possible (see [Wil90] and references therein). In fact, both, spin and statistics are related to symmetries. In the case of spin this is plainly understood in terms of the symmetries of space-time. In the case of statistics the link is more indirect. From the geometry of a configuration space of identical particles [LM77] one is led (in the general case) to the braid group, which acts on it by particle exchange [Wu84]. From the representation theory of the braid group one naturally arrives at the concept of braided categories to describe statistics. While foreign to ordinary quantum field theory, such a general formulation of statistics has already been incorporated into algebraic quantum field theory [FRS89, FG90]. Going further, a reconstruction theorem of quantum group theory tells us that (essentially) every braided category is the category of representations of a quantum group. Thus, for any braid statistics there is a quantum group symmetry of the theory that generates the statistics. The relevant quantum groups for anyonic statistics are known [Maj93a, Maj95b].

After reviewing these facts, we ask, in this chapter, the natural question of whether and how the (quantum) group symmetries behind spin and statistics are related. Such a connection should be expected in the Bose-Fermi case from the spin-statistics theorem [Fie39, Pau40]. In this case, both groups generating spin and statistics turn out to be (essentially) $\mathbb{Z}_2$. Remarkably, the statement of the spin-statistics theorem is found to be precisely equivalent to the requirement that the groups be identified. This leads to a quantum symmetry group that encodes (a) the space-time symmetries, (b) the statistics, and (c) the spin-statistics theorem. Technically, this quantum group is the ordinary space-time symmetry (e.g., Poincaré) group as a Hopf algebra, but equipped with a non-trivial coquasitriangular structure. We proceed to explore the possible relations of spin and statistics in the more general case of fractional spin and anyonic statistics. This amounts (under certain restrictions) to a classification of all possible spin-statistics theorems which could be implemented by a unified quantum group symmetry.

It is essential for our treatment to work with quantum groups of function algebra type and not of enveloping algebra type. The global structure of the (quantum) groups, not visible in the enveloping algebra setting, is crucial in the unified description of spin and statistics. In an enveloping algebra setting one would have to provide the global information "by hand", i.e., by adjoining elements.

Much of the chapter (Sections 5.1–5.4) is devoted to a coherent review of geometrical aspects of spin and statistics. Section 5.1 reviews the geometrical origin of spin while Section 5.2 that of particle exchange statistics. The description of statistics in terms of braided categories is introduced in Section 1.2.1 and the quantum symmetries underlying anyonic statistics are identified in Section 5.4. This finally enables us in Section 5.5 to investigate the unification of the symmetries of spin and statistics.

We work over the complex numbers throughout this chapter.

5.1 Spin

We start by recalling the geometric origin of spin. In classical mechanics we require that observable quantities form a representation of the symmetry group of space-time. In quantum mechanics it is only required that such a representation is projective, i.e., it is a representation "up to a phase" [Wey31]. However, projective representations of a Lie group are in correspondence to ordinary representations of its universal covering group [Bar54].

Suppose we have some connected orientable (pseudo-) Riemannian space-time manifold M. We denote its principal bundle of oriented orthonormal frames by (E, M, G), where E is the total space and G the structure group, i.e., the orientation preserving isotropy group. If M has signature (n, m) then $G = SO(n, m)$. Let $\tilde{G}$ be the universal covering group of G. Denote by $(\tilde{E}, M, \tilde{G})$ the induced lift of (E, M, G) (assuming no global obstructions).[1] Given a representation of $\tilde{G}$ with label j, a field with spin j is described by a section of the corresponding bundle associated with $(\tilde{E}, M, \tilde{G})$. If j labels a representation of G itself, we say that the spin is "integer", otherwise "fractional". Consider the exact sequence

$$\pi_1(G) \hookrightarrow \tilde{G} \twoheadrightarrow G, \tag{5.1}$$

[1] Strictly speaking, we should consider coverings of the global symmetry group. However, if M is a Riemannian homogeneous space, E can be identified with the global isometry group and $\tilde{E}$ with its universal cover (assuming M to be simply connected).

where $\pi_1(G)$ denotes the fundamental group of G. A representation of $\tilde{G}$ is a representation of G if and only if the induced action of $\pi_1(G)$ is trivial. Thus, loosely speaking, the "fractions" of spin are labeled by the irreducible representations of $\pi_1(G)$. In our present context (we assume at most one time direction) there arise only two different cases which we review in the following.

Let M be 3-dimensional Euclidean space. Then $G = SO(3)$ and the exact sequence (5.1) becomes

$$\mathbb{Z}_2 \hookrightarrow SU(2) \twoheadrightarrow SO(3). \tag{5.2}$$

With the usual conventions, irreducible representations of $SU(2)$ are labeled by half-integers and those with an integer label descend to representations of $SO(3)$. $\mathbb{Z}_2$ has just two inequivalent irreducible representations that distinguish between the two choices, integer or non-integer. More generally, $\pi_1(SO(1,n)) = \pi_1(SO(n)) = \mathbb{Z}_2$ for all $n \geq 3$. Thus, if the dimension of space is ≥ 3 we can only have integer and half-integer spins.

Now, let M be 2-dimensional Euclidean space. We obtain the exact sequence

$$\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow SO(2). \tag{5.3}$$

Since the groups are abelian, their unitary irreducible representations form themselves (abelian) groups. In fact these are $SO(2)$, $\mathbb{R}$, and $\mathbb{Z}$. (The sequence (5.3) is Pontrjagin self-dual.) Thus, the unitary irreducible representations of $\mathbb{R}$ are labeled by real numbers and descend to $SO(2)$ if the label is integer. The "fractional" part is labeled by $U(1) = SO(2)$. Since also $\pi_1(SO(1,2)) = \mathbb{Z}$, we conclude that in 2 spatial dimensions continuous real spin is allowed.

Finally, the case of one spatial dimension is degenerate since the orientation preserving spatial isotropy group is trivial. We do not discuss this case further.

5.2 Statistics

In the following we review the various possibilities for exchange statistics arising from the quantisation of a system of identical particles [LM77, Wu84]. We consider particles in d-dimensional Euclidean space. Naively, the configuration space for N particles is $\mathbb{R}^{dN}$. However, due to the particles being identical, configurations which differ only by a permutation of the particle positions are to be considered identical. Furthermore, we exclude the singularities arising from the subspace $D \subset \mathbb{R}^{dN}$ where two or more particle positions coincide. Thus, the true configuration space is $(\mathbb{R}^{dN} - D)/S_N$, where S_N denotes the symmetric group acting by exchanging the particle positions. For more than one particle and more than one dimension it is multiply connected.

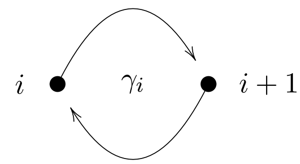


Figure 5.1: Clockwise exchange of particle i with particle $i+1$.

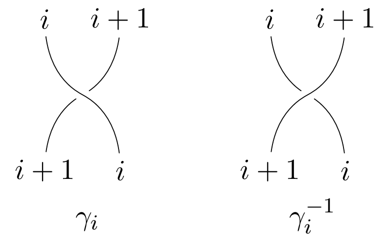


Figure 5.2: Braid generators γ_i and γ_i^{-1} in diagrammatic notation.

Assuming no internal structure for the particles, quantisation can now be performed by constructing a complex line bundle with flat connection over this configuration space. The wave-function is then a section of this bundle. If we parallel transport along a non-contractible loop γ in configuration space, the wave-function picks up a phase factor $\chi(\gamma)$ coming from the holonomy of the connection. This defines a one-dimensional unitary representation of the fundamental group of the configuration space. (Note that this excludes parastatistics here.)

For dimension $d = 2$, the fundamental group of the configuration space is the braid group on N elements, B_N. It is generated by elements $\gamma_1, \ldots, \gamma_{N-1}$ with relations $\gamma_i\gamma_j = \gamma_j\gamma_i$ for $i - j \neq \pm 1$ and

$$\gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1}. \tag{5.4}$$

To understand this more concretely, consider the inequivalent ways of exchanging two particles in the plane without coincidence. They correspond to non-contractible loops in the configuration space and thus to elements of its fundamental group. Indeed, γ_i corresponds to the exchange of particle i and particle $i+1$ in (say) clockwise direction, see Figure 5.1. We represent this by a diagram which can be thought of as depicting the particle trajectories (moving from top to bottom) as they exchange, see Figure 5.2. A counter-clockwise exchange corresponds to the inverse γ_i^{-1}. We can also wind the particles around each other more than once, corresponding to powers of γ_i or its inverse. Figure 5.3 shows the braid relation (5.4) in diagrammatic notation.

The one-dimensional unitary representations of the braid group are labeled by an angle θ

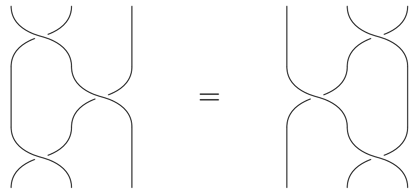


Figure 5.3: Braid relation in diagrammatic notation.

and take the form

$$\chi(\gamma_i) = e^{\mathrm{i}\theta} \quad \forall i. \tag{5.5}$$

This was termed θ-statistics in [Wu84]. More generally, a statistics that is induced by representations of the braid group is called a *braid statistics*.

In dimension $d \geq 3$, the fundamental group of the configuration space is just the symmetric group S_N. It can be obtained from the braid group by imposing the extra relations $\gamma_i = \gamma_i^{-1}$. The geometric meaning of this is that the clockwise and counter-clockwise exchange of particles are equivalent (homotopic), since we can use the extra dimensions to deform one path into the other. Diagrammatically, over- and under-crossings (Figure 5.2) become equivalent. The possible representations (5.5) reduce to just two: bosonic ($\theta = 0$) and fermionic ($\theta = \pi$) statistics.

5.3 Braided Categories and Statistics

In quantum field theory, multi-particle states are usually expressed in a Fock space formalism. That is, they are tensor products of one-particle states. In order to describe a general braid statistics in this context, we define invertible linear maps

$$\psi : V \otimes W \to W \otimes V$$

that exchange particles in state spaces V and W, and represent the elements γ_i of the braid group. ψ is called a *braiding*. We depict it by the same crossing diagram that we used for γ_i (Figure 5.2). The diagram is now interpreted as a map, to be read from top to bottom, where the strands carry the vector spaces V and W respectively (Figure 1.1). In this formulation, we can easily express the statistics between different particles as well, by defining ψ for V and W being different spaces. Also, we are not restricted to one-dimensional representations of the braid group (or symmetric group) as was the discussion in Section 5.2. Furthermore, we can extend ψ to tensor products of multi-particle states by composing in the obvious way (Figure 1.2). In fact, we can forget about the origin from the braid group or symmetric group altogether if we implement the constraints corresponding to their relations. For the braid group this is the braid relation, expressed by the diagram in Figure 5.3, which is now an identity between maps on three-fold tensor products. For the symmetric group we have the additional constraint that ψ and its inverse must be identical, i.e., diagrammatically over- and under-crossings are identified.

If ψ takes the special form

$$\psi(v \otimes w) = q w \otimes v. \tag{5.6}$$

with $q \in \mathbb{C}^*$ it is called an *anyonic statistics*, since particles obeying such statistics are called anyons [Wil82]. If V and W are state spaces of identical particles without internal structure we recover θ-statistics (5.5) with $q = e^{i\theta}$. (Note that we allow θ to be complex here.) The general expression for Bose-Fermi statistics in this formulation is

$$\psi(v \otimes w) = (-1)^{|v|\cdot|w|} w \otimes v, \tag{5.7}$$

where $|v| = 0$ for bosons and $|v| = 1$ for fermions.

In fact, what we have described here is nothing but a *braided category* as introduced in Section 1.2.1. Note that the braid relation (Figure 5.3) follows from Definition 1.2.3.

A description of statistics by braided categories was first employed in the context of algebraic quantum field theory [FRS89, FG90]. However, it can also be integrated into a (generalised) path integral formulation of quantum field theory. This will be the object of Chapter 6.

5.4 Quantum Groups and Statistics

How do quantum groups come into the game? We recall from Section 1.2 that certain module categories of quantum groups are naturally equipped with a braiding, i.e., are naturally braided categories. In particular, this is true for the category of comodules of a coquasitriangular Hopf algebra, see (1.8). This generalises the situation for representations of ordinary groups which come canonically equipped with the trivial braiding $\tau : V \otimes W \to W \otimes V$ defined by the exchange $v \otimes w \mapsto w \otimes v$, and induced from the trivial coquasitriangular structure $\mathcal{R} = \epsilon \otimes \epsilon$.

Remarkably, the converse is also true: Given a braided category, we can (under certain technical conditions) reconstruct a quantum group (coquasitriangular Hopf algebra) so that the given category arises as its category of representations (comodules). This is called Tannaka-Krein reconstruction, see [Maj95b].

Having seen that braided categories can be used to describe statistics, we can say that quantum group theory naturally integrates the notions of “spin” (representation theory) and

quantum group	$\mathbb{R}$	$U(1)$	$\mathbb{Z}_n$	$\mathbb{Z}_2$
representation labels	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{Z}_n$	$\mathbb{Z}_2$
statistics parameter	$q \in \mathbb{C}^*$	$q \in \mathbb{C}^*$	$q^n = 1$	$q = -1$

Table 5.1: Anyonic statistics generating quantum groups.

statistics. More precisely, the reconstruction theorem tells us that for a given braid statistics (given by a braided category) there is an underlying symmetry quantum group, so that the statistics of particles is determined by their representation labels. In the following, we discuss this for anyonic statistics including the special case of Bose-Fermi statistics. The relevant quantum groups were identified by Majid [Maj93a, Maj95b] (in a dual formulation of enveloping algebras).

The quantum group generating general anyonic statistics turns out to be the ordinary group $U(1)$, but with a non-standard coquasitriangular structure. As a quantum group it is the function algebra $\mathcal{C}(U(1))$. A natural basis in terms of the coproduct are the Fourier modes $f_k : \phi \mapsto e^{2\pi \mathrm{i} k\phi}$, labeled by $k \in \mathbb{Z}$. The relations are $f_k f_l = f_{k+l}$, the coproduct is $\Delta\, f_k = f_k \otimes f_k$, the counit is $\epsilon(f_k) = 1$, and the antipode is $\mathrm{S}\, f_k = f_{-k}$. The coquasitriangular structure $\mathcal{R} : \mathcal{C}(U(1)) \otimes \mathcal{C}(U(1)) \to \mathbb{C}$ that generates the braiding is given by

$$\mathcal{R}(f_k \otimes f_l) = q^{kl}. \tag{5.8}$$

The unitary irreducible representations of $U(1)$ are labeled by $k \in \mathbb{Z}$. The braid statistics takes the form

$$\psi(v_k \otimes v_l) = q^{kl} v_l \otimes v_k. \tag{5.9}$$

This reduces to expression (5.6) for particles in the representation $k = l = 1$.

As it will be of relevance later, we remark that the same anyonic statistics can also be generated by the group $\mathbb{R}$. As a quantum group we consider the function algebra $\mathcal{C}(\mathbb{R})$ generated by the periodic functions. The only difference to the $U(1)$ case discussed above is that the basis $\{f_k\}$ is now labeled by real numbers $k \in \mathbb{R}$ and not just integers. Otherwise the algebraic structure is the same and (5.8) and (5.9) still hold in the same form. Representations are also labeled by $k \in \mathbb{R}$ now.

For $q = e^{\mathrm{i}\theta}$ an n-th root of unity, we call the statistics *rational* since $\theta/2\pi$ is rational. In this case we can restrict $U(1)$ to the subgroup $\mathbb{Z}_n$. This corresponds in the quantum group setting to the extra relations $f_k = f_{k+n}$, so that we obtain a finite dimensional quantum group. Irreducible representations are now labeled by $k \in \mathbb{Z}_n$. However, it will be convenient for the following discussion to introduce an alternative fractional labeling by $k \in \frac{1}{n}\mathbb{Z}_n$. Expression

(5.9) is thus modified to

$$\psi(v_k \otimes v_l) = q^{n^2kl} v_l \otimes v_k. \tag{5.10}$$

Expression (5.6) is recovered for $k = l = \frac{1}{n}$. The special case of $\mathbb{Z}_2$ ($\theta = \pi$) is Bose-Fermi statistics (5.7) with $|v_0| = 0$ and $|v_{\frac{1}{2}}| = 1$.

The quantum groups generating anyonic statistics are summarised in Table 5.1 (with the special case of $\mathbb{Z}_2$ listed separately).

5.5 Unifying Spin and Statistics

Having found an underlying "spin" connected with statistics, the natural question arises what possible relation it can have to the geometric spin discussed in Section 5.1. This is in essence the question of what spin-statistics theorems are quantum geometrically realisable. We give a complete answer to this question in the following (under the restricting assumption that integer spin particles behave bosonic in dimension ≥ 3).

We consider first the Bose-Fermi case. Our labelling of the $\mathbb{Z}_2$ representations above by $0, \frac{1}{2}$ is already suggestive of an interpretation as the fractional part of geometric spin. The latter is also described by a $\mathbb{Z}_2$, arising from the universal covering of the isotropy group in dimensions ≥ 3. And in fact, identifying the two groups is precisely equivalent to requiring the usual spin-statistics theorem to hold.

To make this more precise we consider the generic case of 3-dimensional Euclidean space. Translating the exact sequence (5.2) into quantum group language, it takes the arrows reversed form[2]

$$\mathcal{C}(SO(3)) \hookrightarrow \mathcal{C}(SU(2)) \twoheadrightarrow \mathcal{C}(\mathbb{Z}_2). \tag{5.11}$$

Instead of inheriting the trivial coquasitriangular structure canonically associated to ordinary groups, we equip $\mathcal{C}(\mathbb{Z}_2)$ with the non-trivial coquasitriangular structure generating the Bose-Fermi statistics. This induces a non-trivial coquasitriangular structure on $\mathcal{C}(SU(2))$ which precisely exhibits the usual spin-statistics relation. Explicitly, for a group-like basis $\{1, g\}$ of $\mathcal{C}(\mathbb{Z}_2)$ and a Peter-Weyl basis $\{t^l_{ij}\}$, $l \in \frac{1}{2}\mathbb{N}_0$ of $\mathcal{C}(SU(2))$, the right hand map of (5.11) is $t^l_{ij} \mapsto \delta_{ij} g^{2l}$. The coquasitriangular structure $\mathcal{R}(g \otimes g) = -1$ on $\mathcal{C}(\mathbb{Z}_2)$ pulls back to

$$\mathcal{R}(t^l_{ij} \otimes t^m_{kl}) = (-1)^{4lm} \delta_{ij} \delta_{kl}$$

on $\mathcal{C}(SU(2))$. This induces the braiding

$$\psi(v_k \otimes v_l) = (-1)^{4kl} v_l \otimes v_k,$$

[2]Note that this is *not* an exact sequence of vector spaces.

relating spin and statistics for bosons and fermions in the familiar way. The analogous construction can be made in any space-time with spatial dimension ≥ 3, since the only relevant input is that the fundamental group of the isotropy group is $\mathbb{Z}_2$. This ensures that the function algebra of the covering group is $\mathbb{Z}_2$-graded into functions that are symmetric or antisymmetric with respect to exchange of the sheets. This decomposition is also a decomposition into subcoalgebras. Thus, the covering group admits the coquasitriangular structure

$$\mathcal{R}(f \otimes g) = (-1)^{|f|\cdot|g|}\, \epsilon(f)\, \epsilon(g).$$

Note that we can further pull the coquasitriangular structure back to the relevant global space-time symmetry group by the same argument. For a treatment of Bose-Fermi statistics in 2 dimensions see the discussion below.

