Classical General Boundary Field Theory

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— Outline —

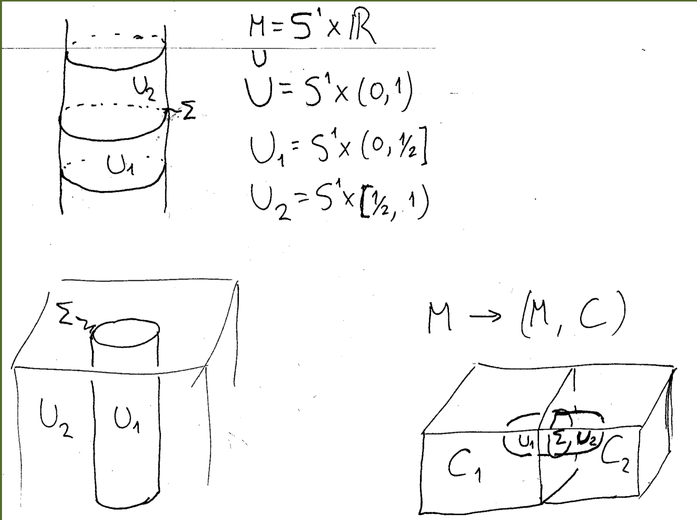
Hypotheses

The arena for classical field theory is spacetime
– a four dimensional differentiable manifold –.

Einstein's theory of General Relativity sets spacetime geometry as a dynamical field interacting with matter fields.

Since the causal structure follows from spacetime geometry, spacetime has no predetermined causal structure (unless gravity is not among the dyn. fields under study).

Consider a spacetime region $U \subset M$ possibly with boundary, which may be subdivided $U = U_1 \#_{\Sigma} U_2$



First-order Lagrangian field theory in $U \subset M$.

Histories:

$$
\begin{aligned}
U \longmapsto \quad & Y|_U, \\
& \textit{Hists}_U \ni \Phi : U \to Y|_U \quad \text{section} \\
\partial U \longmapsto \quad & Y|_{\partial U}, \\
& \textit{Hists}_{\partial U} \ni \phi : \partial U \to Y|_{\partial U}
\end{aligned}
$$

We will display histories as

$$
j\Phi(x) = (x^i, u^\alpha = \Phi^\alpha(x), v_i^\alpha = \partial_i \Phi^\alpha(x), \ldots) \in JY|_U
$$

If truncated to 1st order the notation will be $j^1\Phi \in JY^1|_U$.

First-order Lagrangian field theory in $U \subset M$.

Variations of histories:

Φ_t curve of histories starting at $\Phi_{t=0} = \Phi_0$
with tangent vector at $t = 0$ denoted by $\delta\Phi_V$
and described by an evolutionary vector field $V \in \mathit{Ev}_U$

$$V : Y|_U \to TY|_U$$

(not really a VF) in $Y|_U$ defined in a neighborhood of $\Phi_0(U)$
$V(x, \Phi_0(x)) = 0\frac{\partial}{\partial x^i} + b^\alpha \frac{\partial}{\partial u^\alpha}$
where b^α may depend on Φ_0 and all its partial derivatives.
The variation generator is displayed as

$$jV(j\Phi(x)) = (0, v^a(j\phi(x)), v_i^a(j\phi(x)), ...)$$

First-order Lagrangian field theory in $U \subset M$.

Differentials and Lie derivative:

The differential in the jet splits as $d = d_v + d_h$
where $j\phi^* d_h = dj\phi^*$, $d_v^2 = d_h^2 = 0$, $d_v d_h = d_h d_v$.

The space of k-differential forms splits into a sum of spaces of forms with definite horizontal and vertical degrees
$\Omega^{r\,s}(JY)$ with $r + s = k$,
and d_v, d_h increase the corresp. degree by one.

