BF Theory and Spinfoam Models

Max Dohse

Literature:

talk closely follows J. Baez' paper:

An Introduction to Spin Foam Models of BF Theory and Quantum Gravity [arxiv:gr-qc/9905087]

plus sections 1.1-1.6, 5.1 and 7.2 of R. Oeckl's book:

Discrete Gauge Theory, Imperial College Press, 2005

helpful publications also consulted:

- J. Baez: Spin Foam Models [arxiv:gr-qc/9709052]
- D. Oriti: Spacetime Geometry from Algebra: Spin Foam Models for non-perturbative Quantum Gravity [arxiv:gr-qc/0106091]
- A. Perez: Introduction to Loop Quantum Gravity and Spin Foams [arxiv:gr-qc/0409061]

Outline

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- gauge group: Lie group G,
  with Lie algebra $\mathfrak{g}$ equipped with invariant nondegenerate bilinear form $\langle\cdot\,,\,\cdot\rangle_{\mathfrak{g}}$
- spacetime: smooth, oriented manifold M
- choose principal G-bundle $P \xrightarrow{\pi} M$,
  vector bundle associated to $P \xrightarrow{\pi} M$ via adjoint action of G on $\mathfrak{g}$ is $\mathrm{ad}(P) \xrightarrow{\pi_a} M$
  with $\mathrm{ad}(P) = (P \times \mathrm{Ad}(G))/G$ and $\mathrm{Ad}(G)$ the adjoint representation of G

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- basic fields of theory:
  $A =$ connection on P
  $F = dA + A \wedge A$ is curvature of A: an $\mathrm{ad}(P)$-valued 2-form on M
  $E = \mathrm{ad}(P)$-valued $(n-2)$-form on M

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- pick local trivialization, can think of A, F, E as $\mathfrak{g}$-valued 1,2,(n - 2)-forms on M, with local coordinates $\{x^j\}$ on M and basis $\{e_m\}$ of $\mathfrak{g} \cong T_{\mathbb{1}} G$ we can write

$$
\begin{aligned}
A &= A_a^m \, dx^a \otimes e_m \\
F &= F_{b_1,b_2}^l \, dx^{b_1} \wedge dx^{b_2} \otimes e_l \\
E &= E_{j_1,\ldots,j_{n-2}}^k \, dx^{j_1} \wedge \ldots \wedge dx^{j_{n-2}} \otimes e_k
\end{aligned}
$$

- Lagrangian of BF theory: $\mathcal{L} = \text{Tr}\,(E \wedge F)$

  Tr $(E \wedge F)$ is n-form constructed by taking wedge product of form parts of E and F and using bilinear form $\langle \cdot\,,\,\cdot \rangle_{\mathfrak{g}}$ to pair $\mathfrak{g}$-valued parts

$$\text{Tr}\,(E \wedge F) = E^k_{j_1,\ldots,j_{n-2}}\, F^l_{b_1,b_2}\, \langle e_k, e_l \rangle_{\mathfrak{g}}\, dx^{j_1} \wedge \ldots \wedge dx^{j_{n-2}} \wedge dx^{b_1} \wedge dx^{b_2}$$

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- BF-action invariant under E gauge: $E \mapsto E + d_A\eta$ for some $\mathfrak{g}$-valued $(n-3)$-form η
- since A flat, for any E with $d_A E = 0$ there exists an η such that locally $E = d_A\eta$ since locally all closed forms are exact

- 3d-GR is special case of BF theory: $n = 3$ and $G = \mathsf{SO}(2,1)$
  $\langle \cdot\,,\ \cdot \rangle_{\mathsf{so}(2,1)}$ is minus its Killling form
- if 1-form $E:\ TM \rightarrow \mathsf{ad}(P)$ is bijective, then nondegenerate Lorentzian metric defined by
  $g(x)(v,w) \overset{\text{def.}}{=} \langle E(x)v,\, E(x)w \rangle_{\mathsf{so}(2,1)} \qquad \forall\, v,w \in T_xM$

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- in 3d spacetime vacuum Einstein equations just say: metric is flat
- many different A and E fields correspond to same metric,
  but all related by gauge transformations
- thus in 3d spacetime, BF theory with $G = \mathsf{SO}(2,1)$
  is alternate formulation of Lorentzian GR without matter fields
- with $G = \mathsf{SO}(3)$ we obtain Riemannian GR: much easier to quantize than Lorentzian GR

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  $S =$ smooth, oriented $(n-1)$-dim. manifold representing space
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  point therein consists of connection A on $P|_S$ and ad$(P|_S)$-valued $(n-2)$-form E on S

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- field equations put constraints on initial data A and E on time-zero slice S:
  $0 = B = dA + A \wedge A \qquad d_A E = 0$
- to deal with these constraints: apply symplectic reduction
  to kinematical phase space $T^*\mathcal{A}$ to obtain physical phase space

- constraint $d_A E = 0$ called Gauss law, generates action of gauge transformations on $T^*\mathcal{A}$
- symplectic reduction with respect to Gauss constraint:
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- points in this physical phase space correspond to physical states of *classical* BF theory

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- $\text{Fun}(\mathcal{A})$ = algebra of functionals on space of connections $\mathcal{A}$ of form

$$
\Psi_{f,\Gamma}[A] = f(\{h_{\gamma_k}[A] \mid \gamma_k \in \Gamma\}) \qquad h_{\gamma_k}[A] \stackrel{\text{def.}}{=} \mathcal{P}\exp\int_{\gamma_k} A
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- define inner product on Fun($\mathcal{A}$), complete it, obtain Hilbert space $\mathcal{H}_{\text{kin}} = L^2(\mathcal{A})$

$$\langle \Psi_{f_1,\Gamma_1}, \Psi_{f_2,\Gamma_2} \rangle \overset{\text{def.}}{=} \int \Big(\prod_r^{e_r \in \Gamma_{12}} dh_{e_r} \Big) \overline{f_1(\{h_{\gamma_k}[A] \mid {}^{\gamma_k \in \Gamma_{12}}_{\gamma_k \subseteq \tilde{\gamma}_k \in \Gamma_1}\})}\, f_2(\{h_{\gamma_k}[A] \mid {}^{\gamma_k \in \Gamma_{12}}_{\gamma_k \subseteq \tilde{\gamma}_k \in \Gamma_2}\})$$

$\Gamma_1, \Gamma_2 \subseteq \Gamma_{12}$ and dh = normalized Haar measure of gauge group G

- finite collection of real-analytic paths $\gamma_j : \ [0,1] \ \rightarrow \ S$ = **graph in** S
  if they are embedded in S and intersect only at their endpoints (if at all),
  paths = **edges** and endpoints of paths = **vertices**,
  edge γ_j is **outgoing** from a vertex v if $v \ = \ \gamma_j(0)$ and **incoming** to v if $v \ = \ \gamma_j(1)$