We now proceed to the more complicated case of anyonic statistics. Although we can embed $U(1)$ into $SU(2)$, the coquasitriangular structure (5.8) does not pull back from $\mathcal{C}(U(1))$ to $\mathcal{C}(SU(2))$. The same is true for the other spin-groups. (This is easily seen by embedding through an intermediate $SU(2)$.) Consequently, we cannot relate the statistical "spin" of anyonic statistics to geometric spin in dimension 3 or higher. Even in the rational case this is only possible for $q = \pm 1$, which is the Bose-Fermi case described above. Thus, we must restrict to 2 dimensions, where the covering of the spatial isotropy group is described by the exact sequence (5.3). In contrast to the Bose-Fermi case the statistical group $U(1)$ is different from the group $\mathbb{Z}$ describing the covering. However, we can use the group $\mathbb{R}$ to generate the statistics instead (see Table 5.1) and identify it directly with the universal cover $\mathbb{R}$ of the isotropy group. We obtain a spin-statistics relation between anyonic statistics and continuous geometric spin. However, for $q \neq 1$ we never have the property that representations which descend to the isotropy group have bosonic statistics, i.e., trivial braiding with all other representations.

We can implement this property, however, if we only consider a finite covering of the isotropy group. This leads to the exact sequence

$$\mathbb{Z}_n \hookrightarrow SO(2) \twoheadrightarrow SO(2)$$

instead of (5.3). The spins are restricted from continuous values to n-th fractions. We can now establish a spin-statistics relation by identifying the $\mathbb{Z}_n$ of the covering with the $\mathbb{Z}_n$ of rational anyonic statistics. The braiding is the one described by (5.10). Pullback from $\mathcal{C}(\mathbb{Z}_n)$ to $\mathcal{C}(SO(2))$ corresponds to extending the representation labels from $\frac{1}{n}\mathbb{Z}_n$ to $\frac{1}{n}\mathbb{Z}$. Representations of the covering $SO(2)$ that descend to representations of the covered $SO(2)$ are precisely the ones that are bosonic, i.e., have trivial braiding with all other representations. $n = 2$ is the Bose-Fermi case.

spatial dimension	2	2	≥ 3
statistics quantum group	$\mathbb{R}$	$\mathbb{Z}_n$	$\mathbb{Z}_2$
statistics parameter	$q \in \mathbb{C}^*$	$q^n = 1$	$q = -1$
integer spin bosonic	$-$	$\surd$	$\surd$

Table 5.2: Possible spin-statistics relations.

Conversely, we may ask the question what possible statistics can be attached to the geometric spin, i.e., which coquasitriangular structure is admitted by the relevant (quantum) group. Its turns out that all the relevant groups are abelian. A coquasitriangular structure on the function Hopf algebra of an abelian group is a bicharacter on its Pontrjagin dual, i.e., its group of unitary irreducible representations. In dimension 3 or higher, if we require bosonic statistics for representations descending to the isotropy group, any braiding must be induced by $\mathcal{C}(\mathbb{Z}_2)$ in (5.11). The dual of $\mathbb{Z}_2$ is $\mathbb{Z}_2$ and there are only two bicharacters on it: The trivial one (purely bosonic statistics) and the Bose-Fermi one discussed. In 2 dimensions, the covering group $\mathbb{R}$ of the isotropy group is self-dual and any bicharacter corresponds to (5.8) for some $q \in \mathbb{C}^*$. We also see the reason now why we were not able to induce the braiding from $\mathbb{Z}$: Its dual is $U(1)$ which has only the trivial bicharacter. In 2 dimensions with finite covering the relevant group is $\mathbb{Z}_n$. It is self-dual and the bicharacters correspond to the different n-th roots of unity leading to rational anyonic statistics. Thus, our above discussion has already exhausted the possibilities of attaching statistics to spin. Table 5.2 gives a summary.

Of course, even in the absence of a spin-statistics relation we can encode the space-time symmetries as well as the statistics in terms of just one symmetry (quantum) group. This is then simply the product of the two relevant (quantum) groups.

We make a remark on a geometric explanation of a spin-statistics connection in terms of a rotation. The usual argument (in simplified form) is that rotating two particles about their common center should yield the same phase as the statistical exchange. For general anyonic spin one would thus expect a factor of $q^{\frac{k+l}{2}}$ for particles of spin k and l. This does not in general define a consistent braid statistics, however. The only case in which it does is Bose-Fermi statistics of identical particles ($k = l$). In this case we have the "accident" that $(-1)^{4l^2} = (-1)^{2l}$ for $l \in \frac{1}{2}\mathbb{N}_0$. One could perhaps try to rescue the argument by saying that it is only valid for "elementary" particles and that the different results for higher spin come from "compositeness".

Chapter 6

Braided Quantum Field Theory – Foundations

If we want to realise the promise that quantum geometry appears to hold for physics, we have to generalise our physical theories. More precisely, this means extending the ordinary differential geometric notions on which they are built to quantum geometric ones. For fundamental physical theories this certainly includes quantum field theory with its notion of group symmetry. In other words, we need to extend the framework of quantum field theory to one that admits not only ordinary groups but quantum groups as symmetries.

As discussed in the previous chapter, situations arise where particles obey a more general kind of statistics than Bose or Fermi. To formulate quantum field theories involving such particles, it is desirable to extend the usual commutator/anticommutator formalism to a more general one capable of describing general braid statistics.

In fact, both objectives can be addressed at the same time. This is the subject of the present chapter. We introduce a generalisation of quantum field theory that admits quantum group symmetries as well as braid statistics. This is achieved by the transition from ordinary to braided categories, hence the name *braided quantum field theory*.

We follow the path integral approach, going from (braided) Gaussian path integrals via perturbation theory to Feynman diagrams. In the braided setting this procedure naturally leads us to generalised Feynman diagrams that are braid diagrams, i.e., they have nontrivial over- and under-crossings. In view of Chapter 5 the crossings have a natural interpretation in terms of particle exchanges. We consider the special cases of bosonic and fermionic statistics where we recover the results of quantum field theory in its standard formulation.

Section 6.1 introduces the braided path integral. We start by defining Gaussian integrals on braided spaces based on [Maj93b, KM94]. We obtain a braided generalisation of Wick's theorem that tells us how free $2n$-point functions decompose into propagators. It is shown that for the cases of bosonic and fermionic statistics the braided path integral reproduces the ordinary path integrals with commuting and anticommuting variables respectively.

We consider interactions in Section 6.2 and develop perturbation theory along the lines of ordinary quantum field theory. This leads us to *braided Feynman diagrams* by building on the diagrammatic language of braided categories. The ordinary Feynman rules for bosons and fermions are recovered for bosonic and fermionic statistics respectively.

6.1 The Braided Path Integral

6.1.1 Gaussian Integration

The calculus of differentiation and Gaussian integration on braided spaces that we build on was developed by Majid [Maj93b] and Kempf and Majid [KM94] in an R-matrix setting. However, we need a more abstract and basis free formulation of their formalism so that we redevelop the notions here. Furthermore, our Theorem 6.1.1 goes beyond their result [KM94, Theorem 5.1].

We require the concept of a *rigid* braided category (see the references mentioned at the end of Section 1.2.1). This means, that we have for every object X in the category a dual object X^* and morphisms $\mathrm{ev} : X \otimes X^* \to \Bbbk$ (evaluation) and $\mathrm{coev} : \Bbbk \to X^* \otimes X$ (coevaluation) that compose to the identity on X and X^* in the obvious ways.

Now, suppose we have some rigid braided category $\mathcal{B}$ and a vector space $X \in \mathcal{B}$. Essentially, we want to define the (normalised) integral of functions α in the "coordinate ring" on X multiplied by a Gaussian weight function w, i.e., we want to define

$$\mathcal{Z}(\alpha) := \frac{\int \alpha w}{\int w}. \tag{6.1}$$

First, we need to specify this "coordinate ring". We identify the dual space $X^* \in \mathcal{B}$ as the space of "coordinate functions" on X. This corresponds to the situation in $\mathbb{R}^n$ where a coordinate function is just a linear map from $\mathbb{R}^n$ into the real numbers. The polynomial functions on X are naturally elements of the free unital tensor algebra over X^*,

$$\widehat{X^*} := \bigoplus_{n=0}^{\infty} X^{*n}, \quad \text{with} \quad X^{*0} := \mathbf{1} \quad \text{and} \quad X^{*n} := \underbrace{X^* \otimes \cdots \otimes X^*}_{n \text{ times}},$$

where $\mathbf{1}$ is the one-dimensional space generated by the identity. $\mathbf{1}$ plays the role of the constant functions and the tensor product corresponds to the product of functions. $\widehat{X^*}$ naturally has the structure of a braided Hopf algebra (a Hopf algebra in a braided category, see [Maj95b]) via

$$\Delta\, a = a \otimes 1 + 1 \otimes a, \quad \epsilon(a) = 0, \quad \mathrm{S}\, a = -a$$

for $a \in X^*$ and $\Delta, \epsilon, \mathrm{S}$ extend to $\widehat{X^*}$ as braided (anti-)algebra maps. This encodes translations on X.

To make the notion of "coordinate ring" more precise, one could perhaps consider a kind of symmetrised quotient of $\widehat{X^*}$ in analogy with the observation that coordinates commute in ordinary geometry. There seems to be no obvious choice for such a quotient in the general braided case. Remarkably, however, such a choice is not necessary. In fact, the following treatment is entirely independent of any relations, as long as they preserve the (graded) braided Hopf algebra structure.

The next step is the introduction of differentials [Maj93b]. The space of coordinate differentials should be dual to the space X^* of coordinate functions. We just take X itself and define differentiation on X^* by the pairing $\mathrm{ev} : X \otimes X^* \to \Bbbk$ in $\mathcal{B}$. To extend differentiation to the whole "coordinate ring" $\widehat{X^*}$, we note that the coproduct encodes coordinate translation. This leads to the natural definition that

$$\mathrm{diff} := (\widehat{\mathrm{ev}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) : X \otimes \widehat{X^*} \to \widehat{X^*}$$

is differentiation on $\widehat{X^*}$. Here, $\widehat{\mathrm{ev}}$ is the trivial extension of ev to $X \otimes \widehat{X^*} \to \Bbbk$, i.e., $\widehat{\mathrm{ev}}\,|_{X \otimes X^{*n}} = 0$ for $n \neq 1$. We also use the more intuitive notation $\partial(a) := \mathrm{diff}(\partial \otimes a)$ for $\partial \in X$ and $a \in \widehat{X^*}$. Let $\partial \in X$ and $\alpha, \beta \in \widehat{X^*}$. The definition of $\widehat{\mathrm{ev}}$ gives at once

$$\widehat{\mathrm{ev}}(\partial \otimes \alpha\beta) = \widehat{\mathrm{ev}}(\partial \otimes \alpha)\, \epsilon(\beta) + \widehat{\mathrm{ev}}(\partial \otimes \beta)\, \epsilon(\alpha).$$

Using that the coproduct is a braided algebra map, we obtain the *braided Leibniz rule*

$$\partial(\alpha\beta) = \partial(\alpha)\beta + \psi^{-1}(\partial \otimes \alpha)(\beta). \tag{6.2}$$

Iteration yields

$$\partial(\alpha) = (\mathrm{ev} \otimes \mathrm{id}^{n-2})(\partial \otimes [n]_\psi\, \alpha),$$

where n is the degree of α and $[n]_\psi : X^{*n} \to X^{*n}$ with

$$[n]_\psi := \mathrm{id}^n + \psi \otimes \mathrm{id}^{n-2} + \cdots + \psi_{n-2,1} \otimes \mathrm{id} + \psi_{n-1,1}$$

is a *braided integer*. We adopt the convention of writing $\psi_{n,m}$ for the braiding between X^{*n} and X^{*m} (respectively $\psi^{-1}_{n,m}$ for the inverse braiding).

As in [KM94], we view the Gaussian weight w formally as an element of $\widehat{X^*}$ and define its differentiation via an isomorphism

$$\gamma : X \to X^* \qquad \text{so that} \qquad \partial(w) = -\gamma(\partial)w \qquad \text{for} \quad \partial \in X. \tag{6.3}$$

This expresses the familiar notion that differentiating a Gaussian weight yields a coordinate function times the Gaussian weight. γ should accordingly be thought of as defining a braided analogue of the quadratic form in the exponential of the weight.

Also familiar from ordinary Gaussian integration is the fact that integrals of total differentials vanish. That is, we require

$$\int \partial(\alpha w) = 0 \qquad \text{for} \qquad \partial \in X, \alpha \in \widehat{X^*}. \tag{6.4}$$

It turns out that the three rules (6.2), (6.3), and (6.4) completely determine the integral (6.1).

Remarkably, the statement that the Gaussian integral of a polynomial function can be expressed solely in terms of Gaussian integrals of quadratic functions still holds true in the braided case. This generalises what is known in quantum field theory as Wick's theorem. To state it, we need another set of braided integers $[n]'_\psi : X^{*n} \to X^{*n}$ with

$$[n]'_\psi := \mathrm{id}^n + \mathrm{id}^{n-2} \otimes \psi^{-1} + \cdots + \psi^{-1}_{1,n-1}, \tag{6.5}$$

which are related to the original ones by $[n]'_\psi = \psi^{-1}_{1,n-1} \circ [n]_\psi$. We also require the corresponding *braided double factorials* $[2n-1]'_\psi!! : X^{*2n} \to X^{*2n}$ with

$$[2n-1]'_\psi!! := ([1]'_\psi \otimes \mathrm{id}^{2n-1}) \circ ([3]'_\psi \otimes \mathrm{id}^{2n-3}) \circ \cdots \circ ([2n-1]'_\psi \otimes \mathrm{id}). \tag{6.6}$$

Theorem 6.1.1 (Braided Wick Theorem).

$$\mathcal{Z}\,|_{X^{*2}} = \mathrm{ev} \circ \psi \circ (\mathrm{id} \otimes \gamma^{-1}), \tag{6.7}$$

$$\mathcal{Z}\,|_{X^{*2n}} = (\mathcal{Z}\,|_{X^{*2}})^n \circ [2n-1]'_\psi!!, \quad \mathcal{Z}\,|_{X^{*2n-1}} = 0, \quad \forall n \in \mathbb{N}. \tag{6.8}$$

Proof. For $\alpha \in \widehat{X^*}$ and $a \in X^*$ we have

$$\alpha a w = -\alpha\, \mathrm{diff}(\gamma^{-1}(a) \otimes w) = -\,\mathrm{diff}(\psi(\alpha \otimes \gamma^{-1}(a))w) + (\mathrm{diff} \circ \psi(\alpha \otimes \gamma^{-1}(a)))w,$$

using the differential property (6.3) of w and the braided Leibniz rule (6.2). Applying $\mathcal{Z}$, we can ignore the total differential and obtain

$$\mathcal{Z}(\alpha a) = \mathcal{Z}(\mathrm{diff} \circ \psi(\alpha \otimes \gamma^{-1}(a))). \tag{6.9}$$

This gives us immediately

$$\mathcal{Z}(a) = 0 \qquad \text{and} \qquad \mathcal{Z}(ab) = \mathrm{ev} \circ \psi(a \otimes \gamma^{-1}(b))$$

for $b \in X^*$. We rewrite (6.9) to find

$$\begin{aligned}
\mathcal{Z}\,|_{X^{*n}} &= \mathcal{Z}\,|_{X^{*n-2}} \circ \mathrm{diff} \circ (\gamma^{-1} \otimes \mathrm{id}^{n-1}) \circ \psi_{n-1,1} \\
&= \mathcal{Z}\,|_{X^{*n-2}} \circ (\mathrm{ev} \otimes \mathrm{id}^{n-2}) \circ (\gamma^{-1} \otimes [n-1]_\psi) \circ \psi_{n-1,1} \\
&= (\mathrm{ev} \otimes \mathcal{Z}\,|_{X^{*n-2}}) \circ (\gamma^{-1} \otimes [n-1]_\psi) \circ \psi_{n-1,1} \\
&= (\mathrm{ev} \otimes \mathcal{Z}\,|_{X^{*n-2}}) \circ \psi_{n-1,1} \circ ([n-1]_\psi \otimes \gamma^{-1}) \\
&= (\mathcal{Z}\,|_{X^{*2}} \otimes \mathcal{Z}\,|_{X^{*n-2}}) \circ (\mathrm{id} \otimes \psi_{n-2,1}) \circ ([n-1]_\psi \otimes \mathrm{id}) \\
&= (\mathcal{Z}\,|_{X^{*n-2}} \otimes \mathcal{Z}\,|_{X^{*2}}) \circ \psi^{-1}_{2,n-2} \circ (\mathrm{id} \otimes \psi_{n-2,1}) \circ ([n-1]_\psi \otimes \mathrm{id}) \\
&= (\mathcal{Z}\,|_{X^{*n-2}} \otimes \mathcal{Z}\,|_{X^{*2}}) \circ (\psi^{-1}_{1,n-2} \otimes \mathrm{id}) \circ ([n-1]_\psi \otimes \mathrm{id}) \\
&= (\mathcal{Z}\,|_{X^{*n-2}} \otimes \mathcal{Z}\,|_{X^{*2}}) \circ ([n-1]'_\psi \otimes \mathrm{id}),
\end{aligned}$$

which gives us a recursive definition of $\mathcal{Z}$ leading to the formulas stated. □

6.1.2 Path Integration

Let us rephrase and explain the result in a more familiar notation. To simplify, we consider Gaussian integration with a finite basis $\{\phi_1, \phi_2, \dots\}$ of the space X^* of coordinate functions. The (free) n-point function can be written as[1]

$$\langle \phi_{k_1}\phi_{k_2}\cdots\phi_{k_n}\rangle := \mathcal{Z}(\phi_{k_1}\phi_{k_2}\cdots\phi_{k_n}) = \frac{\int \mathcal{D}\phi\, \phi_{k_1}\phi_{k_2}\cdots\phi_{k_n}\exp(-S(\phi))}{\int \mathcal{D}\phi\, \exp(-S(\phi))}. \tag{6.10}$$

This is to be compared with (6.1), setting $w(\phi) = \exp(-S(\phi))$ and $\alpha(\phi) = \phi_{k_1}\phi_{k_2}\cdots\phi_{k_n}$. We also introduce differentials $\{\partial^1, \partial^2, \dots\}$ forming a basis of X dual to the coordinate functions. The integration rule (6.4) now takes the form

$$\int \mathcal{D}\phi\, \partial^i(\phi_{k_1}\cdots\phi_{k_n}\exp(-S)) = 0. \tag{6.11}$$

Note that $\exp(-S(\phi))$ can not necessarily be given the meaning of an ordinary exponential of a quadratic form in the general braided case. This is the reason for the more abstract notation w chosen above. If it can, we write S as

$$S(\phi) = \frac{1}{2}\sum_{i,j}\phi_i\gamma^{ij}\phi_j. \tag{6.12}$$

We then find that γ^{ij} are precisely the matrix elements of the map γ introduced in (6.3). Explicitly,

$$\partial^i(\exp(-S)) = -\partial^i(S)\exp(-S) = -\sum_j \gamma^{ij}\phi_j\exp(-S). \tag{6.13}$$

[1] The Euclidean signature of the action is chosen for definiteness and does not imply a restriction to Euclidean field theory.