Given an evolutionary vector field V the Lie derivative of $\mu \in \Omega^{r\,s}(JY)$ is

$$\mathcal{L}_{jV}\mu \doteq (d_v \iota_{jV} + \iota_{jV} d_v)\mu.$$

First-order Lagrangian field theory in $U \subset M$.

Dynamics is determined by the action

$$S_U[\phi] = \int_U L \circ j^1\phi, \qquad L(j^1\phi(x)) = L(x^i, \Phi^\alpha(x), \partial_i \Phi^\alpha(x))$$

$$dS_U[\delta\phi_V] = \int_U j\phi^* \mathcal{L}_{jV} L = \int_U j\phi^* \iota_{jV} E(L) + \int_{\partial U} j\phi^* \iota_{jV} \Theta_L,$$

or $d_v L = E(L) + d_h \Theta_L$, where $E(L) = I d_v L$ is an int. by parts op.

Hamilton's principle gives us:
(i) the field equation
$j\phi^* \iota_{jV} E(L) = 0 \, \forall V \ \Rightarrow \ \phi \in \mathit{Sols}_U$ or $j\phi(U) \subset \mathcal{E}_L$ and
(ii) a cons. law for the (pre) symplectic current[2] $\Omega_L \doteq -d_v \Theta_L$.

$$\begin{aligned} d_h \Omega_L|_{\mathcal{E}_L, \mathfrak{F}_L} &= 0 \ \text{or} \\ \omega_{\partial U}(\delta\phi_V, \delta\phi_W) = \int_{\partial U} j\phi^*_W \iota_{jV} \Omega_L &= 0, \quad \forall\, \phi \in \mathit{Sols}_U,\ V, W \in \mathfrak{F}_L. \end{aligned}$$

[2] There is a "corner ambiguity" in Θ_L and in Ω_L.

Summary up to now

$$
\begin{aligned}
U \longmapsto \quad & Y|_U,\ JY|_U \supset \underline{\mathcal{E}_{L\,U}} \quad (\text{fits in } J^2Y) \\
& \mathit{Hists}_U \supset \underline{\mathit{Sols}_U} \\
& \mathit{Ev}_U \supset \underline{\mathfrak{F}_U} \text{ los que preservan } \mathcal{E}_{L\,U} \\
\partial U \longmapsto \quad & Y|_{\partial U},\ JY|_{\partial U} \supset \underline{\mathcal{E}_{L\,\partial U}} = (\mathcal{E}_{L\,U} \cup JY|_{\partial U}) \\
& \mathit{Hists}_{\partial U} \supset \underline{\mathit{Sols}_{\partial U}} \text{ germs of sols. } \underline{\text{1st ord. data}} \text{ for 2PDEs} \\
& \mathit{Ev}_{\partial U} \supset \underline{\mathfrak{F}_{\partial U}} \text{ los que preservan } \mathcal{E}_{L\,\partial U}
\end{aligned}
$$

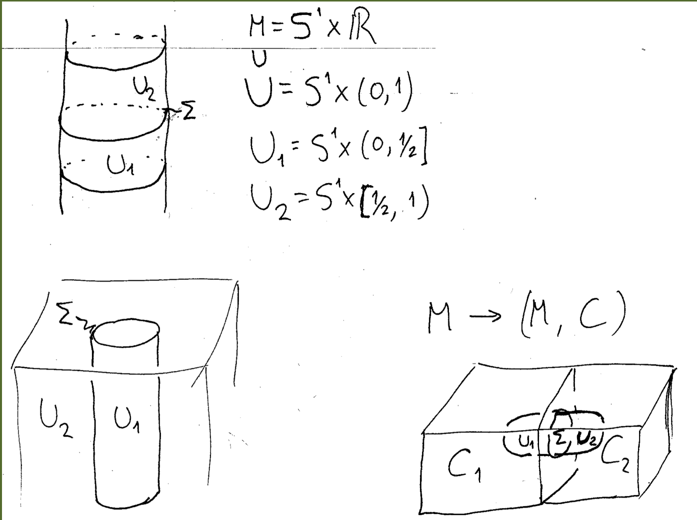
The conservation law for Ω_L can be described as

$$\mathit{Sols}_U \longmapsto (\mathit{Sols}_{\partial U}, \omega_{\partial U}) \text{ as a Lagrangian subspace.}$$

In general there are "constraints" among 1st ord. bdary data: global compat. (as in $\partial U = \Sigma_1 - \Sigma_0$), and local constraints.