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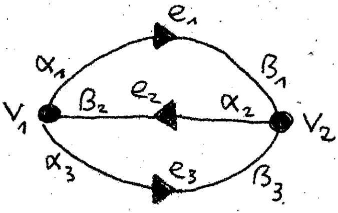
- (closed) **spin network** $N \; = \; (\Gamma, \rho, \iota)$ in S with symmetry group G is triple consisting of:
  1. graph Γ in S
  2. edge-labeling ρ of each edge e of Γ by an irrep ρ_e of G
  3. vertex-labeling ι of each vertex v of Γ by intertwiner ι_v:

$$\iota_v : \quad \left(\rho_{e_1^{\text{in}}(v)} \otimes \ldots \otimes \rho_{e_{n(v)}^{\text{in}}(v)}\right) \; \longrightarrow \; \left(\rho_{e_1^{\text{out}}(v)} \otimes \ldots \otimes \rho_{e_{o(v)}^{\text{out}}(v)}\right)$$

- $\mathrm{Fun}(\mathcal{A}/\mathcal{G})$ = algebra of gauge-invariant functionals in $\mathrm{Fun}(\mathcal{A})$, completing in above norm yields gauge-invariant Hilbert space $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$, this space is spanned by **spin network states**:

$$\Psi_N[A] \stackrel{\text{def.}}{=} \prod_r^{e_r \in N} \rho_{e_r}(h_{e_r}[A])^{\alpha_r}_{\beta_r} \prod_j^{v_j \in N} (\iota_{v_j})^{\beta_{j_1}\ldots\beta_{j_n(j)}}_{\alpha_{j_1}\ldots\alpha_{j_o(j)}}$$



$$\Psi_N[A] = \rho_{e_1}(h_{e_1}[A])^{\alpha_1}_{\beta_1}\ \rho_{e_2}(h_{e_2}[A])^{\alpha_2}_{\beta_2}\ \rho_{e_3}(h_{e_3}[A])^{\alpha_3}_{\beta_3}\ (\iota_{v_1})^{\beta_2}_{\alpha_1\alpha_3}\ (\iota_{v_2})^{\beta_1\beta_3}_{\alpha_2}$$

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- classically imposing no-curvature constraint yields physical phase space $T^*(\mathcal{A}_0/\mathcal{G})$,
  however in most cases no natural measure on $(\mathcal{A}_0/\mathcal{G})$,
  Hilbert space cannot be defined unambiguously
- have to accept mere vector space instead of Hilbert space,
  every functional in $\mathrm{Fun}(\mathcal{A}/\mathcal{G})$ restricts to one in $\mathrm{Fun}(\mathcal{A}_0/\mathcal{G})$,
  the space of gauge-invariant functionals on flat connections,
  call elements of this space **physical states**,
  even if there is no natural measure on $\mathcal{A}_0/\mathcal{G}$

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- reparametrize edge of spin network in S by orientation-preserving diffeomorphism of unit interval
- reverse orientation of an edge, while dualizing irrep and appropriately dualizing intertwiners at its vertices
- subdivide edge into two edges labeled by same irrep, inserting vertex with identity intertwiner
- erase edges labeled by trivial irrep

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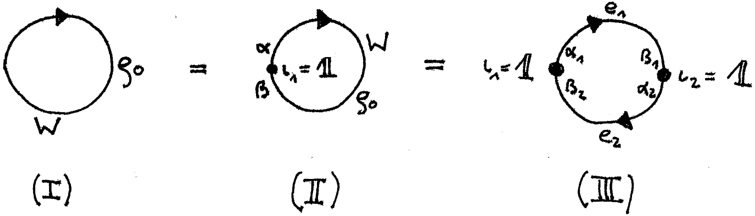
- spin network N defines operator $\widehat{\Psi}_N[A]$
  acting as multiplication by $\Psi_N[A]$ on elements of Fun$(\mathcal{A}/\mathcal{G})$
- $\Psi_N[A] \in$ Fun$(\mathcal{A}/\mathcal{G})$ bounded $\Rightarrow \widehat{\Psi}_N[A]$ = bounded operator on $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$
  called **spin network observable**
- $\Psi_N[A] \in$ Fun$(\mathcal{A}_0/\mathcal{G})$ also bounded $\Rightarrow \widehat{\Psi}_N[A]$ = bounded operator on $\mathcal{H}_{\text{phys}} = L^2(\mathcal{A}_0/\mathcal{G})$
- spin network observables on $\mathcal{H}_{\text{gauge}}$ not invariant under homotopies of graph Γ of N,
  but spin network operators on $\mathcal{H}_{\text{phys}}$ are invariant and satisfy skein relations

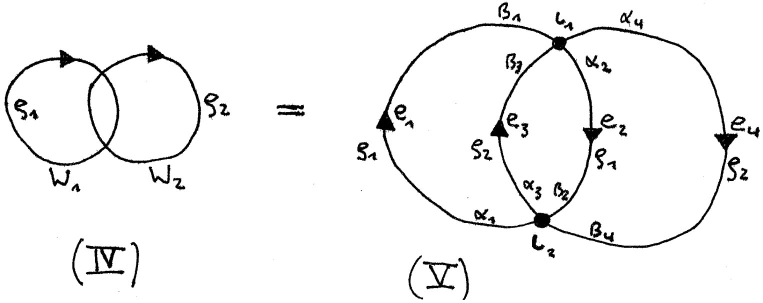
- any product of Wilson loops = finite linear combination of spin network observables
- example: one Wilson loop as spin network observable



$$(\mathrm{I}) \overset{\text{def.}}{=} \mathrm{Tr}\,\rho_0(h_W[A]) \;=\; (\mathrm{II}) \;=\; \rho_0(h_W[A])^{\alpha}_{\beta}\; \overbrace{\delta_{\alpha\beta}}^{\iota_1}$$

$$\begin{aligned}
= (\mathrm{III}) &= \rho_0(h_{e_1}[A])^{\alpha_1}_{\beta_1}\;\rho_0(h_{e_2}[A])^{\alpha_2}_{\beta_2}\;\overbrace{\delta_{\alpha_1\beta_2}}^{\iota_1}\;\overbrace{\delta_{\alpha_2\beta_1}}^{\iota_2}\\
&= \rho_0(h_{e_1}[A])^{\alpha_1}_{\alpha_2}\;\rho_0(h_{e_2}[A])^{\alpha_2}_{\alpha_1}\\
&= \Big(\rho_0(h_{e_1}[A])\,\rho_0(h_{e_2}[A])\Big)^{\alpha_1}_{\alpha_1}\\
&= \Big(\rho_0(h_W[A])\Big)^{\alpha_1}_{\alpha_1}
\end{aligned}$$

- example: two Wilson loops as spin network observable



$$(\text{IV}) = \text{Tr}\ \rho_1(h_{W_1}[A]) \cdot \text{Tr}\ \rho_2(h_{W_2}[A])$$

$$= (\text{V}) = \rho_1(h_{e_1}[A])^{\alpha_1}_{\beta_1}\ \rho_1(h_{e_2}[A])^{\alpha_2}_{\beta_2}\ \rho_2(h_{e_3}[A])^{\alpha_3}_{\beta_3}\ \rho_2(h_{e_4}[A])^{\alpha_4}_{\beta_4}\ \overbrace{\delta_{\alpha_2\beta_1}\ \delta_{\alpha_4\beta_3}}^{\iota_1}\ \overbrace{\delta_{\alpha_1\beta_2}\ \delta_{\alpha_3\beta_4}}^{\iota_2}$$

$$= \Big(\rho_1(h_{e_1}[A])\,\rho_1(h_{e_2}[A])\Big)^{\alpha_1}_{\alpha_1} \cdot \Big(\rho_2(h_{e_3}[A])\,\rho_2(h_{e_4}[A])\Big)^{\alpha_3}_{\alpha_3}$$

$$= \Big(\rho_1(h_{W_1}[A])\Big)^{\alpha_1}_{\alpha_1} \cdot \Big(\rho_2(h_{W_2}[A])\Big)^{\alpha_3}_{\alpha_3}$$