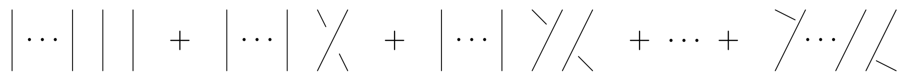


Figure 6.1: Braided integer.

We write the braiding on the space X^* of coordinate functions explicitly as

$$\psi(\phi_i \otimes \phi_j) = \sum_{k,l} \mathsf{R}_{ij}^{kl} \phi_l \otimes \phi_k \tag{6.14}$$

with a matrix R. (Note that the braid relation of Figure 5.3 is now equivalent to the Yang-Baxter equation for R.) The braided Leibniz rule (6.2) takes the explicit form

$$\partial^i \phi_j = \delta^i_j + \sum_{k,l} \mathsf{R}_{jk}^{li} \phi_l \partial^k. \tag{6.15}$$

The three rules completely determining the n-point functions (6.10) are now (6.11), (6.13), and (6.15).

The content of Theorem 6.1.1 can now be written as follows: Equation (6.7) expressing the 2-point function (propagator) in terms of γ takes the form

$$\langle \phi_i \phi_j \rangle = \sum_{k,l} \mathsf{R}_{ij}^{kl} (\gamma^{-1})_{lk}. \tag{6.16}$$

The decomposition of a $2n$-point function into propagators (6.8) becomes

$$\langle \phi_{k_1} \phi_{k_2} \cdots \phi_{k_{2n}} \rangle = \langle \cdot \cdot \rangle^n \circ [2n+1]'_\psi!!(\phi_{k_1} \otimes \phi_{k_2} \otimes \cdots \otimes \phi_{k_{2n}}), \tag{6.17}$$

while an n-point function for odd n vanishes. The right hand side of (6.17) means: Take the tensor product $\phi_{k_1} \otimes \cdots \otimes \phi_{k_{2n}}$, apply the map $[2n+1]'_\psi!!$, then insert the result into n propagators.

The diagrammatic language introduced in Section 1.2.1 makes this more transparent. We represent the braided integer $[n]'_\psi$ by the linear combination of diagrams depicted in Figure 6.1. Each summand of (6.5) is represented by one diagram, containing n strands. As a map, it is to be read from top to bottom. Each strand corresponds to one tensor factor of X^* (i.e., one variable). Crossings correspond to the braiding ψ or its inverse ψ^{-1} (Figure 1.1), while lines that do not cross simply represent the identity map on that tensor factor. The composition of maps as in (6.6) is expressed in terms of diagrams by gluing the strands together, one diagram on top of the other. See for example Figure 6.2, representing the braided double factorial $[5]'_\psi!!$. Sums of diagrams are composed by summing over all compositions of individual diagrams.

With the statistics interpretation of the braiding introduced in Chapter 5 in mind, one can think of the diagrams as representing the paths of an ensemble of particles. Each strand

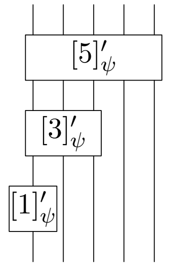


Figure 6.2: The braided double factorial $[5]'_\psi!!$.

is then the track of a particle moving from top to bottom with crossings corresponding to exchanges. Alternatively, if one represents the application of the propagators by connecting the corresponding strands at the bottom one recovers precisely the usual pictures drawn to illustrate the ordinary Wick theorem. However, the pictures obtained here carry additional information encoded in the type of crossing.

We stress again that in contrast to ordinary (path) integrals there are no algebra relations between the ϕ's. However, if compatible relations (see above) are introduced, they commute with the braided Wick theorem in the following sense: Imposing the relations first and then evaluating (6.17) is the same as evaluating (6.17) first and then imposing the relations.

6.1.3 Special Cases: Bosons and Fermions

In this section we show that the ordinary bosonic and fermionic path integrals are recovered given the bosonic respectively fermionic braiding (see Chapter 5).

We restrict first to the more general case where the braiding just permutes variables with an extra factor. That is, we assume $\mathsf{R}^{kl}_{ij} = \rho_{ij}\delta^k_i\delta^l_j$ in (6.14). Explicitly,

$$\psi(\phi_i \otimes \phi_j) = \rho_{ij}\phi_j \otimes \phi_i. \tag{6.18}$$

This is sufficient for considering bosonic, fermionic, and even anyonic statistics. (See Section 7.1 for an example of the latter case.) The braided integers become sums of permutations equipped with extra factors. Consequently, we can express the braided Wick theorem (6.17) in a more familiar way:

$$\langle \phi_{k_1}\phi_{k_2}\cdots\phi_{k_{2n}}\rangle = \sum_{\text{pairings}} \rho(P)\langle \phi_{k_{P_1}}\phi_{k_{P_2}}\rangle\cdots\langle\phi_{k_{P_{2n-1}}}\phi_{k_{P_{2n}}}\rangle, \tag{6.19}$$

where the sum runs over all permutations P of $\{1,\ldots,2n\}$ leading to inequivalent pairings. Generically, ρ is some complicated function of P, built out of the ρ_{ij}. It does not, in general, define a representation of the symmetric group. Note also that the order of the variables in each propagator on the right hand side is relevant. It is such that the two variables are in the same order on both sides of the equation.

Bosonic Path Integral

Recall from Chapter 5 that bosonic statistics is defined by the trivial braiding, i.e., the braiding is just the transposition with $\rho_{ij} = 1$ in (6.18).

The Leibniz rule (6.15) becomes the ordinary one $[\partial^i, \phi_j] = \delta^i_j$ and we recover the relevant bosonic differentiation and integration rules. In expression (6.19) we get $\rho(P) = 1$ and arrive at the bosonic Wick theorem, which merely expresses the combinatorics of grouping variables into pairs. Requiring the propagator to be invariant under the braiding means that γ must be symmetric and we recover the usual bosonic expression

$$\langle \phi_i \phi_j \rangle = \langle \phi_j \phi_i \rangle = (\gamma^{-1})_{ij}$$

from equation (6.16).

As a combinatorial exercise we can count the number of terms in (6.19) by giving each propagator the numerical value 1. This amounts to replacing the braided integers in (6.17) by ordinary integers. The braided double factorial turns into an ordinary double factorial and we obtain the value $(2n-1)!! = (2n)!/(n!2^n)$, which is precisely the number of ways in which we can arrange $2n$ variables into pairs of two.

For the case of conjugated variables $\{\phi_1, \phi_2, \dots\}$ and $\{\bar{\phi}_1, \bar{\phi}_2, \dots\}$ we require S to have the form $S(\phi) = \sum_{i,j} \bar{\phi}_i B_{ij} \phi_j$ for a matrix B. This is the same however, as taking all variables $\{\phi_1, \phi_2, \dots, \bar{\phi}_1, \bar{\phi}_2, \dots\}$ together and requiring γ to have the form

$$\gamma = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix}.$$

The propagator becomes $\langle \phi_k \bar{\phi}_l \rangle = \langle \bar{\phi}_l \phi_k \rangle = B^{-1}_{kl}$ with propagators of two un-barred or two barred variables vanishing. Consequently, Wick's theorem specialises to its familiar form for conjugated bosonic variables

$$\langle \phi_{k_1} \bar{\phi}_{l_1} \cdots \phi_{k_n} \bar{\phi}_{l_n} \rangle = \sum_{\text{permutations } P} \langle \phi_{k_1} \bar{\phi}_{l_{P_1}} \rangle \cdots \langle \phi_{k_n} \bar{\phi}_{l_{P_n}} \rangle,$$

where the sum runs over all permutations P of $\{1, \dots, n\}$.

Fermionic Path Integral

Recall from Chapter 5 that fermionic statistics is given by a braiding that is a transposition with a minus sign, i.e., we set $\rho_{ij} = -1$ in (6.18).

The Leibniz rule (6.15) becomes $\{\partial^i, \phi_j\} = \delta^i_j$. This is indeed the familiar expression for Grassmann variables, which are usually employed to perform the fermionic integration. Furthermore, the other rules (6.11) and (6.13) that we have required to define n-point functions turn out to hold also for Grassmann variables. This is quite obvious for (6.11), since differentiation and integration are the same for Grassmann variables and differentiating twice by the same variable must result in zero. Equation (6.13) follows for Grassmann variables from the observation that the relations $[\partial^i, S] = \gamma_{ij}\phi_j$ and $[\phi_i, S] = 0$ have the same commutator form as in the bosonic case, since S is quadratic. Thus, fermionic braided integration and integration with Grassmann variables must agree. Indeed, $\rho(P)$ in expression (6.19) becomes the signature of the permutation P. This is Wick's theorem for fermions. Requiring the propagator to be invariant under the braiding means that γ must be antisymmetric and we recover the usual fermionic expression

$$\langle \phi_i \phi_j \rangle = -\langle \phi_j \phi_i \rangle = (\gamma^{-1})_{ij}$$

from equation (6.16).

As in the bosonic case, we can play the game to assign each propagator the numerical value 1. This time we count the difference between the number of terms contributing with a plus sign and those with a minus sign in (6.19). In the diagrammatic language, this amounts to replacing any diagram by 1 or -1 depending on whether it contains an even or odd number of crossings. For the braided integers this means that $[n]'_\psi$ takes the value 1 if n is odd and zero if n is even. Since the braided factorial is a product of odd integers it takes the value 1, which is the desired result.

For conjugated variables $\{\phi_1, \phi_2, \dots\}$ and $\{\bar\phi_1, \bar\phi_2, \dots\}$ we write $S(\phi) = \sum_{i,j} \bar\phi_i B_{ij} \phi_j$ for a matrix B. This is the same as requiring γ to have the form

$$\gamma = \begin{pmatrix} 0 & -B^T \\ B & 0 \end{pmatrix}. \tag{6.20}$$

The propagator becomes $\langle \phi_k \bar\phi_l \rangle = -\langle \bar\phi_l \phi_k \rangle = B^{-1}_{kl}$ with propagators of two un-barred or two barred variables vanishing. Wick's theorem specialises to the familiar form

$$\langle \phi_{k_1} \bar\phi_{l_1} \cdots \phi_{k_n} \bar\phi_{l_n} \rangle = \sum_{\text{permutations } P} \operatorname{sign}(P) \langle \phi_{k_1} \bar\phi_{l_{P_1}} \rangle \cdots \langle \phi_{k_n} \bar\phi_{l_{P_n}} \rangle,$$

where $\operatorname{sign}(P)$ takes the value 1 or -1 depending on the signature of the permutation P.

6.2 Braided Feynman Diagrams

In this section we discuss perturbation theory, define the braided generalisation of Feynman diagrams, and consider the special cases of bosonic and fermionic diagrams.

6.2.1 Perturbation Theory

In Theorem 6.1.1 the decomposition of (free) $2n$-point functions into propagators is expressed by first manipulating the input variables (namely by applying $[2n-1]'_\psi!!$) and then inserting them into the propagators. However, we can also think of the n-point functions as independent objects and define their decomposition accordingly. Since we do not specify any input variables we need to indicate the number n explicitly and write $\mathcal{Z}^n$. Since the variables live in X^* we have $\mathcal{Z}^n \in X \otimes \cdots \otimes X$ (n-fold).

Algebraically, the change of point of view corresponds to a dualisation. Diagrammatically, it means turning diagrams upside down. The braided integers (6.5) and braided double factorial (6.6) thus become

$$\begin{aligned} &[n]''_\psi := \mathrm{id}^n + \psi^{-1} \otimes \mathrm{id}^{n-2} + \cdots + \psi^{-1}_{1,n-1} \\ \text{and}\quad &[2n-1]''_\psi!! := (\mathrm{id} \otimes [2n-1]''_\psi) \circ \cdots \circ (\mathrm{id}^{2n-3} \otimes [3]''_\psi) \circ (\mathrm{id}^{2n-1} \otimes [1]''_\psi), \end{aligned}$$

and Theorem 6.1.1 turns into

Corollary 6.2.1. *Let $\mathcal{Z}^k \in X^k$ denote the dual of $\mathcal{Z}|_{X^{*k}}$. Then*

$$\mathcal{Z}^2 = \psi \circ (\gamma^{-1} \otimes \mathrm{id}) \circ \mathrm{coev},$$
$$\mathcal{Z}^{2n} = [2n-1]''_\psi!!\,(\mathcal{Z}^2)^n, \quad \mathcal{Z}^{2n-1} = 0, \quad \forall n \in \mathbb{N},$$

Proof. This is obtained from Theorem 6.1.1 by reversing arrows or equivalently by turning diagrams upside down in the diagrammatic language of braided categories. □

In the standard notation of quantum field theory we write the free n-point function in terms of the path integral

$$\mathcal{Z}^n(x_1, \ldots, x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle_0 = \frac{\int \mathcal{D}\phi\, \phi(x_1) \cdots \phi(x_n) e^{-S_0(\phi)}}{\int \mathcal{D}\phi\, e^{-S_0(\phi)}}$$

with the free action S_0.

To evaluate interacting n-point functions, we use the same perturbation theory as in standard quantum field theory. For $S = S_0 + \lambda S_{\mathrm{int}}$ with coupling constant λ, we expand

$$\begin{aligned} \mathcal{Z}^n_{\mathrm{int}}(x_1, \ldots, x_n) &= \langle \phi(x_1) \cdots \phi(x_n) \rangle \\ &= \frac{\int \mathcal{D}\phi\, \phi(x_1) \cdots \phi(x_n)(1 - \lambda S_{\mathrm{int}}(\phi) + \ldots) e^{-S_0(\phi)}}{\int \mathcal{D}\phi\, (1 - \lambda S_{\mathrm{int}}(\phi) + \ldots) e^{-S_0(\phi)}} \\ &= \frac{\langle \phi(x_1) \cdots \phi(x_n) \rangle_0 - \lambda \langle \phi(x_1) \cdots \phi(x_n) S_{\mathrm{int}}(\phi) \rangle_0 + \ldots}{1 - \lambda \langle S_{\mathrm{int}}(\phi) \rangle_0 + \ldots}. \end{aligned}$$

For S_{int} of degree k we write

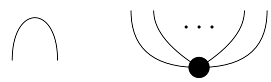
$$\langle \phi(x_1) \ldots \phi(x_n) S_{\mathrm{int}}(\phi) \rangle_0 = ((\mathrm{id}^n \otimes S_{\mathrm{int}})\, \mathcal{Z}^{n+k})(x_1, \ldots, x_n)$$

etc. by viewing S_{int} as a map $X^k \to \Bbbk$. Then, removing the explicit evaluations we obtain

$$\mathcal{Z}^n_{\text{int}} = \frac{\mathcal{Z}^n - \lambda(\mathrm{id}^n \otimes S_{\text{int}})(\mathcal{Z}^{n+k}) + \frac{1}{2}\lambda^2(\mathrm{id}^n \otimes S_{\text{int}} \otimes S_{\text{int}})(\mathcal{Z}^{n+2k}) + \dots}{1 - \lambda S_{\text{int}}(\mathcal{Z}^k) + \frac{1}{2}\lambda^2 (S_{\text{int}} \otimes S_{\text{int}})(\mathcal{Z}^{2k}) + \dots}, \tag{6.21}$$

an expression of the interacting n-point function valid in the general braided case. Vacuum contributions cancel as usual. Note that we have used the ordinary exponential expansion for the interaction and not, say, a certain braided version. The latter might be more natural if, e.g., one wants to look at identities between diagrams of different order. However, we shall not consider this issue here.

6.2.2 The Diagrams



(a) (b)

Figure 6.3: Propagator (a) and vertex (b).

We are now ready to generalise Feynman Diagrams to our braided setting. To do this, we extend the diagrammatic language of braided categories which we have used all along:

- An n-point function is an element in $X \otimes \cdots \otimes X$ (n-fold). Thus, its diagram is closed to the top and ends in n strands on the bottom. Any strand represents an element of X, i.e., a field.
- The propagator $\mathcal{Z}^2 \in X \otimes X$ is represented by an arch, see Figure 6.3.a.
- An n-leg vertex is a map $X \otimes \cdots \otimes X \to \Bbbk$. It is represented by n strands joining in a dot, see Figure 6.3.b. Notice that the order of incoming strands is relevant.
- Over- and under-crossings correspond to the braiding and its inverse, see Figure 1.1.
- Any Feynman diagram is built out of propagators, (possibly different kinds of) vertices, and strands with crossings, connecting the propagators and vertices, or ending at the bottom.

Otherwise the usual rules of braided diagrammatics apply. Notice that in contrast to ordinary Feynman diagrams all external legs end on one line (the bottom line of the diagram) and are ordered. This is necessary due to the possible non-trivial braid statistics.

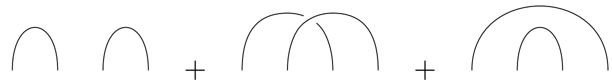


Figure 6.4: Free 4-point function.

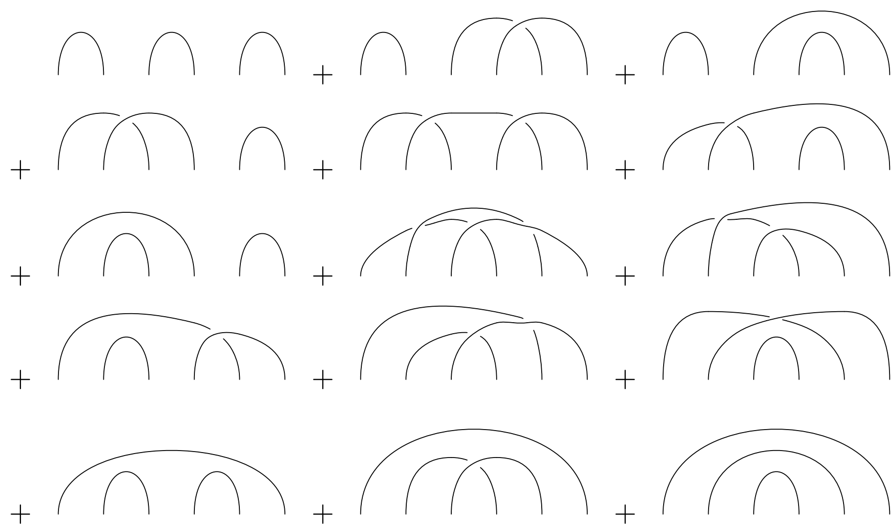


Figure 6.5: Free 6-point function.

The diagrams for the free $2n$-point functions can be read off from Corollary 6.2.1. The crossings are encoded in the braided integers $[n]''_{\psi}$. Figure 6.4 shows for example the free 4-point function and Figure 6.5 the free 6-point function. For the interacting n-point functions we use formula (6.21) to obtain the diagrams. S_{int} gives us the vertices. Consider for example the 2-point function in ϕ^4-theory. To order λ we get

$$\mathcal{Z}^2_{\text{int}} = \mathcal{Z}^2 - \lambda\left((\text{id}^2 \otimes S_{\text{int}})(\mathcal{Z}^6) - \mathcal{Z}^2 \otimes S_{\text{int}}(\mathcal{Z}^4)\right) + \mathcal{O}(\lambda^2). \tag{6.22}$$

S_{int} is just the map $\phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4 \mapsto \int \phi_1\phi_2\phi_3\phi_4$. To obtain the diagrams at order λ we start by drawing the free 6-point function (Figure 6.5) and attach to the 4 rightmost strands of each diagram a 4-leg vertex (Figure 6.3.b). Those are the diagrams generated by the first term in brackets of (6.22). We realize that the first three of our diagrams are vacuum diagrams which are exactly canceled by the second term in the brackets. The remaining 12 diagrams are shown in Figure 6.6. In standard quantum field theory they all correspond to the same diagram: The tadpole diagram, see Figure 6.7. The number 12 would only appear as a combinatorial factor.