Gauge from holography: a sketch of the argument

Consider an imaginary division of a spacetime region
$U = U_1 \#_\Sigma U_2$ and a solution $\phi = \phi_1 \#_\Sigma \phi_2$.



Gauge from holography: a sketch of the argument

A perturbation V_1 of ϕ_1 is coupled to V_2 by means of its **first-order holographic imprint** "$V_1|_\Sigma$".

The **linearized gluing field equation** "LG_Σ" correlating $V_1|_\Sigma$ with $V_2|_\Sigma$ may have a nontrivial null space. Some $V_1 \neq 0$ may be physically irrelevant in U_2.

Newton's **Principle of Determinacy** tells us that if $V_1|_\Sigma \in \mathsf{Ker}(LG_\Sigma)$ for any Σ then physically speaking $V_1 \sim 0$.

> This criterion, together with a locality requirement, leads to the notion of gauge equivalence in first-order Lagrangian field theory.

Details in the argument

Consider a physical field $\phi = \phi_1 \#_\Sigma \phi_2$ over U.
$dS_U[\delta\phi_V] = dS_{U_1}[\delta\phi_{V_1}] + dS_{U_2}[\delta\phi_{V_2}]$ includes the term

$$\int_\Sigma (j\phi_1^* \iota_{jV_1} \Theta_L - j\phi_2^* \iota_{jV_2} \Theta_L)$$

leading to a **gluing equation** for ϕ^a demanding momentum flux matching at Σ.

A physical perturbation of the field $V = V_1 \#_\Sigma V_2$ obeys:
(i) the linearized field equation at U_1 and U_2,
(ii) 0^{th} order continuity along Σ, and
(iii) the **linearized gluing equation** LG_Σ

$$\iota_{jV_1}\Omega_L|_\Sigma \simeq -\iota_{jV_2}\Omega_L|_\Sigma \text{ (apart from bdry terms; no ambg.).}$$

A physical perturbation whose **1st order holographic imprint**

$$jV_1|_\Sigma \in \mathrm{Ker}(LG_\Sigma) \qquad (\Sigma \sim \Sigma' = \Sigma + \partial B)$$

does not carry any transversal information through Σ, hence the reference to a holographic principle.

Locality of measurement

There are currents F defined only up to cohomology class leading to observables through integration

$$f_\Sigma[\phi] = \int_\Sigma j\phi^* F.$$

A class of examples is given by Ω_L contracted with a pair of physical perturbations

$$\omega_\Sigma[\phi, V, W] = \int_\Sigma j\phi^* \iota_{jW}\iota_{jV}\Omega_L.$$

The locality condition amounts to requiring that
for any hypersurface with $\partial\Sigma \subset \partial U$
integrals of this type can be evaluated and are gauge invariant.
In a mutiplisubdivided domain which contains a Cauchy surface
$f_\Sigma = f_{\Sigma_1} + f_{\Sigma_2} + \dots$

Relativity of measurement

Measuring a property of $\phi|_U$
relative to properties of the field outside of U
may be done in the 1st order formalism
using properties of $j\phi|_{\partial U}$ as a reference.

It is convenient to keep the reference gauge invariant.

Gauge equivalence

Definition (Gauge perturbations)

Given a solution ϕ of the field equation over U, a solution V of the linearized field equation around ϕ is declared to be a gauge perturbation if and only if:

(i) V is in the null space of all linearized gluing field equations, i.e.