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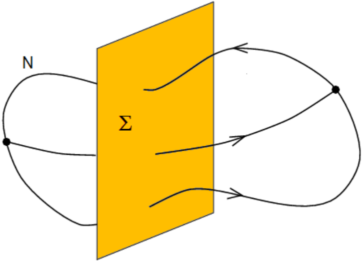
- $e = \mathfrak{g}$-valued function on Σ, $\quad d^{n-2}x = (n-2)$-form on $\Sigma \quad \Rightarrow \quad E|_\Sigma = e\, d^{n-2}x$
  gauge-invariant functional of E provided by

$$E(\Sigma) \overset{\text{def.}}{=} \int_\Sigma \underbrace{d^{n-2}x \, \sqrt{\langle e, e\rangle_{\mathfrak{g}}}}_{|E|}$$

- quantizing $E(\Sigma)$ gives self-adjoint operator $\widehat{E}(\Sigma)$ on $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$

- spin network N in space S intersects $(n-2)$-submanifold Σ in finitely many points:



- $\widehat{E}(\Sigma)$ acts on $\Psi_N \in \mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$ by

$$\widehat{E}(\Sigma)\,\Psi_N = \sum_k \sqrt{C(\rho_k)}\;\Psi_N$$

$C(\rho_k) =$ Casimir of irrep ρ_k

- physical significance of spin network edges: represent quantized flux lines of E-field

- 3-dim. BF theory with $G =$ SU(2), SO(3) is formulation of 3-dim. Riemannian GR, $\widehat{E}(\Sigma)$ can be interpreted as length of curve Σ, in 4-dim. BF theory as area of surface Σ
- irreps of SU(2) correspond to spins $j = 0, \frac{1}{2}, 1, \dots,$
  Casimir of the spin-j irrep is $j(j+1)\mathbb{1}$,
  edge of spin j contributes length/area $\sqrt{j(j+1)}$ to any curve/surface it crosses
- consequence: length/area in 3/4-dim. Riemannian QG have discrete spectra
- application: blackhole entropy:
  associate degrees of freedom of event horizon to points of intersection with spin networks, then derive Bekenstein-Hawking entropy proportional to area of event horizon

Outline

- quantization of BF theory via path integral: consider partition function

$$\mathcal{Z} = \int \mathcal{D}A \int \mathcal{D}E \, \exp i \int_M \mathrm{Tr}\,(E \wedge F)$$

- to give sense to this expression, introduce discretization,
hope path integral becomes well defined
- define theory only on discrete, finite set of points in spacetime,
which we call **vertices**, assume fields to vary slowly between vertices
- connection A, implemented in discrete version by considering parallel transports between vertices, need pathes connecting vertices along which parallel transport takes place, call pathes **edges**
- implement curvature in form of holonomies along closed loops formed by edges,
specify loop by surface bounded by its edges, call surface **face**
- vertices, edges and faces together = **lattice** L on spacetime

- equip L with orientation, define discretized connection A on L by assigning group element g_e to each edge e

$$g_e = \mathcal{P} \exp - \int_e A$$

- measure on space of connections under discretization becomes product over Haar measure of all edges:

$$\mathcal{D}A \rightsquigarrow \prod_e dg_e$$

- define discretized gauge transformation by assigning element h_v of G to each vertex v
- discretized curvature represented by group element g_f (defined up to conjugation)

$$g_f \overset{\text{def.}}{=} g_{e_1}^{o_1} \dots g_{e_k}^{o_k}$$

- under discretized gauge transformations g_f also transforms by conjugation
- thus gauge invariant information about curvature contained in conjugacy class of g_f, to obtain gauge-invariant Lagrangian, apply class function σ to g_f

- g_f expressed by approximately constant curvature 2-form F on f by

$$g_f \approx \exp(-a_f^2 F_{\mu\nu} + \mathcal{O}(a_f^4))$$

- face and dual face in one-to-one correspondence, can associate discretized field $\widetilde{E} = \star E$ to faces f
- for action on M with slowly varying curvature we obtain

$$\begin{aligned}
S &= \int_M \mathrm{Tr}\,(E \wedge F) = \int_M \mathrm{Tr}\,(\star\widetilde{E} \wedge F) \\
&\approx \sum_C a_C^n \sum_{(\mu,\nu)}^{C} \mathrm{Tr}\,(\widetilde{E}_{\mu\nu} F_{\mu\nu}) \\
&= \sum_f \mathrm{Tr}\,(\widetilde{E}^f_{\mu\nu} F^f_{\mu\nu})\, a_f^{n-2} a_f^2 \\
&= \sum_f \mathrm{Tr}\,(\widehat{E}^f_{\mu\nu}\, \underbrace{a_f^2 F^f_{\mu\nu}}_{\widehat{F}^f_{\mu\nu}})
\end{aligned}$$

- in discretization, curvature $a_f^2 F_{\mu\nu}^f = \widehat{F}_{\mu\nu}^f$ on face f becomes group element g_f, thus linear function Tr $(\widehat{E}_{\mu\nu}^f \cdot)$ on Lie algebra $\mathfrak{g}$ must become function on group G, make substitution

$$\exp i\mathrm{Tr}\,(\widehat{E}_{\mu\nu}^f \cdot) \rightsquigarrow \chi_{\rho_f}(\cdot)$$

- thus d.o.f. of E-field appear in discretized BF theory as representation valued d.o.f. attached to faces of lattice
- discretized version of exponentiated action:

$$\exp iS \rightsquigarrow \prod_f \chi_{\rho_f}(g_f)$$

- now discretizing BF theory we naturally have arrived at a lattice setup which is just structure underlying spinfoams defined in next section

- partition function: $\mathcal{Z} = \int \mathcal{D}A \int \mathcal{D}E \, \exp i \int_M \mathrm{Tr}\,(E \wedge F)$
- for one face f: $\int d\widehat{E}^f_{\mu\nu} \, \exp i\mathrm{Tr}\,(\widehat{E}^f_{\mu\nu}\widehat{F}^f_{\mu\nu}) = \delta(\widehat{F}^f_{\mu\nu})$
- discretization: delta function on Lie algebra becomes delta function on group

$$\delta(g_f) = \sum_{\rho_f} \chi_{\rho_f}(g_f) \, \dim \rho_f$$

- all faces

$$\int \mathcal{D}E \, \exp i \int_M \mathrm{Tr}\,(E \wedge F) = \int \left(\prod_f d\widehat{E}^f_{\mu\nu} \right) \exp i \sum_f \mathrm{Tr}\,(\widehat{E}^f_{\mu\nu}\widehat{F}^f_{\mu\nu})$$

$$= \prod_f \delta(\widehat{F}^f_{\mu\nu}) \rightsquigarrow \prod_f \delta(g_f)$$