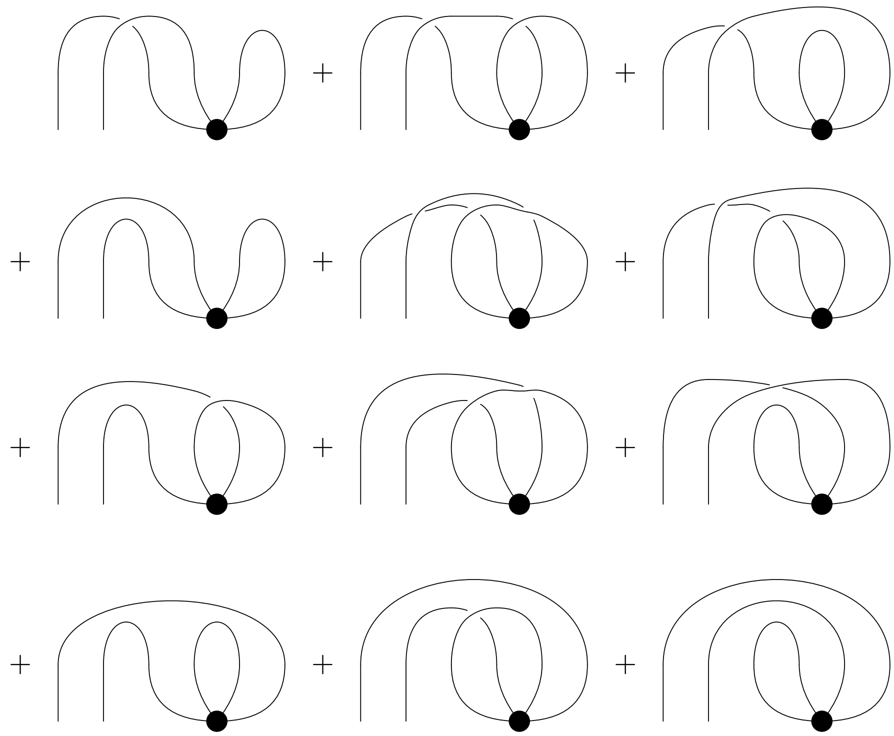


Figure 6.6: Interacting 2-point function of ϕ^4-theory at order 1.

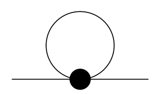


Figure 6.7: Tadpole diagram of standard ϕ^4-theory.

6.2.3 Bosonic and Fermionic Feynman Rules

In braided quantum field theory, no a priori distinction is made in the treatment of fields with different statistics in braided Feynman diagrams. In particular, there is no analogue of the special treatment for fermions in ordinary Feynman diagrams. Nevertheless, the correct Feynman rules both for bosons and fermions emerge from the braiding as we shall show in the following.

For bosonic fields, the braiding is trivial and crossings become ordinary transpositions. A braided Feynman diagram is thus evaluated as an ordinary Feynman diagram and we recover the usual bosonic Feynman rules. For fermions, over- and under-crossings are identical as for bosons, but they introduce a factor of -1. This is the only difference between bosons and fermions in braided Feynman diagrams. At first sight this appears to be at odds with standard quantum field theory, which prescribes no factor for line crossings, but introduces extra rules for fermions instead: (a) Each exchange of external fermion lines introduces a factor of -1,

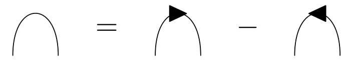


Figure 6.8: Decomposition of the fermionic propagator.

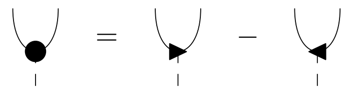


Figure 6.9: Decomposition of fermionic vertices. The dotted lines (drawn downwards for ease of notation) represent other fields.

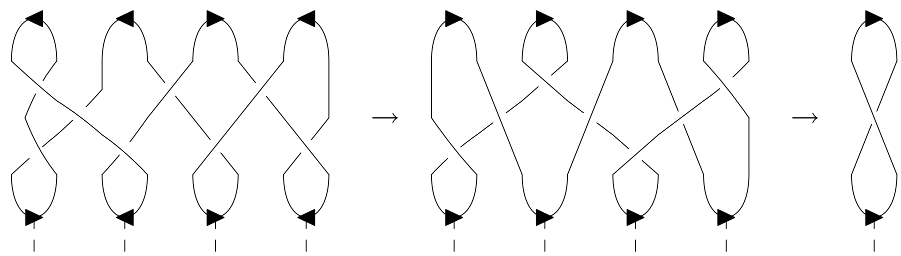
(b) Each internal fermion loop contributes a factor of -1. In fact, both prescriptions are equivalent as we proceed to demonstrate.

It is easy to see how rule (a) comes about. Exchanging external fermion lines is achieved by introducing (or removing) crossings. The number of crossings is necessarily odd, since the exchanged lines cross once, while any other lines are crossed twice (once by each of the two which are to be exchanged). Furthermore, crossings of external lines with loops or of loops with loops do not contribute since they always appear in pairs. It remains to be shown how rule (b) arises.

First, we note that ordinary fermions are described by conjugated variables. Thus, the propagator consists of two components corresponding to the two blocks in (6.20). Usually, one picks out one component and indicates which one it is by an arrow (Figure 6.8). The two components have a relative minus sign as in (6.20) due to fermionic antisymmetry. The same applies to fermion vertices (Figure 6.9).[2] Since only matching orientations contribute, each Fermion line decomposes into two components with consistently chosen orientation of propagators and vertices.

Consider now a fermion loop (see the example in Figure 6.10.a for illustration). We have to sum over both orientations of the loop in general. We consider the contribution of one of the two orientations. Its sign is determined from the various crossings and orientation choices of the propagators and vertices. To simplify, we choose the positive orientation and twist any propagators and vertices with the negative orientation around (Figure 6.10.b). This does not alter the sign, since crossings and orientation changes are introduced in equal number. Now, the sign contribution of the diagram is determined by the number of crossings modulo 2. To find it, we remove propagators and vertices pair-wise by straightening out lines, keeping in mind that we are always allowed to change over- in under-crossings and vice versa. This

[2]Note that the relative choice of positive orientation between propagator and vertex is the choice of sign for the vertex term in the action.



(a) (b) (c)

Figure 6.10: Evaluating the sign of a fermion loop.

removes crossings only pair-wise and leaves the sign invariant. We are left with just one propagator and vertex (Figure 6.10.c). This diagram must have one (or an odd number of) crossing. Thus, the overall factor is -1, in agreement with standard quantum field theory.

Chapter 7

Braided Quantum Field Theory – Special Cases

In this chapter we explore certain special cases of braided quantum field theory which provide either interesting applications in themselves (Section 7.1) or are instrumental for the following chapters (Sections 7.2–7.4).

In Section 7.1 we consider an example of anyonic statistics. We study the *quons* of Greenberg and others [Gre91]. We translate the canonical quon relations into a braid statistics and find that braided quantum field theory provides in this case the path integral counterpart to the canonical approach.

It has been notoriously difficult to incorporate generalised statistics into canonical relations. This is exhibited in the quon case in the necessary absence of any relations between creation or annihilation operators for different momenta. This gross deviation from the usual canonical quantisation approach indicates in our opinion that perhaps canonical relations are not the right way to describe generalised statistics. Instead, the braided approach advocated here appears more flexible.

Section 7.2 shows that a significant simplification occurs in braided quantum field theory when the braiding is symmetric: Braided Feynman diagrams can be replaced by ordinary Feynman diagrams. In particular, this justifies the use of ordinary Feynman diagrams for bosons and fermions in braided quantum field theory. However, it also applies to other situations, as we will see in Chapter 8, where a symmetric statistics occurs that is very different from the bosonic or fermionic one.

Sections 7.3 and 7.4 are particularly relevant for quantum field theory on noncommutative spaces and prepare the ground for Chapter 9. In Section 7.3 we consider braided quantum field theory on quantum homogeneous spaces. Besides obtaining results analogous to ordinary homogeneous spaces we find some surprising simplifications for certain vertices stemming from our algebraic point of view. In Section 7.4 we find that compact quantum spaces provide a situation where space-time quantum group symmetries can be dealt with algebraically rigorously.

7.1 Anyonic Statistics and Quons

In this section, we investigate an example of anyonic statistics (see Chapter 5), i.e., we are interested in a braiding that just exchanges the components but carries an extra factor q.

Since there is no standard quantum field theory of anyons to compare with, we start from a canonical approach. This also sheds new light on the bosonic and fermionic case from this point of view. More specifically, we consider the "quons" which provide an interesting example of anyons studied by Greenberg and others [Gre91].

Consider the relations

$$a_k a_l^\dagger = \delta_{kl} + q a_l^\dagger a_k \tag{7.1}$$

between creation and annihilation operators. Greenberg's treatment of this algebra is motivated by the possibility of small violations of bosonic ($q = 1$) or fermionic ($q = -1$) statistics. However, we need not take this point of view here.

In contrast to the ordinary canonical approach, no relations among a's or $a^\dagger$'s are introduced. In fact, such relations are not needed for normal ordering or the calculation of vacuum expectation values, as was stressed in [Gre91]. We are going one step further by remarking that relation (7.1) is only ever evaluated in one direction: from left to right. Thus, one could interpret (7.1) as defining the exchange statistics between a particle ($a_l^\dagger$) and a "hole" (a_k), where the δ-term just comes from the operator picture, analogous to expression (6.15). The corresponding braiding is

$$\psi^{-1}(a_k \otimes a_l^\dagger) = q a_l^\dagger \otimes a_k. \tag{7.2}$$

The choice of ψ^{-1} over ψ is to conform with the conventions of braided quantum field theory, where only ψ^{-1} appears in (6.6). In fact, we wish to make the whole Hilbert space of states into a braided space, in the spirit of Chapter 5. In order for expressions with an equal number of creation and annihilation operators ("zero particle number") to behave bosonic, we need to impose

$$\psi^{-1}(a_k \otimes a_l) = q^{-1} a_l \otimes a_k \quad \text{and} \quad \psi^{-1}(a_k^\dagger \otimes a_l^\dagger) = q^{-1} a_l^\dagger \otimes a_k^\dagger. \tag{7.3}$$

The particles and holes obey anyonic statistics among themselves.

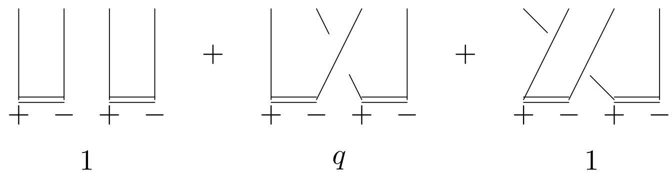


Figure 7.1: The contributions to the quon 4-point function.

We take the statistics generating group according to Table 5.1 to be $U(1)$. Thus, we have the general expression

$$\psi(v \otimes w) = q^{|v|\cdot|w|} w \otimes v \tag{7.4}$$

for the statistics. Particles are in the representation $|a_k^\dagger| = 1$ and holes in the representation $|a_k| = -1$ so that we recover (7.2) and (7.3).

A massive real scalar field is expressed in terms of creation and annihilation operators as

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{\sqrt{(2\pi)^3 2\omega_k}} \left(a_k e^{-\mathrm{i}k\cdot x} + a_k^\dagger e^{\mathrm{i}k\cdot x} \right)$$

with $\omega_k = \sqrt{k^2 + m^2}$. We split it as usual into the components $\phi(x) = \phi^+(x) + \phi^-(x)$, where $\phi^+(x)$ only contains annihilation operators while $\phi^-(x)$ only contains creation operators. We can view this formally as a decomposition of the space of classical fields $X = X^+ \oplus X^-$. The statistics inherited from the canonical picture is given by the $U(1)$ representation labels $|\phi^+(x)| = -1$ and $|\phi^-(x)| = 1$. As a remark, we observe that upon reducing $U(1)$ to $\mathbb{Z}_2$ we have $1 \cong -1$ as representations. This can be seen to be the reason why no analogous splitting of the field was necessary in the fermionic case.

With the braiding defined on the classical field, the path integral description of the quon is now precisely given by the path integral of braided quantum field theory. And indeed, the braided Wick theorem (6.17) takes the anyonic form (6.19) and specialises to the one found by Greenberg in [Gre91]. We consider the example of the free 4-point function. Its decomposition into propagators is given by

$$\begin{aligned} \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \langle\phi(x_1)\phi(x_2)\rangle\langle\phi(x_3)\phi(x_4)\rangle \\ &+ q\langle\phi(x_1)\phi(x_3)\rangle\langle\phi(x_2)\phi(x_4)\rangle + \langle\phi(x_2)\phi(x_3)\rangle\langle\phi(x_1)\phi(x_4)\rangle. \end{aligned} \tag{7.5}$$

This reproduces (37–39) in [Gre91].[1]

To see how (7.5) comes about consider Figure 7.1. The braided double factorial $[3]_\psi!! = [3]_\psi \otimes \mathrm{id}$ produces a sum of three diagrams. At the bottom we have indicated by horizontal

[1]Greenberg uses a complex scalar field. However, it is clear how to obtain (37–39) in [Gre91] from (7.5): Just insert the †'s and remove propagators that are not pairs of a ϕ and a $\phi^\dagger$.

double lines the evaluation by the propagators. In order to see what factors the braidings introduce we note that only the combination $\langle\phi^+(x)\phi^-(y)\rangle$ makes a contribution to the propagator. Accordingly, we have written below each strand the sign indicating the relevant field component carried by the strand. The evaluation is now simply determined by the statistics of the relevant field components: A braiding of a $+$ with a $-$ field gives a factor of q while braidings among $+$ or $-$ fields give a factor of q^{-1}. In this way, any free n-point function can be easily evaluated. Note that the rule for obtaining the q-factors given by Greenberg appears to be slightly different, but is equivalent. If, while fixing the attachments of the strands at the top line we deform the strands (with the attached propagators) so as to minimise the number of intersections, we are only left with intersections between fields with different sign labels. These all introduce factors of q. This is Greenberg's description.

Finally, we consider the issue of the statistics of bound states of quons. It was found in [GH99] that bound states of n quons have a statistics parameter of q^{n^2}. In fact, in our formulation this follows from the knowledge of the (quantum) group symmetry behind the statistics. A quon and its creation operator is in a 1-representation of the statistics generating $U(1)$. A quon hole and the annihilation operator are in the -1-representation. Thus, an n-quon state or operator that increases the quon number by n lives in an n-representation. By formula (7.4) we find that the statistics factor between two such objects is $q^{n\cdot n}$.

Although usually considered in the context of small violations of Bose or Fermi statistics in higher dimensions, our analysis suggests that it would be quite natural to consider quons in 2 dimensions where a spin-statistics relation can be established quantum geometrically, as shown in Section 5.5.

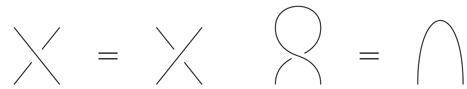
7.2 Symmetric Braided Quantum Field Theory

We have seen that in general it is necessary to impose much stricter rules on the way braided Feynman diagrams are drawn than is the case for ordinary Feynman diagrams. However, if the braiding is symmetric, we can recover the usual freedom in drawing Feynman diagrams. In particular, this applies to bosonic and fermionic statistics.

More precisely: If

- the braiding is symmetric, i.e., $\psi = \psi^{-1}$ (Figure 7.2.a) and
- the propagator is invariant under the braiding, i.e., $\psi(\mathcal{Z}^2) = \mathcal{Z}^2$ (Figure 7.2.b),

then all ways of deforming an ordinary Feynman diagram so that it conforms to the stricter rules of general braided Feynman diagrams are equivalent. This can be seen as follows: To deform a given ordinary Feynman diagram, first bend all external legs downwards and arrange



(a) (b)

Figure 7.2: Symmetric braiding (a) and braid-invariant propagator (b).

them on a line. For crossings introduced in this way it does not matter whether they are over or under. Then lift the propagators to the top. Note that when moving a line past others it can always be done in such a way that the moved line segment is completely "behind" or completely "in front" of the others. Finally, the loops introduced in this process can be "pulled straight" by using the invariance of the propagator.

While in the general braided case a diagram is evaluated strictly from top to bottom, this can be relaxed to the ordinary way of evaluating a diagram for the symmetric case. However, one crucial difference to ordinary Feynman diagrams remains: The line crossings can still be non-trivial. This is the case for fermions for example, where a crossing carries a factor of -1. However, we have seen in Section 6.2.3 that in this case the factor can be replaced by the usual extra rules for fermions.

7.3 Braided QFT on Quantum Homogeneous Spaces

In standard quantum field theory fixing one point of an n-point function still allows to recover the whole n-point function. Thus, we can reduce an n-point function to a function of just $n-1$ variables. This is simply due to the fact that any n-point function is invariant under the isometry group G of the space-time M and G acts transitively on M. In this case M is a homogeneous space under G and we can make the above statement more precise in the following way.

Lemma 7.3.1. *Let G be a group and K a subgroup of G. For any $n \in \mathbb{N}$ there is an isomorphism of coset spaces*

$$\rho_n : (\underbrace{K\backslash G \times \cdots \times K\backslash G}_{n \text{ times}})/G \cong (\underbrace{K\backslash G \times \cdots \times K\backslash G}_{n-1 \text{ times}})/K$$

given by $\rho_n : [a_1, \ldots, a_n] \mapsto [a_1 a_n^{-1}, \ldots, a_{n-1} a_n^{-1}]$ for $a_i \in K\backslash G$. Its inverse is given by $\rho_n^{-1} : [b_1, \ldots, b_{n-1}] \mapsto [b_1, \ldots, b_{n-1}, e]$ for $b_i \in K\backslash G$, where e denotes the equivalence class of the identity in $K\backslash G$. If G is a topological group (i.e., it is a topological space and multiplication and inversion are continuous), then equipping the coset spaces with the induced topologies makes ρ_n into a homeomorphism.

If space-time is an ordinary manifold we can obviously do the same trick in braided quantum field theory. More interestingly, however, we can extend it to noncommutative space-times.

7.3.1 Quantum Homogeneous Spaces

Lemma 7.3.1 generalises to the quantum group case. To see this we first recall the notion of a quantum homogeneous space.

Suppose we have two Hopf algebras A and H together with a Hopf algebra surjection $\pi : A \to H$. This induces coactions $\beta_L = (\pi \otimes \mathrm{id})$ and $\beta_R = (\mathrm{id} \otimes \pi)$ of H on A, making A into a left and right H-comodule algebra. Define ${}^H\!A$ to be the left H-invariant subalgebra of A, i.e., ${}^H\!A = \{a \in A | \beta_L(a) = 1 \otimes a\}$. We have $\Delta\, {}^H\!A \subseteq {}^H\!A \otimes A$ since $(\beta_L \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \beta_L$. This makes ${}^H\!A$ into a right A-comodule (and H-comodule) algebra. Observe also that $\pi(a) = \epsilon(a)1$ for $a \in {}^H\!A$. ${}^H\!A$ is called a right *quantum homogeneous space*. Define the left quantum homogeneous space A^H correspondingly. Due to the anticoalgebra property of the antipode we find $\mathrm{S}\, {}^H\!A \subseteq A^H$ and $\mathrm{S}\, A^H \subseteq {}^H\!A$. If the antipode is invertible, the inclusions become equalities.