$j\phi^* \iota_{jV} \Omega_L|_{\mathfrak{F}_L}$ *is a pure divergence (no ambg.), or equiv.*

$$d_h \iota_{jV} \Omega_L|_{\mathcal{E}_L, \mathfrak{F}_L} = 0$$

and

(ii) $j^1 V|_{\partial U} \simeq 0$.

The space of gauge vector fields will be denoted by $\mathfrak{G}_U \subset \mathfrak{F}_U$.

Gauge equivalence

Te prolongation of a VF in $V \in \mathfrak{F}_U$ to the infinite jet JY leads to an ordinary VF jV in JY.

It is easy to verify that $j(\mathfrak{G}_U) \subset j(\mathfrak{F}_U)$ is a Lie subalgebra.
Gauge VFs generate orbits in Sols_U
i.e. **gauge equiv. classes of solutions.**

We will talk about the gauge group $\mathcal{G}_U$ acting on Sols_U.

Remarks

- The usual definition of gauge perturbations in terms of local symmetries of the Lagrangian implies (i); see for example [2].
- Conditions equivalent to (i) have been conjectured to imply the usual definition [3].
  It is the usual notion of gauge for Yang-Mills and General Relativity.
- The derivation of (i) in terms of the linearized gluing field equation is, to the best of my knowledge, new.
- An alternative to condition (ii) is to add boundary degrees of freedom [1].

End of Part 1

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Beginning of Part 2

Codim 1 surfaces

The importance of codim 1 surfaces such that $\partial\Sigma \subset \partial U$ is that they **separate spacetime into two connected components**

$$U = U_1 \#_\Sigma U_2.$$

"Any communication between U_1 and U_2 must cross Σ".
(Admissible) 1st order data at Σ locally determine solutions of 2nd order PDEs.

Consider M with a Cauchy surf. and a subd. of M into cells $\{U_i\}$.
Due to (the locality) Condition (ii) in the defn. of gauge VFs given $V; W \in \mathfrak{F}_U^{\mathfrak{G}}$

$$\omega_\Sigma(\delta\phi_V, \delta\phi_W) = \int_\Sigma j\phi^* \iota_{jW} \iota_{jV} \Omega_L = \sum_i \omega_{\Sigma_i}(\delta\phi_V|_{U_i}, \delta\phi_W|_{U_i})$$

with each term $\mathfrak{G}_{U_i}$ invariant and the sum $\mathfrak{G}_U$ invariant.
Each term afected by the "corner ambiguity" but not the sum (unless Sup(V) and Sup(W) intersect ∂U).

Summary up to now

$$
\begin{aligned}
U \longmapsto \quad & Y|_U,\; JY|_U \supset \mathcal{E}_{L\,U} \\
& \mathit{Hists}_U \supset \mathit{Sols}_U \to \underline{(\mathit{Sols}/\mathcal{G})_U} \\
& \mathit{Ev}_U \supset \mathfrak{F}_U \supset \mathfrak{F}_U^{\mathcal{G}} \to \underline{(\mathfrak{F}//\mathfrak{G})_U} \\
\Sigma \longmapsto \quad & Y|_\Sigma,\; JY|_\Sigma \supset \mathcal{E}_{L\,\Sigma} \leftrightarrow \mathcal{E}^1_{L\,\Sigma} \subset J^1 Y|_\Sigma \\
& \mathit{Hists}_\Sigma \supset \mathit{Sols}_\Sigma \to \underline{(\mathit{Sols}/\mathcal{G})_\Sigma} \text{ with symp. str. } \omega_\Sigma^{\mathrm{red}} \\
& \mathit{Ev}_\Sigma \supset \mathfrak{F}_\Sigma \supset \mathfrak{F}_\Sigma^{\mathcal{G}} \to \underline{(\mathfrak{F}//\mathfrak{G})_\Sigma}
\end{aligned}
$$

