Algebraic Quantum Field Theory and Category Theory II

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Outline

The general assumptions

(a) Separable Hilbert space $\mathcal{H}$ of state vectors.

(b) Unitary representation $U(a, \Lambda)$ of the Poincaré group $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$

(c) Invariant, normalized state vector $\Omega \in \mathcal{H}$ (vacuum)

(d) A family of $*$-algebras $\mathcal{A}(\mathcal{O})$ of operators on $\mathcal{H}$ (a "field net"), indexed by regions $\mathcal{O} \subset \mathbb{R}^4$

(e) Isotony: $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$

Assumption: Operators are bounded and algebras are closed in the weak operator topology, i.e. $\Rightarrow$ von Neumann algebras.

Axioms (Haag-Kastler Axioms)

(i) ***Local (anti-)commutativity***: $\mathcal{A}(\mathcal{O}_1)$ (anti-)commutes with $\mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ space-like separated.

(ii) ***Covariance***: $U(a,\Lambda)\mathcal{A}(\mathcal{O})U(a,\Lambda)^{-1} = \mathcal{A}(\Lambda\mathcal{O} + a)$.

(iii) ***Spectrum condition***: The energy momentum spectrum, i.e. of the generators of the translations $U(a)$ lies in V^+.

(iv) ***Cyclicity of the vacuum***: $\cup_{\mathcal{O}}\mathcal{A}(\mathcal{O})\Omega$ is dense in $\mathscr{H}$.

Example: Free Field

$$(\Box + m^2)\phi = 0,$$

Algebra of observables generated by

$$\mathcal{A}(\mathcal{O}) := \{e^{i\phi(f)}, suppf \subset \mathcal{O}\}''$$

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Motivation

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Wald[94] defined renormalized energy-momentum tensor

$$T^{ren}_{\mu\nu}(x) = \lim_{y \to x} \left(T_{\mu\nu}(x, y) - t_{\mu\nu}(x, y) \right)$$

where $t_{\mu\nu}$ is the EV w.r.t a quasi-free Hadamard state ω as "reference state"

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$$t_{\mu\nu}(x,y) = \omega(T_{\mu\nu}(x,y))$$

$\Rightarrow$ $T^{ren}_{\mu\nu}(x)$ exists as a well defined op.v.d. in all representations induced by arbitrary Hadamard states.

Motivation

Problem:

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Problem: For renormalized energy-momentum tensor there is a non-uniqueness of reference states!

Wald solved it by imposing as a principle of locality and covariance that energy-momentum tensor only locally depends on the spacetime metric.

Sketch: Assume one has a prescription for $T^{ren}_{\mu\nu}(x)$ on any curved (globally hyperbolic) spacetime $\Rightarrow$ Let κ be an isometric diffeomorphism ($\kappa_* g = g'$) and $\alpha'_\kappa : \mathcal{A}_{M'}(\mathcal{O}') \to \mathcal{A}_M(\mathcal{O})$ is a canonical isomorphism between the local CCR algebras then EMT is covariant and local if:

$$\alpha'_\kappa(T'^{ren}_{\mu\nu}(x')) = \kappa_* T^{ren}_{\mu\nu}(x),$$

where $x' \in \mathcal{O}' \subset M', x \in \mathcal{O} \subset M$.

Motivation

From the prescription two things should be noted:

(i) The neighborhood $\mathcal{O}$ was arbitrary

(ii) Condition uses the fact that QFT can be defined on any globally hyperbolic spacetime and using an algebraic isomorphism α'_κ one can identify QFT's on isometrically diffeomorphic subregions of globally hyperbolic spacetimes

$\Rightarrow$ Main Purpose of AQFT in terms of Category Theory : Formalization of these properties

AQFT in terms of Category Theory

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What the heck???

Outline

Definitions I

Morphism

A structure-preserving map from one mathematical structure to another.

Homomorphism

A structure-preserving map between two algebraic structures of the same type.

Monomorphism

An injective homomorphism or a left-cancellative morphism, that is, an arrow $f : X \to Y$ such that, for all morphisms $g_1, g_2 : Z \to X$,

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2.$$

Definitions II

Category C, (ob(C), arrows)

(i) A class of objects denoted by ob(C)

(ii) A class hom(C) of morphisms, s.t. $\forall f$ has a source a and a target object b where $a, b \in$ ob(C), i.e. $f : a \to b$

(iii) For $a, b, c \in$ ob(C), $\exists$ a binary operation hom$(a, b) \times$ hom$(b, c) \to$ hom(a, c) (composition); s.t.

  (i) (associativity) if $f : a \to b, g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$,

  (ii) (identity) for every object x, $\exists$ morphism $1_x : x \to x$ called identity morphism for x

Definitions III

Functor

Let C and D be categories. A **functor** F from C to D is a mapping that associates to each object X in C an object $F(X)$ in D and associates to each morphism $f : X \to Y$ in C a morphism $F(f) : F(X) \to F(Y)$ in D s.t:

(i) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object X in C,

(ii) $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in C.

Functors must preserve identity morphisms and composition of morphisms.

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Globally Hyperbolic Spacetimes???