Proposition 7.3.2. *In the above setting with invertible antipode the map*

$$\rho_n : (\underbrace{{}^H\!A \otimes \cdots \otimes {}^H\!A}_{n\ times})^A \to (\underbrace{{}^H\!A \otimes \cdots \otimes {}^H\!A}_{n-1\ times})^H$$

given by $\rho_n = (\mathrm{id}^{n-1} \otimes \epsilon)$ for $n \in \mathbb{N}$ is an isomorphism. Its inverse is $(\mathrm{id}^{n-1} \otimes \mathrm{S}) \circ \beta^{n-1}$, where β^{n-1} is the right coaction of A on ${}^H\!A$ extended to the $(n-1)$-fold tensor product.

Proof. Let $a^1 \otimes \cdots \otimes a^n$ be an element of $({}^H\!A \otimes \cdots \otimes {}^H\!A)^A$. In particular,

$$a^1{}_{(1)} \otimes \cdots \otimes a^n{}_{(1)} \otimes a^1{}_{(2)} \cdots a^n{}_{(2)} = a^1 \otimes \cdots \otimes a^n \otimes 1.$$

Applying the antipode to the last component and multiplying with the n-th component we obtain

$$a^1{}_{(1)} \otimes \cdots \otimes a^{n-1}{}_{(1)} \otimes \epsilon(a^n)\, \mathrm{S}(a^1{}_{(2)} \cdots a^{n-1}{}_{(2)}) = a^1 \otimes \cdots \otimes a^n. \tag{7.6}$$

Thus, $(\mathrm{id}^{n-1} \otimes \mathrm{S}) \circ \beta^{n-1} \circ (\mathrm{id}^{n-1} \otimes \epsilon)$ is the identity on $({}^H\!A \otimes \cdots \otimes {}^H\!A)^A$. On the other hand, applying the inverse antipode and then π to the last component of (7.6) we get

$$a^1{}_{(1)} \otimes \cdots \otimes a^{n-1}{}_{(1)} \otimes \epsilon(a^n)\pi(a^1{}_{(2)} \cdots a^{n-1}{}_{(2)}) = a^1 \otimes \cdots \otimes a^{n-1} \otimes \epsilon(a^n)1.$$

This is to say that $a^1 \otimes \cdots \otimes a^{n-1}\, \epsilon(a^n)$ is indeed right H-invariant.

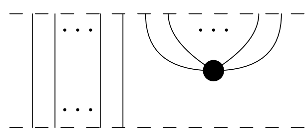


Figure 7.3: Vertex evaluation in a diagram slice.

Conversely, it is clear that $(\mathrm{id}^{n-1} \otimes \epsilon) \circ (\mathrm{id}^{n-1} \otimes \mathrm{S}) \circ \beta^{n-1} = (\mathrm{id}^{n-1} \otimes \epsilon) \circ \beta^{n-1}$ is the identity. Now take $b^1 \otimes \cdots \otimes b^{n-1}$ in $({}^H A \otimes \cdots \otimes {}^H A)^H$. Its image under β^{n-1} is

$$b^1{}_{(1)} \otimes \cdots \otimes b^{n-1}{}_{(1)} \otimes b^1{}_{(2)} \cdots b^{n-1}{}_{(2)}. \tag{7.7}$$

Applying π to the last component we get

$$b^1{}_{(1)} \otimes \cdots \otimes b^{n-1}{}_{(1)} \otimes \pi(b^1{}_{(2)} \cdots b^{n-1}{}_{(2)}) = b^1 \otimes \cdots \otimes b^{n-1} \otimes 1$$

by right H-invariance. Applying $\beta^{n-1} \otimes \mathrm{id}$ we arrive at

$$\begin{aligned} &b^1{}_{(1)} \otimes \cdots \otimes b^{n-1}{}_{(1)} \otimes b^1{}_{(2)} \cdots b^{n-1}{}_{(2)} \otimes \pi(b^1{}_{(3)} \cdots b^{n-1}{}_{(3)}) \\ &\qquad = b^1{}_{(1)} \otimes \cdots \otimes b^{n-1}{}_{(1)} \otimes b^1{}_{(2)} \cdots b^{n-1}{}_{(2)} \otimes 1. \end{aligned}$$

We observe that this is the same as applying $(\mathrm{id}^{n-1} \otimes \beta_R)$ to (7.7). Thus, the last component of (7.7) lives in A^H and the application of the antipode sends it to ${}^H A$ as required. That the result is right A-invariant is also clear by the defining property of the antipode. □

To make use of the result we assume our space X of fields to be a quantum homogeneous space under a quantum group (coquasitriangular Hopf algebra) A of symmetries. (Note that coquasitriangularity implies invertibility of the antipode.) That is, together with A we have another Hopf algebra H and a Hopf algebra surjection $A \to H$. We then assume that the algebra of fields is the right quantum homogeneous space $X = {}^H A$ living in the braided category $\mathcal{M}^A$ of right A-comodules.

7.3.2 Diagrammatic Techniques

Proposition 7.3.2, to which we shall refer as *invariant reduction*, is not only useful to express n-point functions in a more compact way, but can also be applied in the evaluation of braided Feynman diagrams. For this we note that any horizontal cut of a braided Feynman diagram lives in some tensor power of X (since the only allowed strand lives in X) and is invariant (since the diagram is closed at the top). Thus, we can apply invariant reduction to it. We shall give three examples for this, assuming vertices that are evaluated by multiplication and subsequent integration.

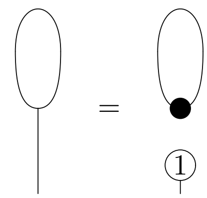


Figure 7.4: Extracting a loop.

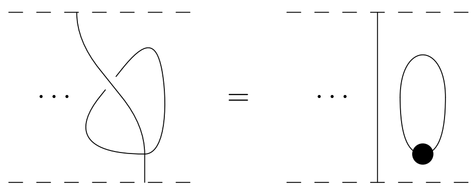


Figure 7.5: Separating a loop in an invariant slice.

Vertex evaluation. Consider the evaluation of an n-leg vertex (the horizontal slice of an invariant diagram depicted in Figure 7.3) with incoming elements $a_1 \otimes \cdots \otimes a_{k+n}$. By invariant reduction this can be expressed in two ways,

$$\begin{aligned} & a_1 \otimes \cdots \otimes a_k \int a_{k+1} \cdots a_{k+n} \\ & = a_{1_{(1)}} \otimes \cdots \otimes a_{k_{(1)}} \, \epsilon(a_{k+1}) \cdots \epsilon(a_{k+n}) \int \mathrm{S}(a_{1_{(2)}} \cdots a_{k_{(2)}}) \end{aligned}$$

Depending on the circumstances each side might be easier to evaluate.

Loop extraction. Assume that the integral on ${}^H A$ is normalised, $\int 1 = 1$. Consider the diagram in Figure 7.4 (left-hand side). It is obviously invariant. Thus, the single outgoing strand carries a multiple of the identity and we can replace it by the integral followed by the identity element (Figure 7.4, right-hand side).

Loop separation. We assume further that the coquasitriangular structure $\mathcal{R}$ is trivial on ${}^H A^H$ in the sense

$$\mathcal{R}(a \otimes b) = \epsilon(a)\, \epsilon(b), \qquad \text{if } a \in {}^H A^H \text{ or } b \in {}^H A^H. \tag{7.8}$$

Consider now the diagram in Figure 7.5 (left-hand side) as a horizontal slice of an invariant diagram. According to invariant reduction we apply the counit to the rightmost outgoing strand. This makes the braiding trivial due to the assumed property of $\mathcal{R}$. We can push the counit up to each of the joining strands and disentangle them. Then proceeding as in the previous example leads to the diagram in Figure 7.5 (right-hand side). Note that this works the same way for an under-crossing.

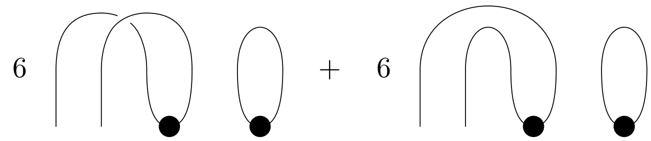


Figure 7.6: Simplified 2-point function of ϕ^4-theory at order 1.

Let us come back to the 2-point function of ϕ^4 theory that we considered at the end of Section 6.2.2. Assuming $\int 1 = 1$ and property (7.8) we can use loop extraction and loop separation to simplify the order 1 diagrams of Figure 6.6 considerably. The result is shown in Figure 7.6. Instead of 12 different diagrams we only have 2 different and much simpler diagrams, each with a multiplicity of 6.

7.4 Braided QFT on Compact Quantum Spaces

7.4.1 Braided Spaces of Infinite Dimension

Up to now we have developed our approach on a formal level insofar, that we have not addressed the question how an infinite dimensional space (of fields) can be treated in a braided category. This is certainly necessary if we want to do quantum field theory (or statistical field theory), i.e., deal with infinitely many degrees of freedom. If only internal symmetries are quantum groups, the solution is rather simple. We decompose the space of fields as a tensor product of scalar functions on space-time and a finite dimensional space of components. Only the latter lives in a braided category.

In general the situation is more involved. An obvious problem is the definition of the coevaluation. It seems that we need at least a completed tensor product for this. In fact, if the braiding is symmetric it is often possible to use functional analytic methods of ordinary (commutative) geometry. However, if the braiding is non-symmetric we might have to rely on purely algebraic methods. This can indeed be sufficient, as we shall see in the following.

Let us assume that the space of (regular) fields X decomposes into a direct sum $\bigoplus_i X_i$ of countably many finite dimensional comodules under the symmetry quantum group A. This corresponds roughly to the classical case of the space-time manifold being compact. In particular, it is the case if the symmetry quantum group A is cosemisimple (or classically the Lie group of symmetries is compact, see Section 7.4.2 below). Denote the projection $X \to X_i$ by τ_i.

We now allow arbitrary sums of elements in X given that any projection τ_i annihilates all but finitely many summands. Similarly, we allow infinite sums in the n-fold tensor product X^n with the restriction that any projection $\tau_{i_1} \otimes \cdots \otimes \tau_{i_n}$ yields a finite sum. To define the dual of X, we take the dual of each X_i and set $X^* = \bigoplus_i X_i^*$. For each component X_i we have an evaluation map $\mathrm{ev}_i : X_i \otimes X_i^* \to \mathbb{k}$ and a coevaluation map $\mathrm{coev}_i : \mathbb{k} \to X_i \otimes X_i^*$ in the usual way. We then formally define $\mathrm{ev} = \sum_i \mathrm{ev}_i \circ (\tau_i \otimes \tau_i^*)$ and $\mathrm{coev} = \sum_i \mathrm{coev}_i$.

Our definition is invariant under coactions of A as it should be, since the projections τ_i commute with the coaction of A. In particular, it is invariant under braidings.

7.4.2 Cosemisimplicity and Peter-Weyl Decomposition

We describe a context in which all comodules over a Hopf algebra decompose into finite dimensional (and even simple) pieces. The discussion here uses results of [Swe69] but is more in the spirit of [CSM95, II.9]. Assume $\mathbb{k}$ to be algebraically closed, e.g., $\mathbb{k} = \mathbb{C}$.

Let C be a coalgebra, V a simple right C-comodule (i.e. V has no proper subcomodules) with coaction $\beta : V \to V \otimes C$. In particular, V is finite dimensional. The dual space V^* is canonically a (simple) left C-comodule. Denote a basis of V by $\{e_i\}$, the dual basis of V^* by $\{f^i\}$. Identify the endomorphism algebra on V, $\mathrm{End}\, V \cong V \otimes V^*$ via $(e_i \otimes f^j)(e_k \otimes f^l) = \delta^j_k (e_i \otimes f^l)$. We denote the dual coalgebra by $(\mathrm{End}\, V)^*$ and identify $(\mathrm{End}\, V)^* \cong V^* \otimes V$ via $\Delta(f^i \otimes e_j) = \sum_k (f^i \otimes e_k) \otimes (f^k \otimes e_j)$.

Now consider the map $(\mathrm{End}\, V)^* \to C$ given by $f^i \otimes e_j \mapsto (f^i \otimes \mathrm{id}) \circ \beta(e_j)$. It is an injective (since V is simple) coalgebra map. We extend this to the direct sum of all inequivalent simple comodules. The resulting map

$$\bigoplus_V (\mathrm{End}\, V)^* \to C$$

is a coalgebra injection. It is an isomorphism of coalgebras if and only if all C-comodules are semisimple (i.e. they are direct sums of simple ones) or equivalently if C is semisimple (i.e. it is a direct sum of simple coalgebras). Assume now that A is a *cosemisimple* Hopf algebra, i.e., A is semisimple as a coalgebra. We write the above decomposition as

$$A \cong \bigoplus_V (V^* \otimes V). \tag{7.9}$$

It is also referred to as the Peter-Weyl decomposition, in analogy to the corresponding decomposition of the algebra of regular functions on a compact Lie group. There is a unique normalised left- and right-invariant integral (Haar functional) on A, given by the induced projection to the unit element in A. Note also that the antipode is invertible.

Consider a second Hopf algebra H with Hopf algebra surjection $\pi : A \to H$. This induces a coaction of H on each A-comodule. For the right quantum homogeneous space we have

$${}^H A \cong \bigoplus_V ({}^H(V^*) \otimes V) \tag{7.10}$$

as right H-comodules.

Chapter 8

Quantum Field Theory on Noncommutative $\mathbb{R}^d$

In this chapter we consider quantum field theory with coordinate commutation relations of the form

$$[x^\mu, x^\nu] = \mathrm{i}\,\theta^{\mu\nu}$$

in d dimensions, where θ is a real-valued antisymmetric matrix. This can also be viewed as equipping the algebra of functions on $\mathbb{R}^d$ with a deformation quantised multiplication known as a Moyal $\star$-product [Moy49]. We refer to this space-time algebra as *noncommutative* $\mathbb{R}^d$. We shall also consider the toroidal compactification known as the *noncommutative torus*. It served as an early example of a noncommutative geometry for Connes [Con80].

Commutation relations of the type considered here were proposed by Doplicher, Fredenhagen and Roberts based on an analysis of the constraints posed by general relativity and Heisenberg's uncertainty principle [DFR95]. (For other approaches at noncommutative space-times see [Maj88, Mad92, PW90, LNR92].) They also initiated the study of quantum field theory on this kind of space-time. (For an alternative approach to quantum field theory with generalised uncertainty relations see [Kem97]). Basic results for Feynman diagrams relating the noncommutative and the commutative setting were obtained by Filk [Fil96]. With the emergence of the noncommutative torus in string theory [CDS98], quantum field theory on such a space has received a much wider interest, see [SW99] and references therein. Recently, the perturbation theory has been of particular interest with the investigation of divergences and renormalisability, see e.g. [CR00, MVS00].

Our starting point is the observation by Watts [Wat00] that ordinary and noncommutative $\mathbb{R}^d$ are related by a certain 2-cocycle. This cocycle is associated with the translation group (which we also denote by $\mathbb{R}^d$) and induces a twist. While the twist turns $\mathbb{R}^d$ into itself as a group, it turns $\mathbb{R}^d$ into noncommutative $\mathbb{R}^d$ as a representation. Importantly, the concepts of cocycle and twist used here are dual to those of ordinary group cohomology and arise only from the quantum group point of view (see Chapter 2). Employing the framework of braided quantum field theory enables us to describe quantum field theory on both, commutative and noncommutative $\mathbb{R}^d$ in a purely algebraic language. This allows for the extension of the twist relating the two spaces to an equivalence between the quantum field theories living on them. Underlying is an equivalence of categories of representations. However, the noncommutative $\mathbb{R}^d$ in this context carries a non-trivial momentum-dependent statistics (in the sense of Chapter 5).

The noncommutative $\mathbb{R}^d$ with ordinary statistics (which is the space considered in the literature) on the other hand is related by the same twist to commutative $\mathbb{R}^d$ with non-trivial statistics. Here as well, we obtain an equivalence of quantum field theories on the two spaces. In this case it is really a duality exchanging noncommutativity and non-trivial statistics. In terms of perturbation theory, the duality exchanges a setting where vertices are noncommutative with a setting where vertices are commutative, but crossings carry an extra Feynman rule. As a byproduct, Filk's results are an immediate consequence. Finally, we investigate further space-time symmetries and gauge symmetry. We find that while they are preserved by the twist (as quantum group symmetries) they are broken by removing the non-trivial statistics from noncommutative $\mathbb{R}^d$. Although the whole discussion is in terms of $\mathbb{R}^d$ for convenience, it applies identically to the torus (except for the extra space-time symmetries).

Our equivalence result also suggests that a noncommutativity of the kind considered here really is too "weak" to be able to regularise a quantum field theory. What one needs for that purpose is a "stronger" noncommutativity in the form of a strict (i.e. non-symmetric) braiding. This is for example provided by q-deformations of Lie groups. See Chapter 9.

Note that the concept of twisting has been used to relate quantum space-times before [Maj94]. Also, 2-cocycles of ordinary group cohomology have been used to obtain noncommutative spaces in the context of matrix theory [HW98].

Section 8.1 looks at noncommutative $\mathbb{R}^d$ from the quantum group point of view and establishes the equivalence with ordinary $\mathbb{R}^d$ via twisting. The torus is treated as a special case. The main part of the chapter is formed by Sections 8.2–8.4, where quantum field theory on noncommutative $\mathbb{R}^d$ is analysed. The twist is extended to quantum field theory, leading to the equivalences mentioned above. Perturbative consequences are investigated. Space-time and gauge symmetry are considered at the end. Appendix 8.A provides some supplementary material required in the main text.

We work over the complex numbers throughout this chapter, except for the appendix.

8.1 Noncommutative $\mathbb{R}^d$ as a Twist

Part of the discussion in this section reproduces [Wat00]. In particular, the 2-cocycle (8.4) was found there, and it was shown to give rise to the deformed product (8.5). However, the full representation theoretic picture essential to our treatment of quantum field theory was lacking. We provide it here.

Although we use the purely algebraic language for convenience, Hopf algebras are to be understood in a topological sense in the following. Tensor products are appropriate completions. One could use the setting of Hopf C^*-algebras for example [VV99]. However, our discussion is independent of the functional analytic details and so we leave them out. When referring to function algebras one should have in mind a class compatible with the functional analytic setting chosen.

Consider $\mathbb{R}^d$ as the group of translations of d-dimensional Euclidean space. In the language of quantum groups, the corresponding object is the Hopf algebra $\mathcal{C}(\mathbb{R}^d)$ of functions on $\mathbb{R}^d$. We can view this as (a certain completion of) the unital commutative algebra generated by the coordinate functions $\{x^1, \ldots, x^d\}$. The product is $(f \cdot g)(x) = f(x) \cdot g(x)$, the unit $1(x) = 1$, the counit $\epsilon(f) = f(0)$, and the antipode $(\mathrm{S}\, f)(x) = f(-x)$. Identifying the (completed) tensor product $\mathcal{C}(\mathbb{R}^d) \otimes \mathcal{C}(\mathbb{R}^d)$ as the functions on the Cartesian product $\mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d)$, the coproduct encodes the group law of translation via $\Delta(f)(x, y) = f(x + y)$. We can formally write this as a Taylor expansion

$$\Delta\, f = \exp\left(x^\mu \otimes \frac{\partial}{\partial x^\mu}\right)(1 \otimes f) = \exp\left(\frac{\partial}{\partial x^\mu} \otimes x^\mu\right)(f \otimes 1).$$

We have the usual $*$-structure $(x^\mu)^* = x^\mu$ making $\mathcal{C}(\mathbb{R}^d)$ into a Hopf $*$-algebra. $\mathcal{C}(\mathbb{R}^d)$ is naturally equipped with the trivial coquasitriangular structure $\mathcal{R} = \epsilon \otimes \epsilon$.