- partition function:

$$\mathcal{Z} = \int \prod_e dg_e \, \prod_f \delta(g_f)$$

$$= \int \prod_e dg_e \prod_f \sum_\rho \chi_\rho(g_f) \, \dim \rho$$

$$= \int \prod_e dg_e \sum_{\{\rho_f\}} \prod_f \chi_{\rho_f}(g_f) \, \dim \rho_f$$

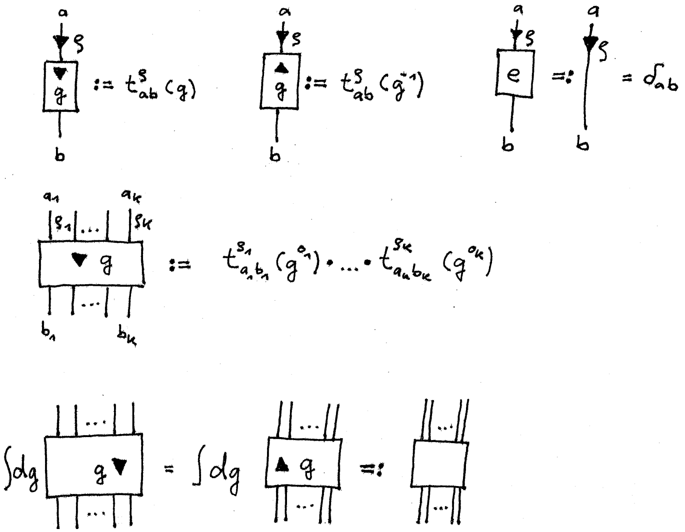
- character $\chi_\rho(g)$ is trace of matrix $t^\rho(g)$ derived from representation matrix via

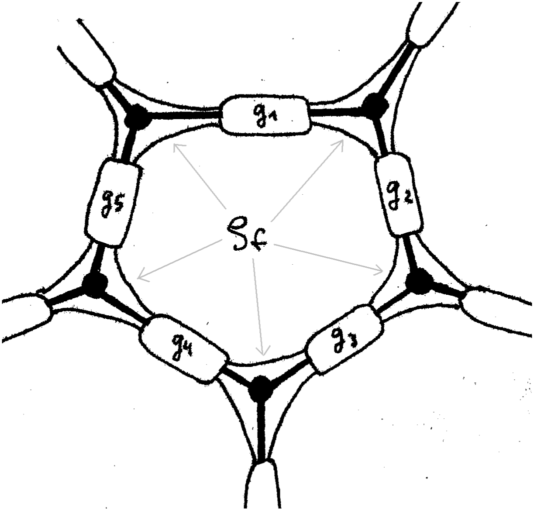
$$\begin{aligned}\chi_\rho(g) &= \mathsf{Tr}\, t^\rho(g) \\ t^\rho_{ab}(g) &= \langle \phi_a | \rho(g) | v_b \rangle\end{aligned}$$

and

$$\begin{aligned}t^\rho(g_1 g_2) &= t^\rho(g_1)\, t^\rho(g_2) \\ \chi_{\rho_f}(g_f) &= \mathsf{Tr}\, t^{\rho_f}(g_{e_1}^{o_1} \ldots g_{e_k}^{o_k}) = \mathsf{Tr}\left(t^{\rho_f}(g_{e_1}^{o_1}) \ldots t^{\rho_f}(g_{e_k}^{o_k})\right)\end{aligned}$$

circuit diagrams:





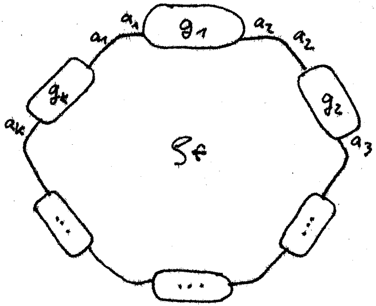
g_1
g_2
g_3
g_4
g_5
g_f

since

$$\prod_f \chi_{\rho_f}(g_f) = \prod_f \mathsf{Tr}\left(t^{\rho_f}(g_{e_1}^{o_1}) \ldots t^{\rho_f}(g_{e_k}^{o_k})\right)$$

$$= \prod_f \sum_{a_1,\ldots,a_k} t^{\rho_f}_{a_1 a_2}(g_{e_1}^{o_1})\, t^{\rho_f}_{a_2 a_3}(g_{e_2}^{o_2}) \ldots t^{\rho_f}_{a_k a_1}(g_{e_k}^{o_k})$$

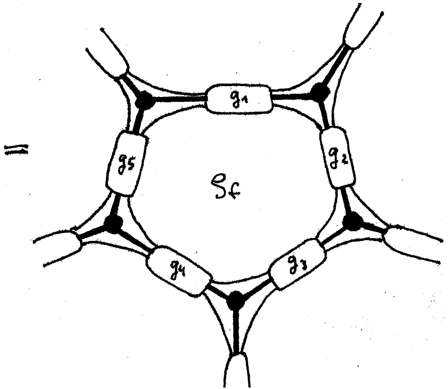
$$= \prod_f$$



since

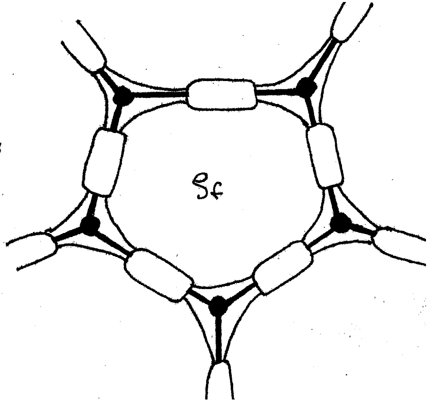
$$\prod_f \chi_{\rho_f}(g_f) = \prod_f \mathrm{Tr}\left(t^{\rho_f}(g_{e_1}^{o_1}) \dots t^{\rho_f}(g_{e_k}^{o_k})\right)$$

$$= \prod_f \sum_{a_1,\dots,a_k} t^{\rho_f}_{a_1 a_2}(g_{e_1}^{o_1})\, t^{\rho_f}_{a_2 a_3}(g_{e_2}^{o_2}) \dots t^{\rho_f}_{a_k a_1}(g_{e_k}^{o_k})$$