Definitions IV

Globally hyperbolic spacetime (M, g)

M a smooth, four-dimensional, orientable and time-orientable MF!

Time-orientability: $\exists\, C^\infty$-VF u on M s.t. $g(u,u) > 0$.

A smooth curve $\gamma : I \to M$, I being a connected subset of $\mathbb{R}$, is **causal** if $g(\dot\gamma, \dot\gamma) \geq 0$. A CC is future directed if $g(\dot\gamma, u) > 0$ and past directed if $g(\dot\gamma, u) < 0$.

For any point $x \in M$, $J^\pm(x)$ denotes the set of all points in M which can be connected to x by a future$(+)$/past $(-)$-directed causal curve.

M is **globally hyperbolic** if for $x, y \in M$ the set $J^-(x) \cap J^+(y)$ is compact if non-empty.

Intuitively: The spacetime has a Cauchy surface!

Advantage of GHST: Cauchy-problem for linear hyperbolic wave-equation is well-posed.

Isometric Embedding

Let (M_1, g_1) and (M_2, g_2) be two globally hyperbolic spacetimes. A map $\psi : M_1 \to M_2$ is called an **isometric** embedding if ψ is a diffeomorphism onto its range $\psi(M)$, i.e. $\bar{\psi} : M_1 \to \psi(M_1) \subset M_2$ is a diffeomorphism and if ψ is an isometry, that is, $\psi_* g_1 = g_2 \upharpoonright \psi(M_1)$.

Outline

Categories

Man: Class of all objects Obj(Man) formed by globally hyperbolic spacetimes (M, g). Given two such objects (M_1, g_1) and (M_2, g_2), the morphisms $\psi \in \text{hom}_{Man}((M_1, g_1), (M_2, g_2))$ are taken to be the isometric embeddings $\psi : (M_1, g_1) \to (M_2, g_2)$ of (M_1, g_1) into (M_2, g_2) as defined above, but with constraint :

> The isometric embedding preserves orientation and time-orientation of the embedded spacetime.

Alg: Category class of objects Obj(Alg) formed by all C^* -algebras possessing unit elements, and the morphisms are faithful (injective) unit-preserving $*$-homomorphisms. For $\alpha \in \text{hom}_{Alg}(\mathcal{A}_1, \mathcal{A}_2)$ and $\alpha' \in \text{hom}_{Alg}(\mathcal{A}_2, \mathcal{A}_3)$ the composition $\alpha \circ \alpha' \in \text{hom}_{Alg}(\mathcal{A}_1, \mathcal{A}_3)$.

Locally covariant quantum field theory

(i) LCQFT is a covariant functor $\mathscr{A}$ between the two categories *Man* and *Alg*, i.e., writing α_ψ for $\mathscr{A}(\psi)$:

$$\begin{array}{ccc} (M,g) & \xrightarrow{\psi} & (M',g') \\ \downarrow{\scriptstyle \mathscr{A}} & & \downarrow{\scriptstyle \mathscr{A}} \\ \mathscr{A}(M,g) & \xrightarrow{\alpha_\psi} & \mathscr{A}(M',g') \end{array}$$

together with the covariance properties

$$\alpha_{\psi'} \circ \alpha_\psi = \alpha_{\psi' \circ \psi}, \qquad \alpha_{\mathrm{id}_M} = \mathrm{id}_{\mathscr{A}(M,g)},$$

for all morphisms $\psi \in \hom_{Man}((M_1, g_1), (M_2, g_2))$, all morphisms $\psi' \in \hom_{Man}((M_2, g_2), (M_3, g_3))$ and all $(M, g) \in \mathrm{Obj}(Man)$.

(ii) A LCQFT described by a covariant functor $\mathscr{A}$ is called causal if: There are morphisms $\psi_j \in \hom_{Man}((M_j, g_j), (M, g)), j = 1, 2$, so that $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally separated in (M, g), then

$$[\alpha_{\psi_1}(\mathscr{A}(M_1, g_1)), \alpha_{\psi_2}(\mathscr{A}(M_2, g_2))] = 0,$$

(iii) We say that a locally covariant quantum field theory given by the functor $\mathscr{A}$ obeys the **time-slice axiom** if

$$\alpha_\psi(\mathscr{A}(M, g)) = \mathscr{A}(M', g')$$

holds for all $\psi \in \hom_{Man}((M, g), (M', g'))$ such that $\psi(M)$ contains a Cauchysurface for (M', g').

