General Boundary Quantum Field Theory in curved space

Daniele Colosi

Centro de Ciencias Matemáticas
UNAM

Seminar *General Boundary Formulation*
16 May 2013

Contents

Outline

General Boundary Formulation

The GBF is an **axiomatic** formulation of quantum theory which combines

- the mathematical framework of Topological Quantum Field Theory (association of algebraic structures to geometric ones) with
- a generalization of the Born's rule to extract probabilities.

Basic structures

In the GBF algebraic structures are associated to geometric ones.

Geometric structures (representing pieces of spacetime):

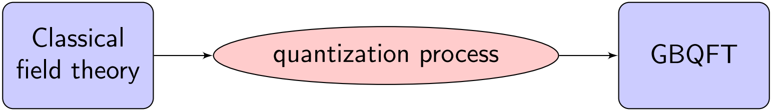
- **hypersurfaces**: oriented manifolds of dimension $d-1$
- **regions**: oriented manifolds of dimension d with boundary

Algebraic structures:

- To each hypersurface Σ associate a Hilbert space $\mathcal{H}_\Sigma$ of states.
- To each region M with boundary ∂M associate a linear amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$
- As in AQFT, observables are associated to spacetime regions: An observable O in a region M is a linear map $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$, called observable map.

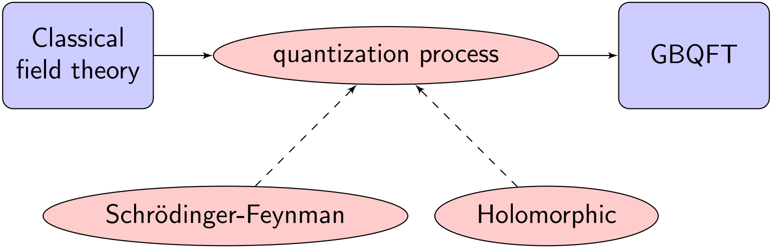
GBQFT

We want to construct a general boundary quantum field theory in curved space.



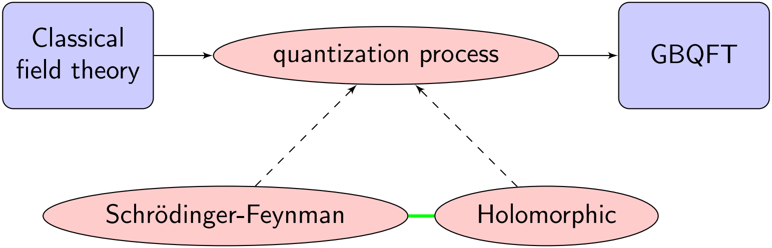
GBQFT

We want to construct a general boundary quantum field theory in curved space.



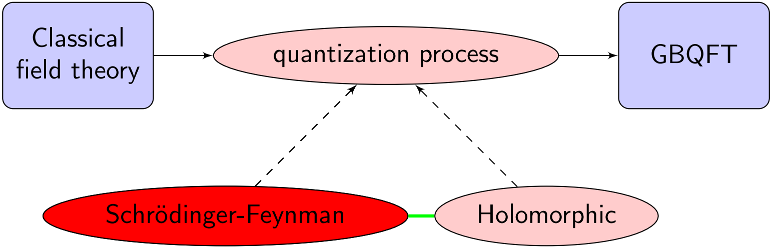
GBQFT

We want to construct a general boundary quantum field theory in curved space.



GBQFT

We want to construct a general boundary quantum field theory in curved space.



Outline

Classical Theory

We consider the linear theory of a massive Klein-Gordon field in a 4d Lorentzian spacetime $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$.