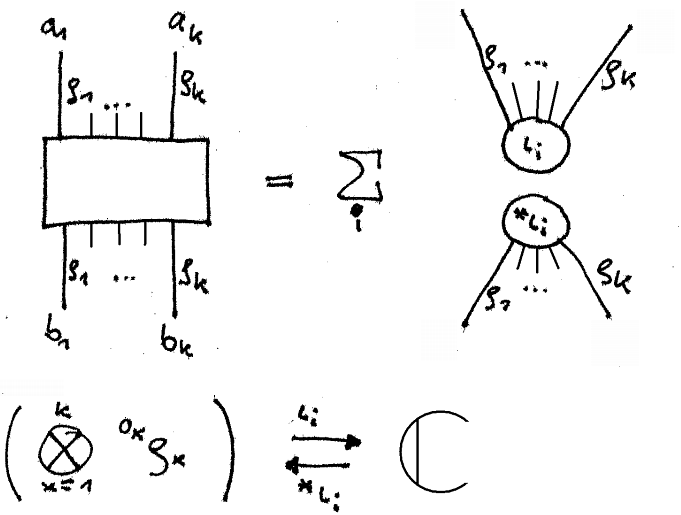


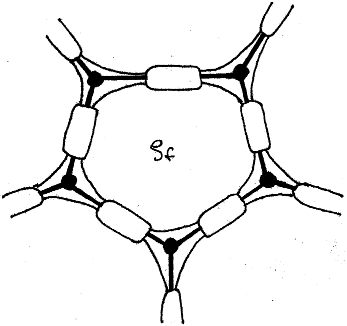
$$\int \prod_{e} dg_e \prod_{f} \chi_{\rho_f}(g_f)$$

$=$

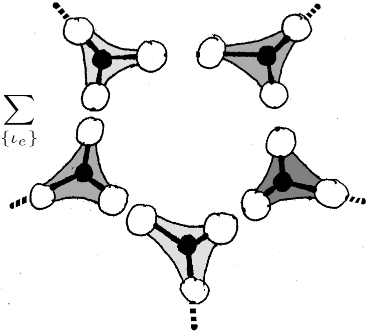


skein relation:





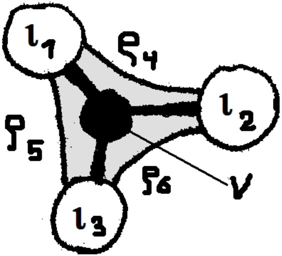
$$= \sum_{\{\iota_e\}}$$



- value of vertex diagram at vertex v depends on irreps labeling incident faces and their orientations, and all intertwiners labeling incident edges, denote value of vertex diagram by

$$A_v(\{\rho_{f_v}\}, \{o_{f_v}\}, \{\iota_{e_v}\})$$

$$=$$



- thus obtain for partition function

$$
\begin{aligned}
\mathcal{Z} &= \sum_{\{\rho_f\}} \left(\prod_f \dim \rho_f\right) \int \left(\prod_e dg_e\right) \left(\prod_f \chi_{\rho_f}(g_f)\right) \\
&= \sum_{\{\rho_f\}} \sum_{\{\iota_e\}} \left(\prod_f \dim \rho_f\right) \left(\prod_v A_v(\{\rho_{f_v}\}, \{o_{f_v}\}, \{\iota_{e_v}\})\right)
\end{aligned}
$$

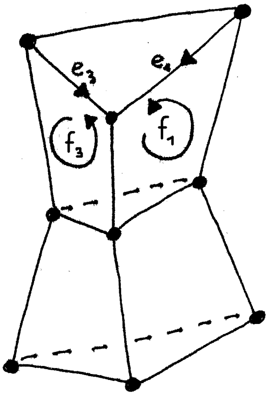
- discretizing BF theory we naturally have arrived at spinfoam defined in next section
- partition function is that of spinfoam model

Outline

- **closed spinfoam** $F = (\kappa, \rho, \iota)$ with symmetry group G is triple consisting of:

  1. oriented 2-complex κ,
  2. face-labeling ρ of each face f of κ by an irrep ρ_f of G and
  3. edge-labeling ι of each edge e of κ by intertwiner ι_e

$$\iota_e : \ \left(\rho_{f_1^{\text{in}}(e)} \otimes \ldots \otimes \rho_{f_{n(e)}^{\text{in}}(e)}\right) \ \rightarrow \ \left(\rho_{f_1^{\text{out}}(e)} \otimes \ldots \otimes \rho_{f_{o(e)}^{\text{out}}(e)}\right)$$

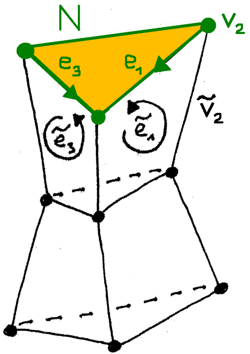


- spinfoam $\big(F = (\kappa, \widetilde{\rho}, \widetilde{\iota})\big) : \ \emptyset \ \to \ \big(N = (\Gamma, \rho, \iota)\big)$ with symmetry group G connecting the empty spin network $\emptyset$ with N is triple consisting of:

1. oriented 2-complex κ bordered by Γ,
2. face-labeling $\widetilde{\rho}$ of each face f of κ by irrep $\widetilde{\rho}_f$ of G and
3. edge-labeling $\widetilde{\iota}$ of each edge e of κ not contained in Γ by intertwiner $\widetilde{\iota}_e$

$$\widetilde{\iota}_e : \ \big(\rho_{f_1^{\mathsf{in}}(e)} \otimes \ldots \otimes \rho_{f_{n(e)}^{\mathsf{in}}(e)}\big) \ \longrightarrow \ \big(\rho_{f_1^{\mathsf{out}}(e)} \otimes \ldots \otimes \rho_{f_{o(e)}^{\mathsf{out}}(e)}\big)$$

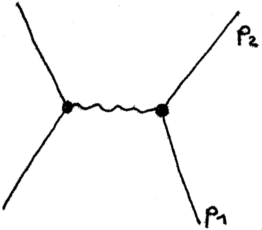
4. such that for any edge e of Γ if $\widetilde{e}$ is incoming to e then $\widetilde{\rho}_{\widetilde{e}} = \rho_e$, but if $\widetilde{e}$ is outgoing from e then $\widetilde{\rho}_{\widetilde{e}} = {}^*\rho_e$ and
5. for any vertex v of Γ after appropriate dualizations: $\widetilde{\iota}_{\widetilde{v}} = \iota_v$.



$\widetilde{\rho}_{\widetilde{e}_1} = \rho_{e_1}$

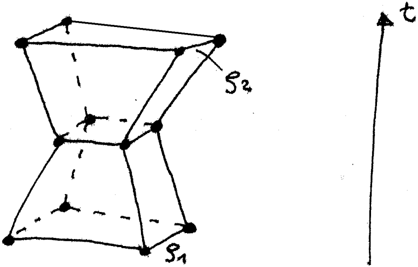
$\widetilde{\rho}_{\widetilde{e}_3} = {}^*\rho_{e_3}$

- **dual spin network** $^*N$ of N: same oriented 1-complex Γ,
  but each edge e labeled by dual irrep $^*\rho_e$
  and each vertex v labeled by appropriately dualized version of intertwiner ι_v.
- **tensor product** $N^1 \otimes N^2$ of two spin networks $N^1 = (\Gamma^1, \rho^1, \iota^1)$ and $N^2 = (\Gamma^2, \rho^2, \iota^2)$:
  disjoint union of N^1 and N^2
- **open spin foam** is spinfoam $F: \; N^1 \to N^2$
  connecting two nonempty spin networks N^1 and N^2,
  defined to be $F: \; \emptyset \to \left({}^*N^1 \times N^2\right)$.