Example KG-field

Global hyperbolicity entails the well-posedness of the Cauchy-problem for the scalar Klein-Gordon equation on (M, g),

$$(\nabla^a \nabla_a + m^2 + \xi R)\varphi = 0$$

Let $E = E_{adv} - E_{ret}$ be the causal propagator of the Klein-Gordon equation and the range of $E(C_0^\infty(M, \mathbb{R}))$ is denoted by $\mathcal{R}$. By defining

$$\sigma(f, Eh) = \int_M f(Eh)d\mu_g, \qquad f, h \in C_0^\infty(M, \mathbb{R})$$

it endows $\mathcal{R}$ with a symplectic form, and thus $(\mathcal{R}, \sigma)$ is a symplectic space. $\Rightarrow$ Weyl-algebra $\mathscr{A}(M, g) = \mathcal{W}(\mathcal{R}, \sigma)$, generated by $W(\phi)$, $\phi \in \mathcal{R}$ satisfying

$$W(\phi)W(\psi) = e^{-i\sigma(\phi,\psi)}W(\phi + \psi).$$

Example KG-field

$(E, \mathcal{R}, \sigma)$ denotes the propagator, the range space and the symplectic form corresponding to a KG-field on (M, g), $(E', \mathcal{R}', \sigma')$ denotes the same for (M', g') and $(E^\psi, \mathcal{R}^\psi, \sigma^\psi)$ for the spacetime $(\psi(M), \psi_* g)$.

$\exists$ C^*-alg. iso., $\tilde{\alpha}_\psi : \mathcal{W}(\mathcal{R}, \sigma) \to \mathcal{W}(\mathcal{R}^\psi, \sigma^\psi)$ so that

$$\tilde{\alpha}_\psi(W(\phi)) = W^\psi(\psi_*(\phi)), \qquad \phi \in \mathcal{R}$$

$\exists$ a symplectic map $T^\psi : (\mathcal{R}^\psi, \sigma^\psi) \to (\mathcal{R}', \sigma')$ assigns to each element $Ef \to E' \iota_{\psi *} f \Rightarrow$ a C^*-alg. endom. $\tilde{\alpha}_{\iota_\psi} : \mathcal{W}(\mathcal{R}^\psi, \sigma^\psi) \to \mathcal{W}(\mathcal{R}', \sigma')$:

$$\tilde{\alpha}_{\iota_\psi}(W^\psi(\phi)) = W'(T^\psi \phi), \qquad \phi \in \mathcal{R}^\psi$$

Theorem

By defining for each $(M, g) \in Obj(Man)$ the C^* -algebra $\mathscr{A}(M, g) = \mathcal{W}(\mathcal{R}, \sigma)$ of the KG equation and for each $\psi \in hom(M, M^{'})$ the C^*-algebraic endomorphism $\alpha_\psi = \tilde{\alpha}_{\iota_\psi} \circ \tilde{\alpha}_\psi : \mathscr{A}(M, g) \to \mathscr{A}(M', g')$, then one obtains a covariant functor $\mathscr{A}$ with the properties of the definitions above. Moreover, this functor is causal and fulfills the time-slice axiom.

In this sense, the free Klein-Gordon FT is a locally covariant QFT!

Outline

Recovering AQFT I

For each element $L \in \mathcal{P}_+^\uparrow$ $\exists$ C^*-algebra $\mathcal{A}$ automorphism $\alpha_L : \mathcal{A} \to \mathcal{A}$:

$$\alpha_{L_1} \circ \alpha_{L_2} = \alpha_{L_1 \circ L_2}, \qquad L_1, L_2 \in \mathcal{P}_+^\uparrow$$

$\mathcal{K}(M, g)$ denotes the set of all subsets in M which are relatively compact and contain with each pair of points x and y also all g-causal curves in M connecting x and y. Given $\mathcal{O} \in \mathcal{K}(M, g)$ we denote $g_{\mathcal{O}}$ the Lorentzian metric restricted to $\mathcal{O}$ so that the injection map $\iota_{M,\mathcal{O}} : (\mathcal{O}, g_{\mathcal{O}}) \to (M, g)$ is the identical map restricted to $\mathcal{O}$.

Recovering AQFT II

Proposition

Let $\mathscr{A}$ be a covariant functor with the properties stated in the Definition of a locally covariant QFT and define a map $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathscr{A}(M,g)$ by setting

$$\mathcal{A}(\mathcal{O}) := \alpha_{\iota_{M,\mathcal{O}}}(\mathscr{A}(\mathcal{O}, g_{\mathcal{O}}))$$

(a) The map fulfills isotony,

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2), \qquad \forall \mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}(M,g)$$

(b) If $\exists$ a group G of isometric diffeomorphisms $\kappa : M \to M$ preserving orientation and time-orientation,then there is a representation of G by a C^*-algebra automorphism $\tilde{\alpha}_\kappa : \mathcal{A} \to \mathcal{A}$ such that

$$\tilde{\alpha}_\kappa(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\kappa(\mathcal{O})), \qquad \mathcal{O} \in \mathcal{K}(M,g)$$

Recovering AQFT II

(c) If the theory given by $\mathscr{A}$ is additionally causal, then it holds

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$$

for all $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}(M, g)$ with $\mathcal{O}_1$ causally separated from $\mathcal{O}_2$.

Outline

Conclusion and Outlook

(i) A locally covariant quantum field theory is an assignment of C^*-algebras to (all) globally hyperbolic spacetimes so that the algebras are identifiable when the spacetimes are isometric, in the indicated way.

(ii) Holds for the Klein-Gordon field on a curved spacetime

(iii) Recovered AQFT in this framework

(iv) Framework possibly allows to define an isomorphism between AQFT and TQFT*

*Joint work with Robert Oeckl

Thank you for your attention!