The action of field in a spacetime region M is

$$S_{M,0}[\phi] = \frac{1}{2}\int_M \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

- the integration is extended over the sapcetime region M
- g is determinant of the metric $g_{\mu\nu}$
- we used the notation $\partial_\mu = \partial/\partial x^\mu$

Assumptions

We assume the following:

- The spacetime region M admits a **foliation** in terms of hypersurfaces, *not necessarily spacelike*, described in terms of a smooth coordinate system $(\tau, \underline{x})$. The coordinates on the leaves of the foliation are denoted by $\underline{x} = (x^1, x^2, x^3) \in I \subset \mathbb{R}^3$ and the leaves are parametrized by the coordinate $\tau \in J \subset \mathbb{R}$. Notice that τ and $\underline{x}$ are not required to be timelike and spacelike coordinates respectively.
- The metric takes a **block-diagonal** form with respect to the coordinates $(\tau, \underline{x})$, i.e. $g^{\tau x^i} = 0 = g^{x^i \tau}$ for all $i \in \{1, 2, 3\}$.
- Let L_Σ be the space of solutions of the e.o.m. in a neighborhood of a hypersurface Σ of the foliation. We define the **symplectic potential** as the one-form on L_Σ,

$$[\xi, \xi'] := -\int_\Sigma \mathrm{d}^3\underline{x}\, \sqrt{g^{(3)}}\, \xi(\partial_\tau \xi'),$$

where $g^{(3)}$ is the determinant of the induced 3-metric on the hypersuface Σ, and $\xi, \xi' \in L_\Sigma$. The corresponding **symplectic form** is

$$\omega_\Sigma(\xi, \xi') = \frac{1}{2}[\xi, \xi'] - \frac{1}{2}[\xi', \xi] \qquad \forall \xi, \xi' \in L_\Sigma.$$

- The Euler-Lagrangian equations are be solved by the method of **separation of variables**, and a solution takes the form

$$\phi(\tau, \underline{x}) = \int \mathrm{d}\underline{k} \left(d_a(\underline{k}) X_{a,\underline{k}}(\tau) Y_{a,\underline{k}}(\underline{x}) + d_b(\underline{k}) X_{b,\underline{k}}(\tau) Y_{b,\underline{k}}(\underline{x}) \right),$$

  where X_a and X_b are the two independent solutions of the part of the equation of motion depending only on the τ variable.
- Solutions can also be written as

$$\phi(\tau, \underline{x}) = \left(X_a(\tau) Y_a\right)(\underline{x}) + \left(X_b(\tau) Y_b\right)(\underline{x}),$$

  where now $X_{a,b}$ are understood as operators from the space of initial data $Y_{a,b}$ to solution on Σ_τ. We assume that $X_{a,b}$ **commute** with each other and are **invertible**.

The most studied quantum field theories on curved spaces are encompassed by these assumptions.

Slice region

- The **slice region** M is defined as the spacetime region is bounded by the disjoint union of two constant-τ hypersurfaces, $\Sigma_1 = \{(\tau, \underline{x}) : \tau = \tau_1\}$ and $\Sigma_2 = \{(\tau, \underline{x}) : \tau = \tau_2\}$, namely $M = [\tau_1, \tau_2] \times \mathbb{R}^3$.
- Let $\varphi_1(\underline{x}) := \phi(\tau_1, \underline{x})$ and $\varphi_2(\underline{x}) := \phi(\tau_2, \underline{x})$ be the **boundary field configurations** on Σ_1 and Σ_2.
- The classical solution can be expressed as

$$\phi(\tau, \underline{x}) = \left(\frac{\Delta(\tau, \tau_2)}{\Delta(\tau_1, \tau_2)} \, \varphi_1 \right)(\underline{x}) + \left(\frac{\Delta(\tau_1, \tau)}{\Delta(\tau_1, \tau_2)} \, \varphi_2 \right)(\underline{x}),$$

where $\Delta(\tau_1, \tau_2) := X_a(\tau_1) X_b(\tau_2) - X_a(\tau_2) X_b(\tau_1)$. All the deltas must be understood as operators acting on the boundary field configurations φ_1 and φ_2, and we assume that these operators are invertible.