D

$\hat{D}: \Psi_{p_1}[\phi] \mapsto \Psi_{p_2}[\phi]$



F

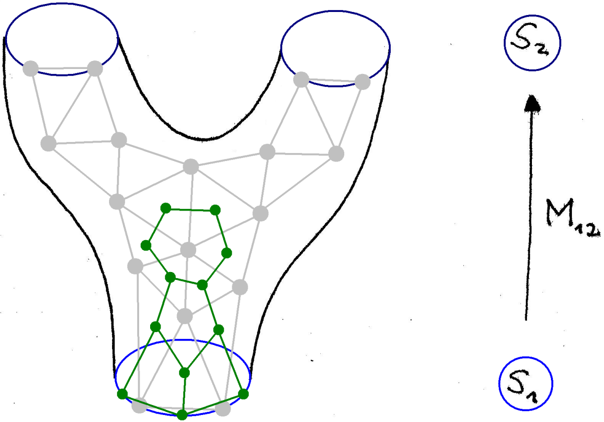
$\hat{F}: \Psi_{\rho_1}[A] \mapsto \Psi_{\rho_2}[A]$

in 4-dim. BF theory with gauge group SU(2):

- spin network edges give area to surfaces they cross,
  since they are slices of spinfoam faces,
  these give area to surfaces they intersect
- spin network vertices give 3-volume to regions of space they lie in,
  since they are slices of spinfoam edges,
  these give volume to 3-surfaces they cross
- spinfoam vertices expected to give 4-volume to regions of spacetime they lie in,
  but computations not finished

Outline

- region of n-dim. spacetime: compact oriented cobordism $M_{12}: \;\; S_1 \;\rightarrow\; S_2$
  with S_1, S_2 being $(n-1)$-dim. compact oriented manifolds representing space
- choose n-dim. triangulation Δ of spacetime M,
  induces $(n-1)$-dim. triangulations $\partial\Delta_1$ and $\partial\Delta_2$ on S_1, S_2 with dual 1-skeletons Γ_1, Γ_2



construction of gauge-invariant Hilbert spaces for S_1 and S_2:

- connection on given graph Γ: assign elements g of gauge group G to edges, space of such connections is $\mathcal{A}_\Gamma$

  assignment can be thought of as representing parallel transport along edge γ, if graph were embedded in space with connection A,

$$g_\gamma[A] \;=\; h_\gamma[A] \overset{\text{def.}}{=} \mathcal{P}\exp\int\limits_\gamma A$$

construction of gauge-invariant Hilbert spaces for S_1 and S_2:

- connection on given graph Γ: assign elements g of gauge group G to edges, space of such connections is $\mathcal{A}_\Gamma$
- gauge transformation on Γ: assign element of G to each vertex, group of these gauge transformations is $\mathcal{G}_\Gamma$
- $\mathcal{H}_{\mathsf{kin}} = L^2(\mathcal{A}_\Gamma) \qquad \mathcal{H}_{\mathsf{gauge}} = L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma)$,

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- $\mathcal{H}_{\text{kin}} = L^2(\mathcal{A}_\Gamma) \qquad \mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma)$,
- ONB of $L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma)$ formed by spin network states Ψ_N with $N = (\Gamma, \rho, \iota)$

construction of gauge-invariant Hilbert spaces for S_1 and S_2:

- connection on given graph Γ: assign elements g of gauge group G to edges, space of such connections is $\mathcal{A}_\Gamma$
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- $\mathcal{H}_{\text{kin}} = L^2(\mathcal{A}_\Gamma) \qquad \mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma)$,
- ONB of $L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma)$ formed by spin network states Ψ_N with $N = (\Gamma, \rho, \iota)$
- if graph Γ embedded in space S, then trivializing G-bundle at vertices gives map from $\mathcal{A}$ to $\mathcal{A}_\Gamma$ and a homomorphism from $\mathcal{G}$ to $\mathcal{G}_\Gamma$, thus

$$L^2(\mathcal{A}_\Gamma) \hookrightarrow L^2(\mathcal{A})$$
$$L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma) \hookrightarrow L^2(\mathcal{A}/\mathcal{G})$$

- use as gauge-invariant Hilbert spaces for S_1 and S_2:

$$\begin{aligned}\mathcal{H}^1_{\text{gauge}} &= L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1}) \\ \mathcal{H}^2_{\text{gauge}} &= L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})\end{aligned}$$

- describe **time evolution** as an operator

$$\widehat{Z}(M): \; L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1}) \;\rightarrow\; L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})$$

- use as gauge-invariant Hilbert spaces for S_1 and S_2:

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- describe **time evolution** as an operator

$$\widehat{Z}(M): \; L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1}) \rightarrow L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})$$

- since spin network states Ψ_{N_Γ} form a basis of $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma)$ sufficient for specifying operator $\widehat{Z}(M)$ to know transition amplitudes

$$\langle \Psi_{N_2}, \widehat{Z}(M)\, \Psi_{N_1} \rangle_2$$

because then we can write with an ONB $\{\Psi_{N_2}\}$

$$\widehat{Z}(M)\, \Psi_{N_1} = \sum_{\Psi_{N_2}}^{\text{ONB}} \Psi_{N_2} \, \langle \Psi_{N_2}, \widehat{Z}(M)\, \Psi_{N_1} \rangle_2$$

- write transition amplitudes as sum over amplitudes $Z(F)$ of spinfoams F going from N_1 to N_2:

$$\langle \Psi_{N_2}, \widehat{Z}(M)\, \Psi_{N_1} \rangle_2 \overset{\text{def.}}{=} \sum_F^{N_1 \to N_2} Z(F)$$

- amplitude for closed spinfoam $F = (\kappa, \rho, \iota)$ is product of amplitudes of its faces f , edges e and vertices v

$$Z(F) = N(F) \prod_{f \in \kappa} Z_f(\rho_f) \prod_{e \in \kappa} Z_e(\rho_{f(e)}) \prod_{v \in \kappa} Z_v(\rho_{f(v)})$$

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- amplitude for closed spinfoam $F = (\kappa, \rho, \iota)$ is product of amplitudes of its faces f , edges e and vertices v

$$Z(F) = N(F) \prod_{f \in \kappa} Z_f(\rho_f) \prod_{e \in \kappa} Z_e(\rho_{f(e)}) \prod_{v \in \kappa} Z_v(\rho_{f(v)})$$

- amplitude formula for open spinfoams differs in three points:
  - ⊕ edges and vertices lying in the spin networks N_1 or N_2 to be connected excluded from the product of amplitudes
  - ⊕ for spinfoam edges $\widetilde{v}$ ending in vertices v of N_1 or N_2 use square root of usual edge amplitude
  - ⊕ for spinfoam faces $\widetilde{e}$ ending in edges e of N_1 or N_2 use square root of usual face amplitude

- reason for these modifications: achieve **product rule for spinfoam amplitudes**

$$Z(F_{13}) = Z(F_{23})\, Z(F_{12})$$

for all spinfoams $F_{13} : \ N_1 \ \rightarrow \ N_3$ obtained by gluing together
$F_{12} : \ N_1 \ \rightarrow \ N_2$ and $F_{23} : \ N_2 \ \rightarrow \ N_3$ along common border N_2
whereby edges and vertices lying in N_2 become erased