Taking the dual point of view, we consider the Lie algebra of translation generators with basis $\{p_1, \ldots, p_d\}$. We denote its universal envelope by $\mathsf{U}(\mathbb{R}^d)$. Expressing elements of $\mathsf{U}(\mathbb{R}^d)$ as functions in the p_μ, we obtain the same Hopf algebra structure as for $\mathcal{C}(\mathbb{R}^d)$. We define the dual pairing by

$$\langle f(p_\mu), g\rangle = f\left(\mathrm{i}\frac{\partial}{\partial x^\mu}\right) g(x)\Big|_{x=0}.$$

The corresponding (left) action of $\mathsf{U}(\mathbb{R}^d)$ on $\mathcal{C}(\mathbb{R}^d)$ that leaves this pairing invariant is given by

$$(p_\mu \triangleright g)(x) = -\mathrm{i}\frac{\partial}{\partial x^\mu} g(x).$$

Viewing $\mathsf{U}(\mathbb{R}^d)$ as momentum space, we have the usual translation covariant Fourier transform

$\hat{}: \mathcal{C}(\mathbb{R}^d) \to \mathsf{U}(\mathbb{R}^d)$ and its inverse given by

$$\hat{f}(p) = \int \frac{\mathrm{d}^d x}{(2\pi)^{d/2}} f(x) e^{-\mathrm{i} p_\mu x^\mu} \quad \text{and} \quad f(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^{d/2}} \hat{f}(p) e^{\mathrm{i} p_\mu x^\mu}. \tag{8.1}$$

Now, let θ be a real valued antisymmetric $d \times d$ matrix. Consider the map $\chi_\theta : \mathcal{C}(\mathbb{R}^d) \otimes \mathcal{C}(\mathbb{R}^d) \to \mathbb{C}$ given by

$$\chi_\theta(f \otimes g) = (\epsilon \otimes \epsilon) \circ \exp\left(\frac{\mathrm{i}}{2}\, \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}\right)(f \otimes g). \tag{8.2}$$

One easily verifies (Definition 2.1.3) that this defines a unital 2-cocycle on $\mathcal{C}(\mathbb{R}^d)$ with inverse $\chi_\theta^{-1} = \chi_{-\theta} = \chi_\theta \circ \tau$ (τ the flip map). Thus, according to Proposition 2.1.4 it gives rise to a twisted Hopf algebra $\mathcal{C}_\theta(\mathbb{R}^d)$. However, the twisted product is the same as the original product, i.e., $\mathcal{C}(\mathbb{R}^d)$ and $\mathcal{C}_\theta(\mathbb{R}^d)$ are identical as Hopf algebras. In other words – the group of translations remains unchanged. In fact, it is easy to see from the formula for the twisted product (Proposition 2.1.4) that this must be so for any product twist on a cocommutative Hopf algebra. The coquasitriangular structure does change on the other hand, and we obtain

$$\mathcal{R}_\theta(f \otimes g) = (\epsilon \otimes \epsilon) \circ \exp\left(-\mathrm{i}\, \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}\right)(f \otimes g).$$

according to (2.1). In particular, this means that the category of comodules of $\mathcal{C}_\theta(\mathbb{R}^d)$ is equipped with a braiding ψ_θ that is not the flip map. Using (1.8) we obtain

$$\psi_\theta(f \otimes g) = \exp\left(-\mathrm{i}\, \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}\right)(g \otimes f). \tag{8.3}$$

In more conventional language this means that the representations of the translation group acquire non-trivial statistics. Note that $\mathcal{R}_\theta^{-1} = \mathcal{R}_\theta \circ \tau$ (with τ the flip map), i.e., $\mathcal{R}_\theta$ is cotriangular (as it must be, being obtained by twisting from a trivial $\mathcal{R}$). Consequently, the braiding is symmetric, i.e., $\psi_\theta^2 = \mathrm{id}$.

By duality, we can equivalently express this twist as an invertible element $\Phi_\theta \in \mathsf{U}(\mathbb{R}^d) \otimes \mathsf{U}(\mathbb{R}^d)$ obeying the axioms of Definition 2.1.1. We get

$$\Phi_\theta = \exp\left(-\frac{\mathrm{i}}{2}\, \theta^{\mu\nu} p_\mu \otimes p_\nu\right). \tag{8.4}$$

This is (3.10) in [Wat00]. As in the above discussion the twisted $\mathsf{U}_\theta(\mathbb{R}^d)$ is the same as $\mathsf{U}(\mathbb{R}^d)$ as a Hopf algebra, but the quasitriangular structure becomes nontrivial.

Now consider d-dimensional Euclidean space with an action of the translation group (from the left say). In quantum group language this means that we take a second copy $\widetilde{\mathcal{C}}(\mathbb{R}^d)$ of $\mathcal{C}(\mathbb{R}^d)$ as a left $\mathcal{C}(\mathbb{R}^d)$-comodule algebra. In contrast to the quantum group $\mathcal{C}(\mathbb{R}^d)$ its algebra structure *is* changed under the twist. This is the situation of Example 2.3.2. Furthermore, we know from Proposition 2.3.3 that the new product on the twisted $\widetilde{\mathcal{C}}(\mathbb{R}^d)$ which we denote by $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ is a deformation quantisation. We find

$$(f \star g)(x) = \exp\left(\frac{\mathrm{i}}{2}\,\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\nu}\right) f(x+\xi)g(x+\eta)\Big|_{\xi=\eta=0}, \tag{8.5}$$

which is known as a Moyal $\star$-product [Moy49]. Note that the inherited $*$-structure is compatible with the new algebra structure making $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ into a $*$-algebra.

According to Theorem 2.2.2, the category of $\mathcal{C}(\mathbb{R}^d)$-comodules and the category of $\mathcal{C}_\theta(\mathbb{R}^d)$-comodules are equivalent. While objects remain the same under twisting, the tensor product does not. In particular, this means that while for $f \in \widetilde{\mathcal{C}}(\mathbb{R}^d)$ the corresponding $f_\theta \in \widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ is just the same function this is not so for functions of several variables. In our context a function of n variables is an element of $\widetilde{\mathcal{C}}(\mathbb{R}^d \times \cdots \times \mathbb{R}^d)$ which we write as the tensor product $\widetilde{\mathcal{C}}(\mathbb{R}^d) \otimes \cdots \otimes \widetilde{\mathcal{C}}(\mathbb{R}^d)$. This is transformed to the tensor product $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d) \otimes \cdots \otimes \widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ via the functor ${\sigma_\chi}^{-1}$ (extend to multiple tensor products by associativity). Explicitly, we obtain

$$f_\theta(x_1,\ldots,x_n) = \exp\left(-\frac{\mathrm{i}}{2}\sum_{l<m}\theta^{\mu\nu}\frac{\partial}{\partial x_l^\mu}\frac{\partial}{\partial x_m^\nu}\right) f(x_1,\ldots,x_n). \tag{8.6}$$

Due to duality, (left) $\mathcal{C}(\mathbb{R}^d)$-comodules are really the same thing as (left) $\mathsf{U}(\mathbb{R}^d)$-modules. In particular, viewing momentum space as a left $\mathsf{U}(\mathbb{R}^d)$-module (coalgebra) denoted by $\widetilde{\mathsf{U}}(\mathbb{R}^d)$, it lives in the same category as $\widetilde{\mathcal{C}}(\mathbb{R}^d)$ and we denote its twisted analogue by $\widetilde{\mathsf{U}}_\theta(\mathbb{R}^d)$. The momentum space version of equation (8.6) reads

$$f_\theta(p^1,\ldots,p^n) = \exp\left(\frac{\mathrm{i}}{2}\sum_{l<m}\theta^{\mu\nu}p^l_\mu p^m_\nu\right) f(p^1,\ldots,p^n). \tag{8.7}$$

The transformation of morphisms (i.e. intertwiners) by $\mathcal{G}_\chi$ is non-trivial only if they transform tensor products to tensor products. In particular, this means that integration and Fourier transform (8.1) are preserved by the twist. Note that even the Fourier transform in several variables survives the twist unchanged, since it factors into Fourier transforms in each variable.

8.1.1 A Remark on the Noncommutative Torus

All constructions we have made for noncommutative $\mathbb{R}^d$ apply equally to the noncommutative torus. We simply restrict to periodic functions. To be more specific, let $\mathbb{T}^d$ denote the group $U(1)^d$ of translations on the d-dimensional torus of unit radius which we also denote by $\mathbb{T}^d$. The Hopf algebra of functions $\mathcal{C}(\mathbb{T}^d)$ on $\mathbb{T}^d$ has a basis of Fourier modes $\{u_k\}$ for $k \in \mathbb{Z}^d$. We can identify u_k as a periodic function in $\mathcal{C}(\mathbb{R}^d)$ via $u_k(x) = \exp(\mathrm{i}\,k_\mu x^\mu)$. For completeness we provide the relevant formulas explicitly: Product and coproduct are given by $u_k u_l = u_{k+l}$

and $\Delta\, u_k = u_k \otimes u_k$. The counit is $\epsilon(u_k) = 1$. Antipode and $*$-structure are $\mathrm{S}\, u_k = u_k^* = u_{-k}$. The twist (8.2) takes the form

$$\chi_\theta(u_k \otimes u_l) = \exp\left(-\frac{\mathrm{i}}{2}\,\theta^{\mu\nu} k_\mu l_\nu\right),$$

and the twisted comodule algebra $\widetilde{\mathcal{C}}_\theta(\mathbb{T}^d)$ satisfies the product rule

$$u_k \star u_l = \exp(-\mathrm{i}\,\theta^{\mu\nu} k_\mu l_\nu)\, u_l \star u_k.$$

8.2 Towards Quantum Field Theory

Let us examine the noncommutative $\mathbb{R}^d$ with a view towards taking it as the space-time of a quantum field theory. Recall that the coordinate functions $x^1, \dots, x^d$ of noncommutative $\mathbb{R}^d$ obey commutation relations of the form

$$[x^\mu, x^\nu] = \mathrm{i}\,\theta^{\mu\nu} \tag{8.8}$$

for θ a real valued antisymmetric $d \times d$ matrix. More precisely, noncommutative $\mathbb{R}^d$ is a deformation quantisation of the algebra of functions on ordinary $\mathbb{R}^d$ satisfying (8.5).

Apart from space-time itself, its group of isometries plays a fundamental role in quantum field theory. After all, fields and particles are representations of this group (or its universal cover) and it leaves a quantum field theory as a whole (i.e., its n-point functions) invariant. What is this group for noncommutative $\mathbb{R}^d$? For general θ, the commutation relations (8.8) are clearly not invariant under rotations or boosts. However, they are invariant under ordinary translations $x^\mu \mapsto x^\mu + a^\mu$. Thus, it appears natural to let the translations play the role of isometries of noncommutative $\mathbb{R}^d$. This is an important ingredient for the following discussion. We later come back to the question of a possible larger group of symmetries.

It was shown in Section 8.1 how ordinary $\mathbb{R}^d$ is turned into noncommutative $\mathbb{R}^d$ by a process of twisting. This is induced by a 2-cocycle χ_θ on the quantum group $\mathcal{C}(\mathbb{R}^d)$ of translations. ("Cocycle" here has the meaning dual to that of ordinary group cohomology.) At the same time $\mathcal{C}(\mathbb{R}^d)$ is turned into the quantum group $\mathcal{C}_\theta(\mathbb{R}^d)$. While this still corresponds to the ordinary group of translations, it is different from $\mathcal{C}(\mathbb{R}^d)$ as a quantum group. The difference is encoded in the coquasitriangular structure $\mathcal{R}_\theta$ which is now non-trivial. It equips noncommutative $\mathbb{R}^d$ with a non-trivial statistics encoded in the braiding ψ_θ. This twist-transformation is represented in Figure 8.1 by the upper arrow. It goes both ways since we can undo the twist by using the inverse 2-cocycle $\chi_{-\theta}$.

commutative ordinary statistics	$\Longleftrightarrow$	noncommutative non-trivial statistics
$\vdots$		$\vdots$
commutative non-trivial statistics	$\Longleftrightarrow$	noncommutative ordinary statistics

Figure 8.1: Relations between quantum field theories on $\mathbb{R}^d$. The arrows indicate equivalences while the dotted lines indicate equality of planar Feynman diagrams.

What about noncommutative $\mathbb{R}^d$ with *ordinary* statistics? After all, this is the space which has been of interest in the literature. Untwisting this space yields the commutative $\mathbb{R}^d$ as before. However, as before, twisting also exchanges ordinary with braid statistics. Only this time the other way round: We obtain commutative $\mathbb{R}^d$ equipped with braid statistics. This is represented by the lower arrow in Figure 8.1. In the language of Section 8.1, we consider $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ (noncommutative $\mathbb{R}^d$) as a comodule of $\mathcal{C}(\mathbb{R}^d)$ (the translation group with ordinary statistics) and apply the twist with the inverse 2-cocycle $\chi_{-\theta}$. We get $\widetilde{\mathcal{C}}(\mathbb{R}^d)$ (ordinary $\mathbb{R}^d$) but as a comodule of $\mathcal{C}_{-\theta}(\mathbb{R}^d)$ (the translation group with braid statistics). The braiding this time is given by $\psi_{-\theta}$ since we have used the inverse twist. Note that the braiding is in both cases symmetric, i.e., ψ^2 is the identity.

We show in Section 8.3 how the twist equivalence between the respective spaces gives rise to an equivalence of quantum field theories on those spaces. Since we need to be able to deal with quantum group symmetries as well as braid statistics the right framework for this is braided quantum field theory. Section 8.4 looks at the perturbative consequences in more detail. Finally, in Section 8.5, we turn to the question of what happens with additional symmetries under twist.

Note that while the whole discussion is solely in terms of $\mathbb{R}^d$ for convenience, everything applies equally to the torus. This follows from the remarks in Section 8.1.1. The only exception are the extra space-time symmetries considered in Section 8.5.1.

8.3 Equivalences for Quantum Field Theory

With the machinery of braided quantum field theory in place we can handle quantum field theory on any of the versions of $\mathbb{R}^d$ represented in Figure 8.1. Recall that the arrows in this figure represent twist transformations between the respective spaces. Now, by Theorem 2.2.2 the twist induces an equivalence between the whole categories of translation covariant objects and maps in which those spaces live. But the whole perturbation expansion takes place in this category, including Feynman diagrams and n-point functions. This is made explicit by using braided Feynman diagrams and associating the space of fields, tensor products and intertwining maps (vertices, the braiding etc.) with elements of those diagrams. Consequently, (braided) quantum field theories on spaces related by twist are equivalent. In particular, the arrows in Figure 8.1 stand for such equivalences. For an n-point function, a Feynman diagram, or a vertex the relation between the commutative quantity G and the noncommutative quantity G_{NC} is in both cases given in momentum space by

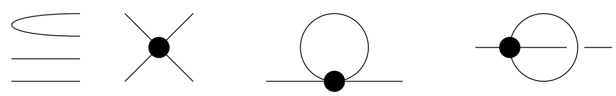
$$G_{NC}(p^1,\ldots,p^n) = \exp\left(\frac{\mathrm{i}}{2}\sum_{l<m}\theta^{\mu\nu}p^l_\mu p^m_\nu\right) G(p^1,\ldots,p^n) \tag{8.9}$$

which is just formula (8.7) from Section 8.1. The corresponding position space version is (8.6).

We would like to stress that our treatment applies to fields in any representation of the translation group and thus to quantum field theory in general. For scalars the space of fields is simply $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ itself. Any other field lives in a bundle associated with the frame bundle (or its universal cover – the spin bundle) which in particular carries an action of the translation group. Choosing a trivialisation "along translation" allows to write the space of sections of the bundle as $V\otimes\widetilde{\mathcal{C}}(\mathbb{R}^d)$ with translations acting trivially on V. Thus, under twist we obtain $V\otimes\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ with the V-component not being affected at all by the twist. In other words: Extra indices like spinor or tensor indices just show the ordinary behaviour and can be considered completely separate from the noncommutativity going on in space-time. This also applies to fermions, since the extra -1 factor of the fermionic braiding is compatible with the braiding ψ_θ and just sits "on top".

Let us make an extra remark about gauge theories. For a gauge bundle there is no canonical action of the translation group. Choosing such an action is the same thing as choosing a trivialisation, i.e., a "preferred gauge". Given such a choice we can treat gauge theory with the above methods. This supposes that we have integrated out the gauge degrees of freedom in the path integral in the usual way, say by the Faddeev-Popov method.

Rigorously speaking, our treatment so far has assumed that quantities encountered in the calculation of Feynman diagrams are finite. Then, the transformation (8.9) between quantum field theories connected by arrows in Figure 8.1 is straightforward. In order to establish the equivalence not only for finite but also for renormalisable quantum field theories, we need to extend the twisting equivalence to the regularisation process involved in the renormalisation. The only condition for the twist transformation to work in this context is that we remain in the translation covariant category, i.e., that the regularisation preserves



(a) (b) (c)

Figure 8.2: Building blocks for the diagrams of the first order contribution to the 2-point function in ϕ^4-theory (a). Resulting tadpole diagram (b). In the cyclic case diagram (c) is non-equivalent.

covariance under translations. This is easily accomplished. For example, a simple momentum-cutoff regularisation would do, or a Pauli-Villars regularisation. (Note however, that the popular dimensional regularisation can not be used here.) Using such a regularisation, the twisting equivalence holds at every step of the renormalisation procedure, and in particular for the renormalised quantities at the end.

As a further remark, the equivalences should also hold non-perturbatively, since the n-point functions (perturbative or not) naturally live in the respective categories. However, for lack of a general non-perturbative method, we can obviously not demonstrate this explicitly. Turning the argument round, one could say that for a well defined theory on one side the transformation (8.9) *defines* the respective equivalent theory.

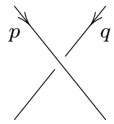
8.4 Perturbative Consequences

Let us explore the consequences of the equivalences in terms of perturbation theory. Note first that the braiding is symmetric and the propagator braid invariant. Thus, we have symmetric braided quantum field theories (Section 7.2) and are allowed to use ordinary Feynman diagrams.

We start by discussing the issue of vertex symmetry. It has been observed that vertices which are totally symmetric under an exchange of legs retain only a cyclic symmetry on noncommutative $\mathbb{R}^d$ (with ordinary statistics). Following the upper arrow in Figure 8.1 from left to right we retain total symmetry. For a transposition this takes the form

$$G_{NC}(p^1,\ldots,p^i,p^{i+1},\ldots,p^n) = \exp\left(-\mathrm{i}\,\theta^{\mu\nu}p^i_\mu p^{i+1}_\nu\right) G_{NC}(p^1,\ldots,p^{i+1},p^i,\ldots,p^n).$$

However, "stripping off" the non-trivial braiding, i.e., considering ordinary transpositions by flip, leaves only a cyclic symmetry. Following the lower arrow to the left, we have the opposite situation. Vertices are now ordinarily totally symmetric, but we have a non-trivial braiding with respect to which they are only cyclic symmetric.



$$= \quad \exp(\mathrm{i}\,\theta^{\mu\nu} p_\mu q_\nu)$$

Figure 8.3: Extra Feynman rule for crossings with braid statistics. The arrows indicate the direction of the momenta p and q.