- The free action in the slice region takes the form

$$S_{[\tau_1,\tau_2],0}(\phi) = \frac{1}{2}\int \mathrm{d}^3\underline{x}\left(\sqrt{|g^{(3)}_{\tau_2} g^{\tau\tau}_{\tau_2}|}\,\phi(\tau_2,\underline{x})(\partial_\tau\phi)(\tau_2,\underline{x}) - \sqrt{|g^{(3)}_{\tau_1} g^{\tau\tau}_{\tau_1}|}\,\phi(\tau_1,\underline{x})(\partial_\tau\phi)(\tau_1,\underline{x})\right),$$

where $g^{(3)}_{\tau_1}$ and $g^{(3)}_{\tau_2}$ denote the metric restricted to the hypersurfaces Σ_1 and Σ_2 respectively.

- In terms of the boundary field configuration,

$$S_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2) = \frac{1}{2}\int \mathrm{d}^3\underline{x}\ \begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix} W_{[\tau_1,\tau_2]} \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},$$

where the $W_{[\tau_1,\tau_2]}$ is a 2×2 matrix with elements $W^{(i,j)}_{[\tau_1,\tau_2]}, (i,j=1,2)$, given by

$$W^{(1,1)}_{[\tau_1,\tau_2]} = -\sqrt{|g^{(3)}_{\tau_1} g^{\tau\tau}_{\tau_1}|}\,\frac{\Delta_1(\tau_1,\tau_2)}{\Delta(\tau_1,\tau_2)},\quad W^{(1,2)}_{[\tau_1,\tau_2]} = -\sqrt{|g^{(3)}_{\tau_1} g^{\tau\tau}_{\tau_1}|}\,\frac{\Delta_2(\tau_1,\tau_1)}{\Delta(\tau_1,\tau_2)},$$

$$W^{(2,1)}_{[\tau_1,\tau_2]} = \sqrt{|g^{(3)}_{\tau_2} g^{\tau\tau}_{\tau_2}|}\,\frac{\Delta_1(\tau_2,\tau_2)}{\Delta(\tau_1,\tau_2)},\qquad W^{(2,2)}_{[\tau_1,\tau_2]} = \sqrt{|g^{(3)}_{\tau_2} g^{\tau\tau}_{\tau_2}|}\,\frac{\Delta_2(\tau_1,\tau_2)}{\Delta(\tau_1,\tau_2)},$$

where

$$\Delta_1(\tau_1,\tau_2) := \partial_\tau \Delta(\tau,\tau_2)\big|_{\tau=\tau_1} \qquad \Delta_2(\tau_1,\tau_2) := \partial_\tau \Delta(\tau_1,\tau)\big|_{\tau=\tau_2}.$$

- The symplectic form on the space of smooth solutions in M is

$$\omega(\phi_1, \phi_2) = \frac{1}{2} \int_\Sigma \mathrm{d}^3\underline{x} \sqrt{|g^{(3)} g^{\tau\tau}|} \, (\phi_1 \, \partial_\tau \phi_2 - \phi_2 \, \partial_\tau \phi_1) \, ,$$

which is independent of the choice of leaf Σ of the foliation.

- Consequently the operator

$$\mathcal{W} := \sqrt{|g_\tau^{(3)} g_\tau^{\tau\tau}|} \, \Delta_2(\tau, \tau) = -\sqrt{|g_\tau^{(3)} g_\tau^{\tau\tau}|} \, \Delta_1(\tau, \tau)$$

is independent of τ.