- reason for these modifications: achieve **product rule for spinfoam amplitudes**

$$Z(F_{13}) = Z(F_{23})\, Z(F_{12})$$

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- this assures **composition property (gluing rule)** (provided sum over spinfoam amplitudes converges) for composable cobordisms $M_{12}: \ S_1 \ \rightarrow \ S_2$ and $M_{23}: \ S_2 \ \rightarrow \ S_3$:

$$\widehat{Z}(M_{23}M_{12}) = \widehat{Z}(M_{23})\, \widehat{Z}(M_{12})$$

- for spinfoam $F_{12}: \;\; N_1 \;\to\; N_2$ define operator $\widehat{F}_{12}: \;\; L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1}) \;\to\; L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})$ acting on spin network states $\Psi_1 \in L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1})$ as

$$\widehat{F}_{12}\,\Psi_1 \overset{\text{def.}}{=} \Psi_{N_2}\,\langle \Psi_{N_1}\,,\,\Psi_1\rangle_1$$

$$\Rightarrow \qquad \widehat{F}_{12}\,\Psi_{N_1} \;=\; \Psi_{N_2}\,\|\,\Psi_{N_1}\,\|^2$$

- for spinfoam $F_{12}: \; N_1 \to N_2$ define operator $\widehat{F}_{12}: \; L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1}) \to L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})$ acting on spin network states $\Psi_1 \in L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1})$ as

$$\widehat{F}_{12}\,\Psi_1 \overset{\text{def.}}{=} \Psi_{N_2}\,\langle \Psi_{N_1}, \Psi_1\rangle_1$$

$$\Rightarrow \quad \widehat{F}_{12}\,\Psi_{N_1} = \Psi_{N_2}\,\|\,\Psi_{N_1}\,\|^2$$

- thus for arbitrary spin network states $\Psi_{1,2} \in L^2(\mathcal{A}_{\Gamma_{1,2}}/\mathcal{G}_{\Gamma_{1,2}})$

$$\langle \Psi_2, \widehat{F}_{12}\,\Psi_1\rangle_2 = \langle \Psi_2, \Psi_{N_2}\rangle_2\,\langle \Psi_{N_1}, \Psi_1\rangle_1$$

- for spinfoam $F_{12}: \ N_1 \ \rightarrow \ N_2$ define operator $\widehat{F}_{12}: \ L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1}) \ \rightarrow \ L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})$ acting on spin network states $\Psi_1 \in L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1})$ as

$$\widehat{F}_{12}\,\Psi_1 \overset{\text{def.}}{=} \Psi_{N_2}\,\langle\Psi_{N_1},\Psi_1\rangle_1$$
$$\Rightarrow \quad \widehat{F}_{12}\,\Psi_{N_1} = \Psi_{N_2}\,\|\,\Psi_{N_1}\,\|^2$$

- thus for arbitrary spin network states $\Psi_{1,2} \in L^2(\mathcal{A}_{\Gamma_{1,2}}/\mathcal{G}_{\Gamma_{1,2}})$

$$\langle\Psi_2,\widehat{F}_{12}\,\Psi_1\rangle_2 = \langle\Psi_2,\Psi_{N_2}\rangle_2\,\langle\Psi_{N_1},\Psi_1\rangle_1$$

- can write time evolution operator consistently with previous equations as

$$\begin{aligned}
\widehat{Z}(M_{12})\Psi_{N_1} &= \sum_{N_2}^{\Gamma_2}\sum_{F_{12}}^{N_1\rightarrow N_2} Z(F_{12})\,\widehat{F}_{12}\,\Psi_{N_1}\,/\,\|\,\Psi_{N_1}\,\|^2 \\
&= \sum_{N_2}^{\Gamma_2}\Psi_{N_2}\sum_{F_{12}}^{N_1\rightarrow N_2} Z(F_{12}) \\
&= \sum_{N_2}^{\Gamma_2}\Psi_{N_2}\,\langle\Psi_{N_2},\widehat{Z}(M_{12})\,\Psi_{N_1}\rangle_2
\end{aligned}$$

Outline

- Palatini formulation of 4-dim. GR: spacetime M = 4-dim. oriented smooth manifold
- choose bundle $\mathcal{T}$ over M, isomorphic to TM but not in canonical way, **internal space**, equip $\mathcal{T}$ with orientation and metric η, either Lorentzian or Riemannian
- P = oriented orthonormal frame bundle of M = a principal G-bundle with G either SO(3,1) or SO(4) corresponding to metric η

- basic fields in Palatini formulation:
  - a $\mathcal{T}$-valued 1-form e on M: $\quad e: \; TM \rightarrow \mathcal{T}$
  - a connection A on P
  - curvature F of A: an ad(P)-valued 2-form

- basic fields in Palatini formulation:
  - a $\mathcal{T}$-valued 1-form e on M: $\quad e: \; TM \to \mathcal{T}$
  - a connection A on P
  - curvature F of A: an ad(P)-valued 2-form
- ad(P) isomorphic to $\Lambda^2\mathcal{T}$, thus can think of F as $\Lambda^2\mathcal{T}$-valued 2-forms, also $(e \wedge e)$, with local coordinates $\{x^k\}$ on M and basis $\{b_m\}$ of $T^*\mathcal{T}$:

$$
\begin{aligned}
e \wedge e &= e^{l_1}_{k_1} e^{l_2}_{k_2} \, dx^{k_1} \wedge dx^{k_2} \otimes b_{l_1} \wedge b_{l_2} \\
F &= F^{m_1 m_2}_{j_1 j_2} \, dx^{j_1} \wedge dx^{j_2} \otimes b_{m_1} \wedge b_{m_2} \\
e \wedge e \wedge F &= e^{l_1}_{k_1} e^{l_2}_{k_2} F^{m_1 m_2}_{j_1 j_2} \, dx^{k_1} \wedge dx^{k_2} \wedge dx^{j_1} \wedge dx^{j_2} \otimes b_{l_1} \wedge b_{l_2} \wedge b_{m_1} \wedge b_{m_2} \\
&= \underbrace{f(x) \, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4}_{\mathrm{Tr}\,(e \wedge e \wedge F)} \otimes \underbrace{b_1 \wedge b_2 \wedge b_3 \wedge b_4}_{\mathrm{vol_{int}}}
\end{aligned}
$$

- Palatini Lagrangian: $\mathcal{L} = \mathrm{Tr}\,(e \wedge e \wedge F)$
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- pull back connection A to metric connection Γ on TM, if $e: \ TM \ \to \mathcal{T}$ bijective, then $d_A e = 0$, Γ torsion-free, thus Γ is Levi-Civita connection of spacetime metric g
- rewriting $e \wedge F$ in terms of Riemann curvature tensor, one sees it is proportional to Einstein tensor, thus $e \wedge F = 0$ is vacuum Einstein equation

- setting $E = e \wedge e$ makes Palatini Lagrangian look like BF Lagrangian,
  difference: not every ad(P)-valued 2-form E is of form $e \wedge e$,
  thus allowed variations of E field in Palatini GR more restricted than in BF theory,
  thus Palatini field equations weaker than BF equations:

$$
\begin{aligned}
F &= 0 & \qquad\qquad e \wedge F &= 0 \\
d_A E &= 0 & \qquad\qquad d_A E &= 2e \wedge d_A e = 0
\end{aligned}
$$