$\lessdot$ Multisympl.: The cons. law for Ω_L can be understood as a compatibility property of $\Sigma \overset{\Omega_L}{\longmapsto} (\Gamma(J^1 Y|_\Sigma), \omega_\Sigma)$

$$\mathit{Sols}_U \longmapsto (\Gamma(J^1 Y|_{\partial U}), \omega_{\partial U}) \text{ as a Lagrangian subspace.}$$

In general there are “constraints” among 1st ord. bdary data: global compat. (as in $\partial U = \Sigma_1 - \Sigma_0$), and local constraints.

There are gauge perts. at $U = U_1 \#_\Sigma U_2$ not vanishing over Σ. Considered over U_i these perts. are nontrivial symm. generators, but considered over U they are gauge perts.

The neglected gauge perturbations before gluing are $\mathfrak{G}_\Sigma = \mathfrak{G}_U / (\mathfrak{G}_{U_1} \#_\Sigma \mathfrak{G}_{U_2})$.

Proposition

$$(\mathit{Sols}/\mathcal{G})_U = ((\mathit{Sols}/\mathcal{G})_{U_1} \#_\Sigma (\mathit{Sols}/\mathcal{G})_{U_2}) \,/\, \mathcal{G}_\Sigma,$$

Observables

S_U and the variational principle lead to a model of dynamics **in the absence of** measuring devices disturbing the system.

$$S_U + \lambda f_U^n \mapsto \phi_\lambda = \phi_0 + \lambda \delta\phi_{V_{f^n}} + O(\lambda^2)$$

models the system together with a "minimally-disturbing apparatus" in U measuring the property of the system

$$f_U^n : \mathit{Hists}_U \to \mathbb{R}$$

(defined in a certain open $\text{Domain}_f \subset \mathit{Hists}_U$).
The Peierls bracket uses this idea to construct a map

$$f_U^n \mapsto V_{f^n} \in \mathcal{F}_U.$$

The boundary conditions used in the construction of $V_{f^n} = V_{f^n}^+ - V_{f^n}^-$ are advanced and retarded respectively. Conditions of this type are not immediately extendible to boundaries $\partial U \neq \Sigma_1 - \Sigma_0$.

This point of view motivates: formal deformation quantization and strict deformation quantization.

Observables

$\sphericalangle$ Multisympl.: $\Sigma \overset{\Omega_L}{\longmapsto} (\Gamma(J^1 Y|_\Sigma), \omega_\Sigma)$.
Observables $f_\Sigma^{n-1} : \Gamma(J^1 Y|_\Sigma)\mathbb{R}$

$$f_\Sigma^{n-1}[\phi] = \int_\Sigma j\phi^* F$$

may follow a conservation law compatible with the compat. cond. of Ω_L (inv. under $\Sigma \to \Sigma' = \Sigma + \partial B$) and they play a double role:

(i) measure a local property of the system (above formula)
(ii) generate transformations in $\Gamma(J^1 Y|_\Sigma)$ by $X_f \in \mathfrak{F}_\Sigma^{\mathrm{LH}}$ determ. by $df_\Sigma^{n-1}[\phi] = -\iota_{X_f}\omega_\Sigma[\phi] = -\int_\Sigma j\phi^* \iota_{X_f}\Omega_L$ or by

$$d_v F = -\iota_{X_F}\Omega_L + d_h \sigma_F.$$

Warning: An interesting enough algebra of VFs in $\Gamma(J^1 Y|_\Sigma)$ requires X_F to be an evolutionary VF in $J^1 Y$ or a VF in JY. This forces F to be a conserved current in $JY|U$.

Observable Currents

A *locally defined* current $F \in \Omega^{n-1,0}(JY|_U)$
which is conserved and gauge invariant
is called an observable current, $F \in \mathrm{OC}_U$.