- This implies that

$$W^{(1,2)}_{[\tau_1, \tau_2]} = W^{(2,1)}_{[\tau_1, \tau_2]} .$$

Outline

Schrödinger-Feynman quantization

- The state space $\mathcal{H}_\Sigma$ for a hypersurface Σ is the space of functions on field configurations K_Σ on Σ.
- Inner product,

$$\langle\psi_2|\psi_1\rangle = \int_{K_\Sigma} \mathcal{D}\varphi\, \psi_1(\varphi)\overline{\psi_2(\varphi)}.$$

- Amplitude for a region M, $\psi \in \mathcal{H}_{\partial M}$,

$$\rho_M(\psi) = \int_{K_{\partial M}} \mathcal{D}\varphi\, \psi(\varphi) Z_M(\varphi),$$

where Z_M is the field propagator given by the Feynman path integral,

$$Z_M(\varphi) = \int_{K_M, \phi|_{\partial M}=\varphi} \mathcal{D}\phi\, e^{\mathrm{i}S_M(\phi)}, \quad \forall\varphi \in K_{\partial M}.$$

The integral is over the space K_M of space-time field configurations ϕ in the interior of M which agree with φ on the boundary ∂M.

Free theory: field propagator

The field propagator in the region M results to be

$$Z_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2) = \int_{\phi|_{\Sigma_{1,2}}=\varphi_{1,2}} \mathcal{D}\phi \, e^{\mathrm{i}S_{[\tau_1,\tau_2],0}(\phi)}.$$

The integral is evaluated by shifting the integration variable by a classical solution matching the boundary configurations φ_1 and φ_2 at $\tau=\tau_1$, $\tau=\tau_2$ respectively,

$$Z_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2) = N_{[\tau_1,\tau_2],0} \, e^{\mathrm{i}S_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2)},$$

where $S_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2)$ is the free action and the normalization factor is formally given by

$$N_{[\tau_1,\tau_2],0} = \int_{\phi|_{\Sigma_{1,2}}=0} \mathcal{D}\phi \, e^{\mathrm{i}S_{[\tau_1,\tau_2],0}(\phi)} = \left(\det \frac{\mathrm{i}W^{(1,2)}_{[\tau_1,\tau_2]}}{2\pi}\right)^{-1/2}.$$

The field propagator satisfies the composition rule

$$Z_{[\tau_1,\tau_3],0}(\varphi_1,\varphi_3) = \int \mathcal{D}\varphi_2 \, Z_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2) Z_{[\tau_2,\tau_3],0}(\varphi_2,\varphi_3)$$

and the identity

$$\int \mathcal{D}\varphi_2 \, \overline{Z_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2)} Z_{[\tau_1,\tau_2],0}(\tilde{\varphi}_1,\varphi_2) = \delta\left(\varphi_1,\tilde{\varphi}_1\right),$$

where φ_1 and $\tilde{\varphi}_1$ are field configurations on Σ_1. The above identity can be interpreted in terms of the unitarity of the evolution implemented by the field propagator.

Vacuum state

The vacuum wave functional has the form of a Gaussian,

$$\psi_{\Sigma,0}(\varphi) = C_\Sigma \exp\left(-\frac{1}{2}\int \mathrm{d}^3 s\, \varphi(s)(A_\Sigma \varphi)(s)\right),$$

s being a generic coordinate system on Σ, A_Σ is some operator

$$A_\Sigma = -\mathrm{i}\sqrt{|g_\Sigma^3|}\,\frac{\partial_n(\Upsilon(s^0))}{\Upsilon(s^0)},$$

where g_Σ^3 is the determinant of the 3-metric on Σ, $\partial_n = \sqrt{|g_\Sigma^{00}|}\,\partial/\partial s^0$ is the normal derivative to Σ and

$$\Upsilon(s^0) := c_a X_a(s^0) + c_b X_b(s^0),$$

where $c_{a,b}$ are complex numbers s.t. $\overline{c_a} c_b - \overline{c_b} c_a \neq 0$, and s^0 is the parameter indexing the foliation. The normalization factor is

$$C_\Sigma = \det\left(\mathrm{i}\,\frac{\sqrt{|g_\Sigma^3\, g_\Sigma^{00}|}(\overline{c_a} c_b - \overline{c_b} c_a)\Delta_1(s^0, s^0)}{2\pi\Upsilon^2(s^0)}\right)^{1/4}.$$