- relation between Palatini GR and BF theory suggests
  one could develop spinfoam model of QG
  by taking spinfoam model of BF theory and imposing quantum analogues of constraint
  that E is of form $e \wedge e$

- consider at classical level constraints that must hold
  in order to have E field of form $e \wedge e$:
  pick spin structure for spacetime and take Spin(4) as gauge group

- locally can think of E field as taking values in $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$,
  thus can write $E = E^+ + E^-$ as sum of left-handed and right-handed parts
  taking values in $\mathfrak{so}(3)$

- if $E = e \wedge e$ then constraint $\| E^+(v, w) \| = \| E^-(v, w) \|$
  holds for all vector fields v, w on M
  constraint guarantees $E = e \wedge e$ (up to sign and Hodge star on $\Lambda^2 \mathcal{T}$)

- recall facts of 4-dim. BF theory with gauge group SU(2):
  spinfoam is dual 2-skeleton of triangulated 4-manifold,
  each dual face labeled by spin, each dual edge by intertwiner,
  corresponds to labeling each triangle by spin
  and each tetrahedron by 4-valent intertwiner ($=$ connecting four spins)

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- general trivalent intertwiners: tensor product of any pair ρ_1, ρ_2 of irreps
  can be written as direct sum of irreps ρ_k, pick one $\rho_3 \in \{\rho_k\}$,
  projection from $\rho_1 \otimes \rho_2$ to ρ_3 is trivalent intertwiner ι
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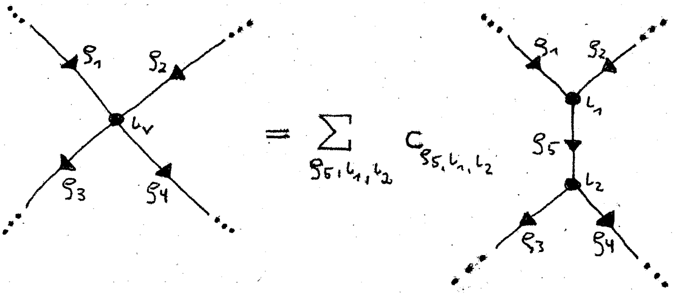
- if irrep ρ_3 appears more than once in direct sum decomposition $(\rho_1 \otimes \rho_2)$
  then more than one intertwiner of above form,
  can always choose ONB $\{\iota_k\}$ of such intertwiners: Tr $(\iota_k\ {}^*\iota_l) = \delta_{kl}$

- can break any 4-valent intertwiner into trivalent ones using skein relation:



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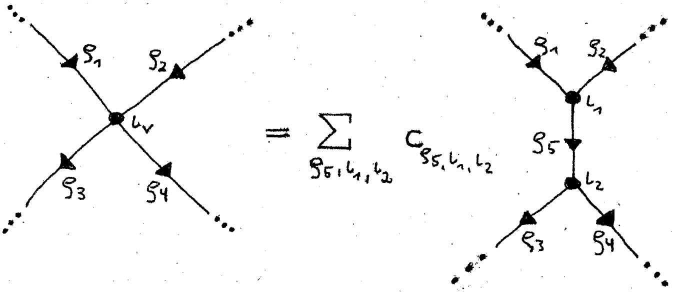

- SU(2) irreps: $j_1 \otimes j_2 \cong |j_1 - j_2| \oplus ... \oplus j_1 + j_2$,
  thus each basis of intertwiners $\iota : \ (j_1 \otimes j_2) \to j_3$
  consists of exactly one (normalized) element iff $|j_1 - j_2| \leq j_3 \leq (j_1 + j_2)$

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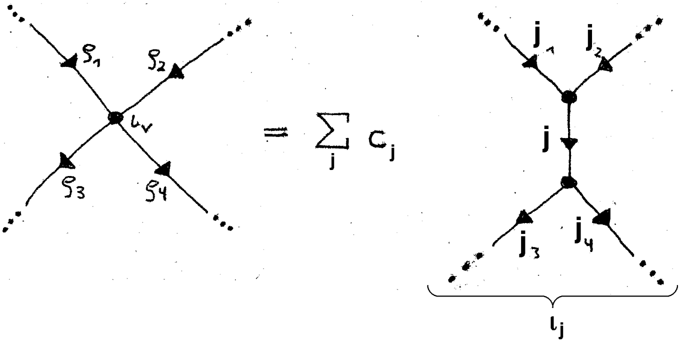

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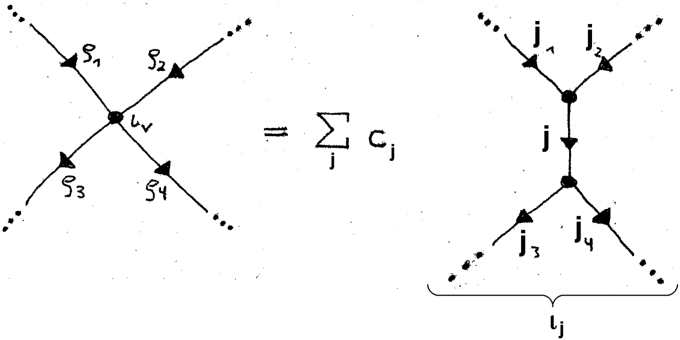

- 4-dim. triangulation with each tetrahedron labeled by a 4-valent intertwiner:
  skein relation = chopping each tetrahedron in half by parallelogram
  labeled by sum over spins j on right-hand side of skein relation,
  thus all data encoded in spins labeling surfaces,
  each spin describing integral of $\| E \|$ over its surface

- describe 4-dim. Riemannian QG as BF theory with gauge group Spin(4) with extra constraint, since Spin(4) $\cong$ SU(2) $\times$ SU(2), irreps are of form $j^+ \otimes j^-$ for arbitrary spins $j^\pm$, thus label triangles and parallelograms by pairs (j^+, j^-) of spins, describing integral of $\| E^\pm \|$ over surface
- in order to on quantum level impose constraint $\| E^+(v,w) \| = \| E^-(v,w) \|$, restrict to labeling surfaces with equal spins

- label each triangle a by irrep of form $j_a \otimes j_a$
  and each tetrahedron by intertwiner of form $\sum_j c_j \, \iota_j \otimes \iota_j$
  with $\iota_j : \;\; j_{a_1} \otimes j_{a_2} \xrightarrow{j} j_{a_3} \otimes j_{a_4}$

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- however exist three ways of splitting tetrahedron in half by parallelogram P,
  want constraint $\int_P \| E^+ \| = \int_P \| E^- \|$ to hold for all three,
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  which maintain this form during switching to different splitting
- unique solution: $\iota = \sum_j (2j+1) \, \iota_j \otimes \iota_j$

Outline

- state:
  - ⊕ have proposal for spinfoam model of QG with quantized values for area and volume
  - ⊕ quantum state of space is linear combination of spin networks
  - ⊕ transition amplitudes computed as sum over spinfoams connecting spin networks
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- tasks:
  - develop spinfoam models of Lorentzian QG
  - determine role which triangulations should play in spinfoam models with local d.o.f.
  - develop computational techniques for studying large-scale limit