The deeper reason for the retention of cyclic symmetry is a property of the coquasitriangular structure $\mathcal{R}_\theta$ defining the braiding. As a consequence of this property, cyclic symmetry with respect to ordinary and the braid statistics is the same for translation invariant objects like vertices. This is due to Lemma 8.6.1 in Appendix 8.A. For perturbation theory the use of vertices that are only cyclic symmetric means that diagrams which would be the same for total symmetry may now differ. Consider for illustration the 2-point function in ϕ^4-theory at 1-loop order. Assembling the building blocks (Figure 8.2.a) in all possible ways (noting that the legs of the propagator are to be considered identical) results in 8 times diagram (b) plus 4 times diagram (c), given only cyclic symmetry of the vertex (see Figure 8.2). A total symmetry would imply that both diagrams are equal, leading to the usual factor of 12.

Now, recall from Section 7.2 that although we are allowed to draw ordinary Feynman diagrams, one difference in the evaluation remains: The braiding ψ instead of the trivial exchange is associated with each crossing. This is a kind of "extra" Feynman rule. In fact, this is the only effect of the (symmetric) braiding in perturbation theory. It follows immediately that planar Feynman diagrams are identical in theories that differ only by their (symmetric) braid statistics. This is indicated by the dotted lines in Figure 8.1.

Filk's result [Fil96] for planar diagrams follows: We evaluate a planar diagram in the commutative setting and follow the lower arrow in Figure 8.1 to the right. Diagrams are simply related by the equivalence formula (8.9). For non-planar diagrams we also use the commutative setting. We only have to take into account the crossing factors from the non-trivial statistics. They are given by the extra Feynman rule in Figure 8.3. This is the momentum space version of formula (8.3) with opposite sign for θ. If we aggregate the factors for a given diagram by encoding all the crossings into an intersection matrix, we obtain an overall factor

$$\exp\left(\mathrm{i}\sum_{k>l} I_{kl}\theta^{\mu\nu}p^k_\mu p^l_\nu\right). \tag{8.10}$$

Here, the indices k, l run over all lines of the diagram and I_{kl} counts the oriented number of intersections between lines k and l. Then again, relation (8.9) leads to the noncommutative theory. This is Filk's result for non-planar diagrams [Fil96]. Note that it was already observed in [MVS00] that (8.10) can be obtained by assigning phase factors to crossings.

As a further remark, it has been observed that quadratic terms in the Lagrangian are not modified in the noncommutative setting. This follows from a property of the twisting cocycle. Any invariant object with 2 components (like a 2-leg vertex, a free propagator etc.) remains unchanged by the twist. This is Lemma 8.6.2 in Appendix 8.A.

8.5 Additional Symmetries

In this final section of the chapter we consider the effect of twisting on additional symmetries. We follow the upper arrow in Figure 8.1 from left to right.

8.5.1 Space-Time Symmetry

As mentioned before, the commutation relations (8.8) are not invariant under rotations. However, ordinary Euclidean space is, and since the noncommutative version is a twist of the commutative one, there should be an analogue of those symmetries. This is indeed the case. Consider the group of (orientation preserving) rotations $SO(d)$ in d dimensions. In quantum group language we consider the algebra of functions $\mathcal{C}(SO(d))$ generated by the matrix elements t^μ_ν of the fundamental representation. We have relations $t^\mu_\rho t^\nu_\rho = \delta^\mu_\nu = t^\rho_\mu t^\rho_\nu$ (summation over ρ implied) and $\det(t) = 1$, coproduct $\Delta\, t^\mu_\nu = t^\mu_\rho \otimes t^\rho_\nu$, counit $\epsilon(t^\mu_\nu) = \delta^\mu_\nu$, and antipode $\mathrm{S}\, t^\mu_\nu = t^\nu_\mu$. We have a $*$-structure given by $(t^\mu_\nu)^* = t^\nu_\mu$.

We extend the translation group $\mathbb{R}^d$ to the full group $E := \mathbb{R}^d \rtimes SO(d)$ of (orientation preserving) isometries of Euclidean space. I.e., we consider the Hopf algebra $\mathcal{C}(E) = \mathcal{C}(\mathbb{R}^d \rtimes SO(d)) \cong \mathcal{C}(\mathbb{R}^d) \rtimes \mathcal{C}(SO(d))$. (For the semidirect product of Hopf algebras see the end of Appendix 8.A.) The rotations coact on the translations from the left by $x^\mu \mapsto t^\mu_\nu \otimes x^\nu$. The resulting semidirect product Hopf algebra is generated by x^μ and t^μ_ν with the given relations. The coproduct of t^μ_ν remains the same but for x^μ we now obtain $\Delta\, x^\mu = x^\mu \otimes 1 + t^\mu_\nu \otimes x^\nu$. (Use (8.11).) This also determines the left coaction on $\widetilde{\mathcal{C}}(\mathbb{R}^d)$.

The cocycle χ_θ on $\mathcal{C}(\mathbb{R}^d)$ extends trivially to a cocycle on the larger quantum group $\mathcal{C}(E)$, i.e., we let χ_θ just be the counit on the generators of $\mathcal{C}(SO(d))$. The twist *does* change the algebra structure now. This was to be expected since we have already seen that ordinary rotation invariance is lost. What do we have instead? Using the twist (8.2) in Proposition 2.1.4 we find that the relations for the x^μ become

$$x^\mu \bullet x^\nu - x^\nu \bullet x^\mu = i\,\theta^{\mu\nu} - i\,\theta^{\rho\sigma} t^\mu_\rho \bullet t^\nu_\sigma,$$

while the t^μ_ν still commute with the other generators. Thus, the twisted space-time symmetries $\mathcal{C}_\theta(E)$ form a genuine quantum group (noncommutative Hopf algebra), no longer corresponding to any ordinary group.

When dealing with the translation group alone, we were able to remove the non-trivial coquasitriangular structure responsible for the braid statistics and replace it by a trivial one (follow the dotted line on the right in Figure 8.1 downwards). However, this is no longer possible for the whole Euclidean motion group. A genuine quantum group as the one obtained here does not admit a trivial coquasitriangular structure. Thus, removing the braid statistics really breaks the symmetry for the quantum field theory.

Note that the argument applies to Minkowski space and the Poincaré group in the identical way.

8.5.2 Gauge Symmetry

Let us consider a gauge theory with gauge group G. The gauge transformations are the maps $\mathbb{R}^d \to G$. We denote the group of such maps by $\Gamma = \{\mathbb{R}^d \to G\}$. The symmetry group generated by translations and gauge transformations is the semidirect product $\Omega := \Gamma \rtimes \mathbb{R}^d$, where we have chosen an action of the translation group on the gauge bundle. (Note that the inclusion of further space-time symmetries does not modify the argument.) While the group Ω "forgets" about the trivialisation of the gauge bundle corresponding to the chosen action, we do need the trivialisation to extend the twisting cocycle from $\mathbb{R}^d$ to Ω. This is in accordance with our remark on gauge theories above. In quantum group language we have the semidirect product of Hopf algebras $\mathcal{C}(\Omega) = \mathcal{C}(\Gamma \rtimes \mathbb{R}^d) \cong \mathcal{C}(\Gamma) \rtimes \mathcal{C}(\mathbb{R}^d)$. The cocycle χ_θ extends trivially from $\mathcal{C}(\mathbb{R}^d)$ to $\mathcal{C}(\Gamma)$. Applying Proposition 2.1.4 with (8.11) results in a noncommutative product

$$f \bullet \gamma = \chi_\theta(f_{(1)} \otimes \gamma_{(1)})\, f_{(2)}\gamma_{(\underline{2})},$$
$$\gamma \bullet \omega = \chi_\theta(\gamma_{(1)} \otimes \omega_{(1)})\, \gamma_{(\underline{2})}\omega_{(\underline{2})},$$

while $f \bullet g = fg$ for $f, g \in \mathcal{C}(\mathbb{R}^d)$ and $\gamma, \omega \in \mathcal{C}(\Gamma)$. Thus, the group of gauge transformation does not survive the twist as an ordinary group. As for the case of rotations we find that we obtain a genuine quantum group. Again, the removal of the braid statistics would break the symmetry. Note that this does not exclude the possibility of a different kind of gauge symmetry. See [CR87] and more recently [SW99] for discussions of gauge theory on noncommutative $\mathbb{R}^d$.

8.A Appendix

This appendix collects two small lemmas and a known formula for the semidirect product of commutative Hopf algebras.

Let $H, \mathcal{R}$ be a coquasitriangular Hopf algebra. If $\mathcal{R}$ satisfies a certain condition, then for H-invariant elements the braiding ψ (given by (1.8)) is just the same as the flip map:

Lemma 8.6.1. *Let H be a Hopf algebra with coquasitriangular structure $\mathcal{R} : H \otimes H \to \Bbbk$ satisfying the property $\mathcal{R}(\mathrm{S}\, a_{(1)} \otimes a_{(2)}) = \epsilon(a)$. Then for left comodules V and W and $v \otimes w \in V \otimes W$ H-invariant we have $\psi(v \otimes w) = w \otimes v$.*

Proof. $w_{(1)} \otimes v_{(1)} \otimes w_{(\underline{2})} \otimes v_{(\underline{2})} = \mathrm{S}\, v_{(1)} \otimes v_{(2)} \otimes w \otimes v_{(\underline{3})}$ due to invariance. Inserting this into (1.8) gives the desired result. □

We note that this property extends to cyclic permutations of invariant elements in multiple tensor products (just replace V or W by a multiple tensor product).

Given a Hopf algebra H with coquasitriangular structure $\mathcal{R}$, recall from Chapter 2 that a 2-cocycle $\chi : H \otimes H \to \Bbbk$ (Definition 2.1.3) induces a twisted Hopf algebra H_χ with twisted coquasitriangular structure $\mathcal{R}_\chi$ (Proposition 2.1.4). This gives then rise to a twist of comodule categories according to Theorem 2.2.2. In this context, we remark that an H-invariant element of a 2-fold tensor product remains the same under twist if χ satisfies an extra property. This is the following lemma.

Lemma 8.6.2. *In the context of Theorem 2.2.2 let V and W be H-comodules and $v \otimes w \in V \otimes W$ be H-invariant. Assume further that χ satisfies $\chi(a_{(1)} \otimes \mathrm{S}\, a_{(2)}) = \epsilon(a)$. Then ${\sigma_\chi}^{-1}(v \otimes w) = v \otimes w$.*

Proof. First observe that the mentioned property of χ is automatically satisfied by χ^{-1} as well. Then use invariance in the form $v_{(1)} \otimes w_{(1)} \otimes v_{(\underline{2})} \otimes w_{(\underline{2})} = v_{(1)} \otimes \mathrm{S}\, v_{(2)} \otimes v_{(\underline{3})} \otimes w$ and apply ${\sigma_\chi}^{-1}$. □

Finally, let us mention that for commutative Hopf algebras C and H with C a left H-comodule algebra and coalgebra there is a commutative semidirect product Hopf algebra $C \rtimes H$. It is freely generated by C and H as a commutative algebra. Its coproduct on elements of H is the given one, while the coproduct on elements of C is modified to

$$\Delta_{\rtimes}\, c = c_{(1)} c_{(2)\,[1]} \otimes c_{(2)\,[\underline{2}]}. \tag{8.11}$$

Here, brackets denote the coaction to distinguish it from the coproduct. This is the straightforward equivalent to a semidirect product of groups in quantum group language. For the general theory of crossed products of Hopf algebras see [Maj95b].

Chapter 9

ϕ^4-Theory on the Quantum 2-Sphere

The idea that noncommutative geometry might serve as a regulator for quantum field theory is quite old [Sny47]. One approach that has been quite successful is similar in spirit to lattice field theory. The infinite dimensional algebra of functions is replaced by a finite dimensional approximation, so that the path integral is manifestly well-defined and finite. A considerable advantage over a lattice discretisation is the fact that space-time symmetries are not broken. See e.g. [Mad92, Mad95, GKP96, Haw99].

The disadvantage as compared to other conventional quantum field theoretic regularisation methods is the problem of the existence and identification of a "continuum limit". It would thus be desirable to find a regularisation that is both continuous *and* preserves the symmetries. While such a regularisation is probably non-existent, quantum geometry does seem to offer an approach that "almost" delivers on this. The idea is to replace symmetry groups by quantum deformations, so that for a certain value of the deformation parameter the ordinary group (and theory) is recovered. This was advocated e.g. in [Maj90b].

Braided quantum field theory makes possible a systematic realisation of this idea. This is the subject of the current chapter. We consider ϕ^4-theory on the quantum 2-sphere [Pod87] with $SU_q(2)$ symmetry. Ordinary ϕ^4-theory in 2 dimensions is super-renormalisable and has just one basic divergence: the tadpole diagram (Figure 6.7 in Chapter 6). (See e.g. [Zin96] for a treatment of standard ϕ^4-theory.) We demonstrate that this diagram becomes finite for $q > 1$.

We show how the conventional divergence is converted into a divergence in q-parameter space. The form of the divergence leads us to speculate that conventional divergences irrespective of degree could be regularised in this manner. Finally, we consider renormalisation of the theory and obtain a nice diagrammatic interpretation of the renormalisation process.

Since $SU_q(2)$ is compact we make use of the Peter-Weyl decomposition and have a well-defined theory in the sense of Section 7.4. Furthermore, like the commutative counterparts, S^2_q is a quantum homogeneous space under $SU_q(2)$. Thus, we also have the methods of Section 7.3 at our disposal.

Although we deal with a real scalar field theory we work over the complex numbers. This is necessary since the standard q-deformations viewed as deformations of complexifications of compact Lie groups do not restrict to real subalgebras for $q \neq 1$. However, viewing q-deformation purely as a regularisation tool we can always restrict to $\mathbb{R}$ when considering physical quantities living at $q = 1$.

Throughout this chapter, we adopt the convention to denote a Hopf algebra of regular functions by the name of the respective (quantum) group or space.

9.1 The Decomposition of $SU_q(2)$ and S^2_q

To prepare the ground we need to recall the construction of S^2_q as a quantum homogeneous space under $SU_q(2)$ and the Peter-Weyl decomposition of the latter [Koo89, MMN+91]. This will enable us to apply the machinery of the previous chapters.

Recall that $SU_q(2)$ is the compact real form of $SL_q(2)$. It is cosemisimple and there is one simple (right) comodule V_l for each integer dimension, conventionally labelled by a half-integer l such that the dimension is $2l+1$. Thus, the Peter-Weyl decomposition (7.9) is

$$SU_q(2) \cong \bigoplus_{l \in \frac{1}{2}\mathbb{N}_0} (V_l^* \otimes V_l).$$

There is a Hopf $*$-algebra surjection $\pi : SU_q(2) \to U(1)$ corresponding to the diagonal inclusion in the commutative case. This defines the quantum 2-sphere S^2_q as the right homogeneous $*$-space ${}^{U(1)}SU_q(2)$. Under the coaction of $U(1)$ induced by π the comodules V_l decompose into inequivalent one-dimensional comodules. Those are classified by integers i such that the coaction takes the form $v \mapsto v \otimes g^i$. This determines up to normalisation a basis $\{v^{(l)}_n\}$ for V_l, where n are half-integers such that the coaction of $U(1)$ is $v^{(l)}_n \mapsto v^{(l)}_n \otimes g^{-2n}$. It turns out that the indices n take the values $-l, -l+1, \dots, l$. In particular, we find that $V_l^{U(1)}$ is one-dimensional if l is integer and zero-dimensional otherwise. Thus, (7.10) simplifies to

$$S^2_q \cong \bigoplus_{l \in \mathbb{N}_0} V_l$$

as right $SU_q(2)$-comodules. We write the induced (normalisation independent) basis vectors of $SU_q(2)$ as $t^{(l)}_{i\,j} = (f^{(l)}_i \otimes \mathrm{id}) \circ \beta(e^{(l)}_j)$ where $f^{(l)}_n$ is dual to $e^{(l)}_n$ and $\beta : V_l \to V_l \otimes SU_q(2)$

is the coaction of $SU_q(2)$ on V_l. As a subalgebra S_q^2 has the basis $\{t^{(l)}_{0\,i}\}$. The bi-invariant subalgebra ${S_q^2}^{U(1)} = {}^{U(1)}SU_q(2)^{U(1)}$ has the basis $\{t^{(l)}_{0\,0}\}$.

Note that by construction

$$\epsilon\left(t^{(l)}_{m\,n}\right) = \delta_{m,n} \quad \text{and} \quad \Delta\, t^{(l)}_{m\,n} = \sum_k t^{(l)}_{m\,k} \otimes t^{(l)}_{k\,n}.$$

The antipode and $*$-structure of $SU_q(2)$ in this basis are

$$\mathrm{S}\, t^{(l)}_{m\,n} = (-q)^{m-n} t^{(l)}_{-n\,-m}, \quad \left(t^{(l)}_{m\,n}\right)^* = \mathrm{S}\, t^{(l)}_{n\,m} = (-q)^{n-m} t^{(l)}_{-m\,-n},$$

as can be verified by direct calculation from the formulas in [KS97, 4.2.4]. The normalised invariant integral (Haar functional) is simply $\int t^{(l)}_{i\,j} = \delta_{l,0}$. We also need its value on the product of two basis elements

$$\int t^{(l)}_{m\,n} t^{(l')}_{m'\,n'} = \frac{(-1)^{m-n} q^{m+n}}{[2l+1]'_q} \delta_{l,l'} \delta_{m+m',0} \delta_{n+n',0}. \tag{9.1}$$

This can be easily worked out considering the equation $\epsilon(a) = \int a_{(1)}\, \mathrm{S}\, a_{(2)}$ and using the invariance of the integral in the form $b_{(1)} \int a b_{(2)} = \mathrm{S}\, a_{(1)} \int a_{(2)} b$ and $\mathrm{S}\, b_{(2)} \int a b_{(1)} = a_{(2)} \int a_{(1)} b$ on basis elements. We need q-integers for $q \in \mathbb{C}^*$ defined as

$$[n]'_q := \sum_{k=0}^{n-1} q^{n-2k-1} = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

(The second expression is only defined for $q^2 \neq 1$).

Denoting a dual basis of $\{t^{(l)}_{m\,n}\}$ by $\{\tilde{t}^{(l)}_{m\,n}\}$, we observe that $SU_q(2)^*$ becomes an object in $\mathcal{M}^{SU_q(2)}$ with evaluation $\mathrm{ev} : SU_q(2) \otimes SU_q(2)^* \to \mathbb{C}$ and coevaluation $\mathrm{coev} : \mathbb{C} \to SU_q(2)^* \otimes SU_q(2)$ by the coaction $\tilde{t}^{(l)}_{m\,n} \mapsto \sum_k \tilde{t}^{(l)}_{m\,k} \otimes \mathrm{S}^{-1}\, t^{(l)}_{n\,k}$.