Coherent states

Coherent states are defined, in the interaction picture, in terms of a complex function ξ,

$$\psi_{\tau,\xi}(\varphi) = K_{\tau,\xi} \exp\left(\int \mathrm{d}^3\underline{x}\, \frac{\xi(\underline{x})}{\Upsilon(\tau)}\, \varphi(\underline{x}) \right) \psi_{\tau,0}(\varphi),$$

and the normalization factor results to be

$$K_{\tau,\xi} = \exp\left(-\frac{1}{2} \int \mathrm{d}^3\underline{x}\, \frac{1}{\mathrm{i} W^{(1,2)}_{[\tau_1,\tau_2]} (\overline{c_a}\, c_b - \overline{c_b}\, c_a)\, \Delta(\tau_1,\tau_2)} \left(\frac{\overline{\Upsilon(\tau)}}{\Upsilon(\tau)} \xi^2(\underline{x}) + |\xi(\underline{x})|^2 \right) \right).$$

Free amplitude

The free amplitude for the coherent state $\psi_{\tau_1,\xi_1} \otimes \overline{\psi_{\tau_2,\xi_2}} \in \mathcal{H}_1 \otimes \mathcal{H}_2^*$ is

$$\rho_{[\tau_1,\tau_2],0}(\psi_{\tau_1,\xi_1} \otimes \overline{\psi_{\tau_2,\xi_2}})$$
$$= \int \mathcal{D}\varphi_1\, \mathcal{D}\varphi_2\, \overline{\psi_{\tau_2,\xi_2}(\varphi_2)}\, \psi_{\tau_1,\xi_1}(\varphi_1)\, Z_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2),$$
$$= \exp\left(-\frac{1}{2}\int \mathrm{d}^3\underline{x}\, \frac{1}{\mathrm{i}W^{(1,2)}_{[\tau_1,\tau_2]}(\overline{c_a}\, c_b - \overline{c_b}\, c_a)\,\Delta(\tau_1,\tau_2)}\left(|\xi_1(\underline{x})|^2 + |\xi_2(\underline{x})|^2 - 2\overline{\xi_2(\underline{x})}\,\xi_1(\underline{x})\right)\right).$$

This amplitude does not depend on τ_1 and τ_2, since the product $W^{(1,2)}_{[\tau_1,\tau_2]}\Delta(\tau_1,\tau_2)$ does not depend on τ_1 and τ_2. This was expected since we are considering the free evolution of states in the interaction picture.

Interacting theory

Consider the interaction of the scalar field with a real source field μ described by the action

$$S_{M,\mu}(\phi) = S_{M,0}(\phi) + \int_M \mathrm{d}^4x\, \sqrt{-g(x)}\, \mu(x)\, \phi(x).$$

We assume that the field μ is confined in the interior of the region M, i.e. $\mu(x) = 0$ for $x \in \partial M$ and $x \notin M$.
In the region $M = [\tau_1, \tau_2] \times \mathbb{R}^3$ the action reads

$$S_{[\tau_1,\tau_2],\mu}(\varphi_1, \varphi_2) = S_{[\tau_1,\tau_2],0}(\varphi_1, \varphi_2) + \int \mathrm{d}^3\underline{x} \big(\mu_1(\underline{x})\, \varphi_1(\underline{x}) + \mu_2(\underline{x})\, \varphi_2(\underline{x})\big),$$

where

$$\mu_1(\underline{x}) := \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \sqrt{-g(\tau, \underline{x})}\, \frac{\Delta(\tau, \tau_2)}{\Delta(\tau_1, \tau_2)}\, \mu(\tau, \underline{x}),$$
$$\mu_2(\underline{x}) := \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \sqrt{-g(\tau, \underline{x})}\, \frac{\Delta(\tau_1, \tau)}{\Delta(\tau_1, \tau_2)}\, \mu(\tau, \underline{x}).$$