- Observables induced by observable currents separate points in $(\mathit{Sols}/\mathcal{G})_U$.
- The assignment $\mathrm{OC}_U \supset \mathrm{OC}_U^{\mathrm{H}} \ni F \longmapsto X_F \in \mathfrak{F}_U^{\mathrm{LH}}$ yields a Lie algebra structure in $\mathrm{OC}_U = \mathrm{OC}_U^{\mathrm{H}}$.

Gluing spacetime localized subsystems

There are gauge perts. at $U = U_1 \#_\Sigma U_2$ not vanishing over Σ. Considered over U_i these perts. are nontrivial symm. generators, but considered over U they are gauge perts.

The neglected gauge perturbations before gluing are $\mathfrak{G}_\Sigma = \mathfrak{G}_U / (\mathfrak{G}_{U_1} \#_\Sigma \mathfrak{G}_{U_2})$.

Proposition

$$(\mathit{Sols}/\mathcal{G})_U = ((\mathit{Sols}/\mathcal{G})_{U_1} \#_\Sigma (\mathit{Sols}/\mathcal{G})_{U_2}) \,/\, \mathcal{G}_\Sigma,$$

$$\mathrm{OC}_U = \mathrm{Inv}_{\mathcal{G}_\Sigma}(\mathrm{OC}_{U_1} \#_\Sigma \mathrm{OC}_{U_2}) \, .$$

Looking for interesting gravitational observables

- Any function of $\phi|_{\partial U}$ evaluated on solutions is an observable.
  The origin of this wealth of observables is our choice of having a gauge invariant reference at ∂U.
- We will look for interesting observables guided by the geometric structure of the field theory.
  We will study locally Hamiltonian perturbations and "locally defined" observables related to them.

Observable currents from vector fields

The integration of an observable current $F \in \mathrm{OC}_U$

$$f_\Sigma[\phi] = \int_\Sigma j\phi^* F.$$

is an observable $f_\Sigma : \mathrm{Sols}_U \to \mathbb{R}$ if $\partial\Sigma \subset \partial U$,
which is invariant under $\Sigma \to \Sigma' = \Sigma + \partial U'$.

Locally defined observable currents of this type, $F_V \in \mathrm{OC}_U$, may be constr. from locally Ham. vector fields V in Sols_U:
$V \xmapsto{\Omega_L} d_v F_V \mapsto F_V$ (up to a "constant").

Also $F_{VW} = \iota_{jW}\iota_{jV}\Omega_L \in \mathrm{OC}_U$, defined for any pair of physical perturbations V, W yields the observable

$$\omega_\Sigma[\phi, V, W] = \int_\Sigma j\phi^* F_{VW}.$$

Gravitational observables related to would be gauge d.o.f.

In GR on a confined spacetime domain $U \subset M$
gauge perts. are those generating diffeomorphisms fixing ∂U,

$$g_{ab} \mapsto g_{ab} + h^X_{ab} \quad with \quad h^X_{ab} = \nabla_{(a} X_{b)} \quad and \quad X = X(jg(x)).$$

It is a fact that $h^X_{ab} \overset{\Omega_L}{\longmapsto} j\phi^* d_v F^X$ is a pure divergence (field eq.).

Then the corresp. obs. are holographic (i.e. bdary obs. [1])

$$f_\Sigma[g] = \int_\Sigma j\phi^* F^X = \int_{\partial\Sigma} j\phi^* \tilde{F}^X + \text{ const.}$$

Holographic gravitational observables on localized domains

Anderson and Torre proved that **the only vector fields** in $\mathrm{Sols}_U^{\mathrm{GR}}$
$h_{ab} = h_{ab}(jg(x)) = h_{ab}(x, g, \partial_x g, \dots)$ depending on an arbitrary, but finite, number of partial derivatives of the metric **are** [3]

$$h_{ab} = h_{ab}^X + c\, g_{ab}.$$

It would seem that (apart from rescaling gens.)
all vector fields in $\mathrm{Sols}_U^{\mathrm{GR}}$ are gauge <u>or would be gauge</u>.
($\not\leftrightarrow$ free initial data sets for GR.)