In the commutative case $q = 1$, the basis $\{t^{(l)}_{m\,n}\}$ becomes the usual basis of regular functions (i.e., matrix elements of representations) on $SU(2)$ (see e.g. [VK91, Chapter 6] to whose conventions we conform in this case). The restriction to $\{t^{(l)}_{0\,n}\}$ recovers nothing but (a version of) the spherical harmonics on S^2. In particular, we notice that the zonal spherical functions can be expressed in terms of Legendre polynomials $t^{(l)}_{0\,0}(\phi, \theta, \psi) = P_l(\cos\theta)$, where ϕ, θ, ψ are the Euler angles on $SU(2)$ (see [VK91, Chapter 6]). From the orthogonality relation of the Legendre polynomials, the fact that their only common value is at $P_l(1) = 1$, and considering that $\theta = 0$ denotes a pole of $SU(2)$, we find that the delta function at the identity of $SU(2)$ restricted to S^2 can be represented as

$$\delta_0(\phi, \theta) = \sum_l (2l+1)\, P_l(\cos\theta) = \sum_l (2l+1)\, t^{(l)}_{0\,0}(\phi, \theta). \tag{9.2}$$

For calculations we need the functionals u and v defined for a coquasitriangular structure $\mathcal{R}$ as (see e.g. [Maj95b])

$$u(a) := \mathcal{R}(a_{(2)} \otimes \mathrm{S}\, a_{(1)}), \quad v(a) := \mathcal{R}(a_{(1)} \otimes \mathrm{S}\, a_{(2)}). \tag{9.3}$$

For $SU_q(2)$ in our basis they are

$$u(t^{(l)}_{m\,n}) = \delta_{m,n}\, q^{-2l(l+1)+2m}, \quad v(t^{(l)}_{m\,n}) = \delta_{m,n}\, q^{-2l(l+1)-2m}. \tag{9.4}$$

We also note that property (7.8) is satisfied, i.e.,

$$\mathcal{R}\left(t^{(l)}_{0\,0} \otimes t^{(l)}_{i\,j}\right) = \delta_{i,j} = \mathcal{R}\left(t^{(l)}_{i\,j} \otimes t^{(l)}_{0\,0}\right). \tag{9.5}$$

See Appendix 9.A for a derivation of (9.4) and (9.5).

9.2 The Free Propagator

In ordinary quantum field theory the free propagator is defined by the free action. For a Euclidean massive real scalar field theory on a manifold M it takes the form

$$S_0(\phi) = \frac{1}{2}\int_M dx\, \phi(x)(m^2 - \Delta_M)\phi(x),$$

where Δ_M is the Laplace operator on M and m is the mass of the field. Define $\mathsf{L} := m^2 - \Delta_M$. Applying equation (6.13) we obtain in the more abstract notation of Section 6.1.1

$$\gamma = \left(\mathrm{id} \otimes \int_M\right) \circ (\mathrm{id} \otimes \cdot) \circ (\mathrm{coev} \otimes \mathsf{L}), \tag{9.6}$$

which we take as the defining equation for γ.

In the noncommutative case $q \neq 1$ we still have an integral on our "manifold" $M = S^2_q$. We further need an analogue of the Laplace operator. We note that by the duality of $SU_q(2)$ with the quantum enveloping algebra $\mathsf{U}_q(\mathfrak{sl}_2)$, a central element of the latter defines an invariant operator on $SU_q(2)$-comodules. A natural choice is the quantum Casimir element which we define as

$$C_q = EF + \frac{(K-1)q^{-1} + (K^{-1}-1)q}{(q-q^{-1})^2}.$$

Here K, K^{-1}, E, and F are the generators of $\mathsf{U}_q(\mathfrak{sl}_2)$ (see Appendix 9.A). C_q differs from quantum Casimir elements considered elsewhere (see e.g. [MMN+91] or [KS97]) only by a q-multiple of the identity. The eigenvalue of C_q on V_l is $[l]'_q[l+1]'_q$ so that we get exactly the (negative of the) usual Laplace operator for $q = 1$. Including a mass term we set

$$\mathsf{L} = C_q + m^2.$$

Thus, the eigenvalue of L on V_l is

$$\mathsf{L}_l = [l]'_q[l+1]'_q + m^2.$$

We determine γ^{-1} according to (9.6). Using (9.1) we find

$$\gamma\left(t^{(l)}_{0\,i}\right) = \sum_{m,j} \tilde{t}^{(m)}_{0\,j} \int t^{(m)}_{0\,j}\, \mathsf{L}\left(t^{(l)}_{0\,i}\right) = [2l+1]'^{\,-1}_q\, \mathsf{L}_l\,(-q)^{-i}\,\tilde{t}^{(l)}_{0\,-i}.$$

Inverting we obtain

$$\gamma^{-1}\left(\tilde{t}^{(l)}_{0\,i}\right) = [2l+1]'_q\, \mathsf{L}^{-1}_l\,(-q)^{-i}\, t^{(l)}_{0\,-i}.$$

Now, we are ready to determine the free propagator according to Corollary 6.2.1.

$$\begin{aligned}
\mathcal{Z}^2 &= \sum_{l,k} (\mathrm{id}\otimes\gamma^{-1})\circ\psi\left(\tilde{t}^{(l)}_{0\,k}\otimes t^{(l)}_{0\,k}\right)\\
&= \sum_{l,i,j,k} t^{(l)}_{0\,i}\otimes\gamma^{-1}\left(\tilde{t}^{(l)}_{0\,j}\right)\mathcal{R}\left(\mathrm{S}^{-1}\,t^{(l)}_{k\,j}\otimes t^{(l)}_{i\,k}\right)\\
&= \sum_{l,i,j} t^{(l)}_{0\,i}\otimes\gamma^{-1}\left(\tilde{t}^{(l)}_{0\,j}\right)\,u\left(t^{(l)}_{i\,j}\right)\\
&= \sum_{l,i} [2l+1]'_q\,\mathsf{L}^{-1}_l\,q^{-2l(l+1)}\,(-q)^i\,t^{(l)}_{0\,i}\otimes t^{(l)}_{0\,-i}.
\end{aligned}$$

Using invariant reduction (Proposition 7.3.2) we find

$$\widetilde{\mathcal{Z}}^2 = \sum_l [2l+1]'_q\,\mathsf{L}^{-1}_l\,q^{-2l(l+1)}\,t^{(l)}_{0\,0} \tag{9.7}$$

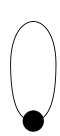
to be the reduced form of the propagator as an element of $S^{2\,U(1)}_q$. In the commutative case ($q=1$) we can rewrite (9.7) as

$$\widetilde{\mathcal{Z}}^2\,|_{q=1} = (m^2-\Delta)^{-1}\delta_0$$

by comparison with (9.2). This is the familiar expression from ordinary quantum field theory.

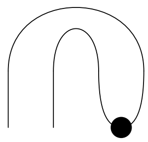
9.3 Interactions

We proceed to evaluate the order 1 contribution of the ϕ^4-interaction to the 2-point function. The corresponding diagrams are depicted in Figure 6.6 (see Section 6.2.2). Since the property (7.8) holds in $SU_q(2)$ the diagrams simplify to those of Figure 7.6 (see Section 7.3.2). The disconnected loop comes out as

$\delta_{\text{loop}} :=$ 

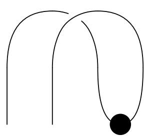
$$= \sum_l \frac{[2l+1]'_q}{[l]'_q[l+1]'_q+m^2}\,q^{-2l(l+1)}. \tag{9.8}$$

(Just apply the counit to (9.7).) The connected diagram in the right-hand summand of Figure 7.6 is (in reduced form)



$$
\begin{aligned}
&= \left(\mathrm{id} \otimes \epsilon \otimes \int\right) \circ (\mathrm{id}^2 \otimes \cdot) \circ (\mathrm{id} \otimes \mathcal{Z}^2 \otimes \mathrm{id}) \circ \mathcal{Z}^2 \\
&= \sum_{l,m,i,j} \alpha_l \alpha_m \, t^{(l)}_{0\,i} \, \epsilon\left(t^{(m)}_{0\,j}\right) \int \mathrm{S}\, t^{(m)}_{j\,0} \; \mathrm{S}\, t^{(l)}_{i\,0} \\
&= \sum_l \alpha_l^2 \, [2l+1]_q'^{\,-1} \, t^{(l)}_{0\,0},
\end{aligned}
$$

with $\alpha_l := [2l+1]_q' \, \mathsf{L}_l^{-1} \, q^{-2l(l+1)}$. We have used $\mathcal{Z}^2$ as reconstructed from its reduced form (9.7), the property $\int \circ \mathrm{S} = \int$ of the integral, and (9.1). The connected diagram in the left-hand summand of Figure 7.6 is (in reduced form)



$$
\begin{aligned}
&= \left(\mathrm{id} \otimes \epsilon \otimes \int\right) \circ (\mathrm{id}^2 \otimes \cdot) \circ (\mathrm{id} \otimes \psi^{-1} \otimes \mathrm{id}) \circ (\mathcal{Z}^2 \otimes \mathcal{Z}^2) \\
&= \sum_{l,m,i,j,k,n} \alpha_l \alpha_m \, t^{(l)}_{0\,i} \, \epsilon\left(t^{(m)}_{0\,k}\right) \int \mathrm{S}\, t^{(l)}_{n\,0} \; \mathrm{S}\, t^{(m)}_{j\,0} \, \mathcal{R}^{-1}\left(t^{(m)}_{k\,j} \otimes \mathrm{S}\, t^{(l)}_{i\,n}\right) \\
&= \sum_{l,m,i,j,n} \alpha_l \alpha_m \, t^{(l)}_{0\,i} \int t^{(m)}_{j\,0} \, t^{(l)}_{n\,0} \, \mathcal{R}\left(t^{(m)}_{0\,j} \otimes t^{(l)}_{i\,n}\right) \\
&= \sum_{l,m,i,j,k} \alpha_l \alpha_m \, t^{(l)}_{0\,i} \int t^{(m)}_{k\,0} \, t^{(l)}_{i\,0} \, \mathcal{R}\left(t^{(m)}_{0\,j} \otimes \mathrm{S}\, t^{(m)}_{j\,k}\right) \\
&= \sum_{l,m,i,k} \alpha_l \alpha_m \, t^{(l)}_{0\,i} \int t^{(m)}_{k\,0} \, t^{(l)}_{i\,0} \, v\left(t^{(m)}_{0\,k}\right) \\
&= \sum_l \alpha_l^2 \, [2l+1]_q'^{\,-1} \, q^{-2l(l+1)} \, t^{(l)}_{0\,0}.
\end{aligned}
$$

We have also used the invariance of the integral in the form $(\int ab_{(2)})b_{(1)} = (\int a_{(2)}b) \, \mathrm{S}\, a_{(1)}$ in the third equality. Thus, the (reduced) 2-point function up to order 1 comes out as

$$
\begin{aligned}
\widetilde{\mathcal{Z}}^2_{\mathrm{int}} = \sum_l & [2l+1]_q' \, \mathsf{L}_l^{-1} \, q^{-2l(l+1)} \, t^{(l)}_{0\,0} \\
& \left(1 - 6\,\lambda\,\delta_{\mathrm{loop}} \, \mathsf{L}_l^{-1} q^{-2l(l+1)} (1 + q^{-2l(l+1)}) + \mathcal{O}(\lambda^2)\right).
\end{aligned}
\tag{9.9}
$$

In the commutative case ($q = 1$), we know that the order 1 contribution (given by the tadpole diagram in Figure 6.7) is divergent. We can easily see where this divergence comes from. The loop contribution (9.8)

$$
\delta_{\mathrm{loop}}|_{q=1} = \sum_l \frac{2l+1}{l(l+1)+m^2} \tag{9.10}
$$

is infinite. However, at $q > 1$ it becomes finite! We are truly able the regularise the tadpole diagram. Let us identify the divergence in q-space. For $q > 1$ we can find both an upper and a lower bound for (9.8) of the form

$$\mathsf{const} + \int_1^\infty dl\, \frac{2}{l} q^{-2l^2},$$

where const does not depend on q (but may depend on m^2). Setting $q = e^{2h^2}$ with $h > 0$ we find

$$\delta_{\text{loop}}|_{q>1} = \frac{1}{h} + \mathcal{O}(1).$$

The conventional divergence of (9.10) is only logarithmic in l. What would happen with higher divergences? It seems natural to assume that they would give rise to terms like

$$\sum_l [l]_q'^{\,n}\, q^{-2l(l+1)}.$$

But this converges in the domain $q > 1$ for any n. We can even apply the very same discussion of the divergence in q-space as above. The nature of the divergence in q-space does not seem to be affected by the degree of the ordinary (commutative) divergence at all. This suggests that q-regularization in our framework is powerful indeed. Compare this, e.g., to dimensional regularization which can only handle logarithmic divergences.

Reviewing our calculations of $\mathcal{Z}^2$ and $\mathcal{Z}^2_{\text{int}}$ we find that the crucial factor of $q^{-2l(l+1)}$ is caused by the braiding. Thus, the braiding and not the mere noncommutativity appears to be essential for the regularisation.

9.4 Renormalisation

Ordinarily, ϕ^4-theory in dimension 2 is super-renormalisable. The only basic divergent diagram is the tadpole (Figure 6.7). Thus, if in a given diagram we separate out all loops from "tadpole vertices" according to Figures 7.4 and 7.5 (as we did in the last section for the tadpole), the remaining diagram is finite at $q = 1$. The "stripped" vertices have just become ϕ^2-vertices. Notice however, that from a rigorous point of view this can only be done if the whole diagram is finite. While we have seen that the tadpole diagram alone becomes finite for $q > 1$, it is conceivable that certain diagrams that converge at $q = 1$ would diverge at $q > 1$. This might be due to the introduction of factors like $q^{2l(l+1)}$ into summations over l. The expression (9.9) suggests, however, that this does not happen, but rather that all q-factors introduced in summations have negative exponent. We shall assume this in the following.

Let us now see what renormalisation looks like in our framework. To renormalise the coupling we set

$$\lambda' := \lambda\,\delta_{\mathrm{loop}}$$

at $q > 1$ and fix λ' as our new coupling. Then, given any diagram we perform the stripping of tadpole loops as described above and the separated loops cancel with $\delta_{\mathrm{loop}}^{-1}$ in λ'. As q goes to 1 those loops are just ϕ^2 vertices. However, in this limit, any other vertex vanishes due to a factor of $\delta_{\mathrm{loop}}^{-1}$. Thus, we remain with only ϕ^2-vertices corresponding to an additional mass term. The effective shift in mass is

$$m^2 \to m^2 + 12\lambda'.$$

What one usually does in ϕ^4-theory is to renormalise the mass to cancel the divergence. Thus, instead of redefining λ we set

$$\tilde{m}^2 := m^2 - 12\lambda\,\delta_{\mathrm{loop}}.$$

to be the new mass. Of course this equation must not be understood literally in our framework since we cannot write the free and the interacting part of the action together. Instead, we introduce (at $q > 1$) a ϕ^2 vertex with a factor of $-6\,\lambda\,\delta_{\mathrm{loop}}$ in front. Writing all diagrams of a given order in λ, all tadpoles are canceled in the limit $q \to 1$, and only finite non-tadpole vertices remain.

9.A Appendix: Coquasitriangular Structure of $SU_q(2)$

In this appendix we provide the required formulas for the coquasitriangular structure of $SU_q(2)$ in our Peter-Weyl basis. We use the context of Section 9.1. Definitions and results that are just stated are standard and can be found e.g. in [Maj95b] or [KS97].

The Hopf algebra $\mathsf{U}_q(\mathfrak{sl}_2)$ is defined over $\mathbb{C}$ for $q \in \mathbb{C}^*$ and $q^2 \neq 1$ with generators E, F, K, K^{-1} and relations

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$
$$KK^{-1} = K^{-1}K = 1, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}},$$
$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$
$$\Delta(K) = K \otimes K, \quad \epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0,$$
$$\mathrm{S}(K) = K^{-1}, \quad \mathrm{S}(E) = -EK^{-1}, \quad \mathrm{S}(F) = -KF.$$

$\mathsf{U}_q(\mathfrak{sl}_2)$ and $SU_q(2)$ are non-degenerately paired. Thus, actions of $\mathsf{U}_q(\mathfrak{sl}_2)$ and coactions of $SU_q(2)$ on finite dimensional vector spaces are dual to each other. In particular, the simple comodule V_l of $SU_q(2)$ is a simple module of $\mathsf{U}_q(\mathfrak{sl}_2)$. By the representation theory of $\mathsf{U}_q(\mathfrak{sl}_2)$ it has a basis $\{w_i\}$, $i = -l, -l+1, \dots, l$ such that

$$\begin{gathered} K \triangleright w_m = q^{2m} w_m, \quad E \triangleright w_m = ([l-m]'_q [l+m+1]'_q)^{1/2} w_{m+1} \\ F \triangleright w_m = ([l+m]'_q [l-m+1]'_q)^{1/2} w_{m-1}. \end{gathered} \tag{9.11}$$

$\mathsf{U}_q(\mathfrak{sl}_2)$ has an h-adic version $\mathsf{U}_h(\mathfrak{sl}_2)$ defined over $\mathbb{C}[[h]]$ correspondingly with $q = e^h$ and an additional generator H so that $q^H = K$. It has the quasitriangular structure

$$R = q^{(H \otimes H)/2} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(1-q^{-2})^n}{[n]'_q!} E^n \otimes F^n. \tag{9.12}$$

The elements (define $R^{(1)} \otimes R^{(2)} = R$)

$$u' = (\mathrm{S}\, R^{(2)}) R^{(1)}, \quad v' = R^{(1)}\, \mathrm{S}\, R^{(2)} \tag{9.13}$$

act on V_l as [Maj95b, Proposition 3.2.7]

$$u' \triangleright w_m = q^{-2l(l+1)+2m} w_m, \quad v' \triangleright w_m = q^{-2l(l+1)-2m} w_m. \tag{9.14}$$

The coquasitriangular structure $\mathcal{R}$ of $SU_q(2)$ is given by the duality with $\mathsf{U}_q(\mathfrak{sl}_2)$ from the quasitriangular structure R of $\mathsf{U}_h(\mathfrak{sl}_2)$. Using

$$u(a_{(1)}) a_{(2)} = \mathrm{S}^2\, a_{(1)} u(a_{(2)}) \quad \text{and} \quad v(a_{(1)})\, \mathrm{S}^2\, a_{(2)} = a_{(1)} v(a_{(2)})$$

we find

$$u\left(t^{(l)}_{m\,n}\right) = \delta_{m,n}\, q^{2(m-k)} u\left(t^{(l)}_{k\,k}\right), \quad v\left(t^{(l)}_{m\,n}\right) = \delta_{m,n}\, q^{2(k-m)} v\left(t^{(l)}_{k\,k}\right). \tag{9.15}$$

Since the definitions (9.3) and (9.13) are dual to each other we can use

$$g \triangleright v_n = \sum_m v_m \langle g, t^{(l)}_{m\,n} \rangle, \quad g \in \mathsf{U}_q(\mathfrak{sl}_2)$$

to compare (9.14) with (9.15). We find (9.4) and infer that w_i is (a multiple of) v_i. With the latter, the pairing between $\mathsf{U}_q(\mathfrak{sl}_2)$ and $SU_q(2)$ comes out from (9.11) as

$$\begin{gathered} \langle K, t^{(l)}_{m\,n} \rangle = \delta_{m,n}\, q^{2n}, \quad \langle E, t^{(l)}_{m\,n} \rangle = \delta_{m,n+1} ([l-n]'_q [l+n+1]'_q)^{1/2}, \\ \langle F, t^{(l)}_{m\,n} \rangle = \delta_{m,n-1} ([l+n]'_q [l-n+1]'_q)^{1/2}. \end{gathered}$$

Note also $\langle H, t^{(l)}_{m\,n} \rangle = \delta_{m,n} 2n$ in the h-adic version. With this pairing and (9.12) we easily verify the property (9.5).

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