Field propagator

The field propagator can be expressed in terms of the free one as

$$Z_{[\tau_1,\tau_2],\mu}(\varphi) = Z_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2) \times \frac{N_{[\tau_1,\tau_2],\mu}}{N_{[\tau_1,\tau_2],0}} \exp\left(\mathrm{i}\int \mathrm{d}^3\underline{x}\,(\mu_1(\underline{x})\,\varphi_1(\underline{x}) + \mu_2(\underline{x})\,\varphi_2(\underline{x}))\right),$$

The normalization factor $N_{[\tau_1,\tau_2],\mu}$ is formally equal to

$$N_{[\tau_1,\tau_2],\mu} = \int_{\phi|_{\tau_{1,2}}=0} \mathcal{D}\phi\, e^{\mathrm{i}S_{[\tau_1,\tau_2],\mu}(\phi)} = N_{[\tau_1,\tau_2],0} \exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\sqrt{-g(x)}\,\alpha(x)\,\mu(x)\right).$$

The integral has been evaluated by shifting the integration variable by a solution α of the inhomogeneous Klein-Gordon equation,
$(\Box + m^2)\,\alpha(x) = \mu(x)$, with vanishing boundary conditions,
$\alpha(\tau_1,\underline{x}) = \alpha(\tau_2,\underline{x}) = 0$.
α can be written as

$$\alpha(x) = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau'\,\sqrt{-g(\tau',\underline{x})}\left(\theta(\tau'-\tau)\frac{\Delta(\tau_1,\tau)\Delta(\tau',\tau_2)}{W^{(1,2)}_{[\tau_1,\tau_2]}\,\Delta^2(\tau_1,\tau_2)} + \theta(\tau-\tau')\frac{\Delta(\tau_1,\tau')\Delta(\tau,\tau_2)}{W^{(1,2)}_{[\tau_1,\tau_2]}\,\Delta^2(\tau_1,\tau_2)}\right)\mu(\tau',\underline{x}).$$

Unitary

It can be shown that the field propagator satisfies the identity

$$\int \mathcal{D}\varphi_2 \, \overline{Z_{[\tau_1,\tau_2],\mu}(\varphi_1, \varphi_2)} Z_{[\tau_1,\tau_2],\mu}(\tilde{\varphi}_1, \varphi_2) = \delta\left(\varphi_1, \tilde{\varphi}_1\right),$$

representing the unitarity of the quantum evolution implemented by the field propagator $Z_{[\tau_1,\tau_2],\mu}$.

Amplitude

The amplitude for the boundary state $\psi_{\tau_1,\xi_1} \otimes \overline{\psi_{\tau_2,\xi_2}}$ in the presence of the source field μ is

$$\rho_{[\tau_1,\tau_2],\mu}(\psi_{\tau_1,\xi_1} \otimes \overline{\psi_{\tau_2,\xi_2}}) = \int \mathcal{D}\varphi_1 \, \mathcal{D}\varphi_2 \, \overline{\psi_{\tau_2,\xi_2}(\varphi_2)} \, \psi_{\tau_1,\xi_1}(\varphi_1) \, Z_{[\tau_1,\tau_2],\mu}(\varphi_1,\varphi_2),$$

and can be expressed in terms of the free amplitude

$$\begin{aligned} \rho_{[\tau_1,\tau_2],\mu}(\psi_{\tau_1,\xi_1} \otimes \overline{\psi_{\tau_2,\xi_2}}) = {} & \rho_{[\tau_1,\tau_2],0}(\psi_{\tau_1,\xi_1} \otimes \overline{\psi_{\tau_2,\xi_2}}) \\ & \times \exp\left(\int \mathrm{d}^4x \, \sqrt{-g(x)} \, \hat{\xi}(x) \, \mu(x) \right) \\ & \times \exp\left(\frac{\mathrm{i}}{2} \int \mathrm{d}^4x \, \mathrm{d}^4x' \, \sqrt{g(x)g(x')} \, \mu(x) \, G_F(x,x') \, \mu(x') \right). \end{aligned}$$