In our context, a corollary of Torre [4] says that
all gravitational observables (dep. on fin. many p. ders. of g) obtained integrating currents **are holographic**.

Thank you for your attention!

References

't Hooft, Susskind, Bousso, Maldacena, Witten, Marolf, Strominger, ...

- T. Regge and C. Teitelboim, "Role of surface integrals in the Hamiltonian formulation of general relativity," Ann. Phys. **88**, 286 (1974).
- J. Lee and R. M. Wald, "Local symmetries and constraints," J. Math. Phys. **31**, 725 (1990).
- E. G. Reyes, "On Covariant Phase Space and the Variational Bicomplex," Int. J. Theor. Phys. **43**, 1267 (2004).
- L. Vitagliano, J. Geom. Phys. **59**, 426 (2009) [arXiv:0809.4164 [math.DG]].

References

W. Donnelly and L. Freidel, "Local subsystems in gauge theory and gravity," JHEP **1609**, 102 (2016) [arXiv:1601.04744 [hep-th]].

M. Geiller, "Edge modes and corner ambiguities in 3d Chern–Simons theory and gravity," Nucl. Phys. B **924**, 312 (2017) [arXiv:1703.04748 [gr-qc]].

I. M. Anderson and C. G. Torre, "Classification of generalized symmetries for the vacuum Einstein equations," Commun. Math. Phys. **176**, 479 (1996) [gr-qc/9404030].

C. G. Torre, "Gravitational observables and local symmetries," Phys. Rev. D **48**, R2373 (1993) [gr-qc/9306030].

Thank you for your attention!

Solutions mod gauge on glued spacetime regions

It can be shown that gauge perturbations form a Lie subalgebra of evolutionary vector fields.
The Lie algebra of gauge perturbations in a given domain and some of its subalgebras are relevant below.
In the space of solutions these Lie algebras induce flows.

The "space of solutions modulo gauge" $(\mathrm{Sols}/\mathcal{G})_U$
is constructed in two steps from $(\mathrm{Sols}/\mathcal{G})_{U_1}$, $(\mathrm{Sols}/\mathcal{G})_{U_2}$:

1) Consider "the diagonal" of the cartesian product obtained by imposing the Σ-gluing field equation, $(\mathrm{Sols}/\mathcal{G})_{U_1} \#_\Sigma (\mathrm{Sols}/\mathcal{G})_{U_2}$.

2) Take a quotient by the group generated by the neglected gauge perturbations, $\mathcal{G}_\Sigma = \mathcal{G}_U / (\mathcal{G}_{U_1} \#_\Sigma \mathcal{G}_{U_2})$.

Gluing observable current algebras of adjacent domains

Now consider a domain that is composed by two subdomains that intersect along a hypersurface $U = U_1 \#_\Sigma U_2$. There are maps $\mathrm{OC}_U \to \mathrm{OC}_{U_i}$; additionally, the following definition shows how to glue compatible observable currents of the subdomains to produce any observable current in OC_U.

Definition (Gluing algebras of adjacent domains)

- $\mathrm{OC}_{U_1} \#_\Sigma \mathrm{OC}_{U_2} = \{(F_1, F_2) : F_i \in \mathrm{OC}_{U_i}$ *with* $F_1|_\Sigma = F_2|_\Sigma\}$.
- $\mathrm{Inv}_{\mathcal{G}_\Sigma}(\cdot)$ *denotes those OCs in* $\cdot$ *whose Lie derivative along* Σ*-gauge perturbations vanish.*

Proposition

$$\mathrm{OC}_U = \mathrm{Inv}_{\mathcal{G}_\Sigma}(\mathrm{OC}_{U_1} \#_\Sigma \mathrm{OC}_{U_2})\,.$$