where the function $\hat{\xi}$ is

$$\hat{\xi}(x) = \frac{1}{W^{(1,2)}_{[\tau_1,\tau_2]}(\overline{c_a}\, c_b - \overline{c_b}\, c_a)\, \Delta(\tau_1,\tau_2)} \left(\left(\overline{\Upsilon(\tau)} \xi_1 \right)(\underline{x}) + \left(\Upsilon(\tau) \overline{\xi_2} \right)(\underline{x}) \right),$$

and G_F is given by

$$G_F(x,x') = \int \mathrm{d}^3\underline{k}\, \frac{1}{W^{(1,2)}_{[\tau_1,\tau_2]}\, \Delta(\tau_1,\tau_2)(\overline{c_a}\, c_b - \overline{c_b}\, c_a)}$$

$$\left(\theta(\tau' - \tau) \left(\Upsilon(\tau)\varphi_{\underline{k}}\right)(\underline{x})\, \overline{\left(\Upsilon(\tau')\varphi_{\underline{k}}\right)}(\underline{x}') + \theta(\tau - \tau') \left(\Upsilon(\tau')\varphi_{\underline{k}}\right)(\underline{x}')\, \overline{\left(\Upsilon(\tau)\varphi_{\underline{k}}\right)}(\underline{x}) \right)$$

In Minkowski spacetime, in a slice region where τ represents the Minkowski time, G_F coincides with the standard Feynman propagator. The same happens in de Sitter and Rindler spaces with τ equal to de Sitter conformal time and Rindler time respectively.

G_F plays the rôle of the **Feynman propagator** for the scalar field theory defined in the slice region and satisfies the inhomogeneous Klein-Gordon equation in both variables x and x'

$$\left(\Box + m^2\right) G_F(x,x') = (-g(x))^{-1/2}\, \delta^4(x - x').$$

This expression is independent of τ_1 and τ_2. The limit of the amplitude for asymptotic values of τ_1 and τ_2 is trivial, and we can interpret it as the S-matrix for the scalar theory in the presence of a source field.

General interaction

Consider the general interacting theory

$$S_{M,V}(\phi) = S_{M,0}(\phi) + \int_M \mathrm{d}^4x\, V(x, \phi(x)).$$

where V is an arbitrary potential. The exponential of i times this action can be written as

$$e^{\mathrm{i}S_{M,V}(\phi)} = \exp\left(\mathrm{i}\int_M \mathrm{d}^4x\, \sqrt{-g(x)}\, V\left(x, -\mathrm{i}\frac{\delta}{\delta\mu(x)}\right)\right) e^{\mathrm{i}S_{M,\mu}(\phi)}\bigg|_{\mu=0},$$

where $S_{M,\mu}$ is the action in the presence of a source interaction. We assume that the potential V vanishes outside the region M. The corresponding field propagator is

$$Z_{M,V}(\varphi) = \exp\left(\mathrm{i}\int_M \mathrm{d}^4x\, \sqrt{-g(x)}\, V\left(x, -\mathrm{i}\frac{\delta}{\delta\mu(x)}\right)\right) Z_{M,\mu}(\varphi)\bigg|_{\mu=0},$$

and the amplitude for the general interacting theory is

$$\rho_{M,V}(\psi) = \exp\left(\mathrm{i}\int_M \mathrm{d}^4x\, \sqrt{-g(x)}\, V\left(x, -\mathrm{i}\frac{\delta}{\delta\mu(x)}\right)\right) \rho_{M,\mu}(\psi)\bigg|_{\mu=0}.$$