Schrödinger-Feynman quantization for general boundary QFT

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Classical theory

We consider the linear theory of a massive real Klein-Gordon field in a 4d Lorentzian spacetime $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

The action of field in a spacetime region M is

$$S_{M,0}[\phi] = \frac{1}{2} \int_M d^4x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - (m^2 + \xi R) \phi^2 \right)$$

where the integration is extended over the sapcetime region M; g is determinant of the metric $g_{\mu\nu}$ and we used the notation $\partial_\mu = \partial / \partial x^\mu$.
Assumptions:

- The spacetime region M is foliated by hypersurfaces, *not necessarily spacelike*, described by coordinates $(\tau, \underline{x})$. The coordinates on the leaves of the foliation are denoted by $\underline{x} = (x^1, x^2, x^3) \in I \subset \mathbb{R}^3$ and the leaves are parametrized by the coordinate $\tau \in J \subset \mathbb{R}$. Notice that τ and $\underline{x}$ are not required to be timelike and spacelike coordinates respectively.

- The metric takes a block-diagonal form with respect to the coordinates $(\tau, \underline{x})$, i.e. $g^{\tau x^i} = 0 = g^{x^i \tau}$ for all $i \in \{1, 2, 3\}$.
- The Euler-Lagrangian equations are solved by separation of variables: Let X_a and X_b be the two independent solutions of the $\tau-$part of the e.o.m.

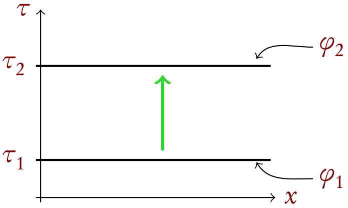
$$\begin{aligned}\phi(\tau, \underline{x}) &= \int \mathrm{d}\underline{k} \left(d_a(\underline{k}) X_{a,\underline{k}}(\tau) Y_{a,\underline{k}}(\underline{x}) + d_b(\underline{k}) X_{b,\underline{k}}(\tau) Y_{b,\underline{k}}(\underline{x}) \right), \\ &= (X_a(\tau) Y_a)(\underline{x}) + (X_b(\tau) Y_b)(\underline{x}),\end{aligned}$$

where now $X_{a,b}$ are understood as operators from the space of initial data $Y_{a,b}$ to solution on Σ_τ. We assume that $X_{a,b}$ commute with each other and are invertible.

The most studied quantum field theories on curved spaces are encompassed by these assumptions.

Slice region

Slice region M bounded by the disjoint union of two constant-τ hypersurfaces, $\Sigma_1 = \{(\tau, \underline{x}) : \tau = \tau_1\}$ and $\Sigma_2 = \{(\tau, \underline{x}) : \tau = \tau_2\}$, namely $M = [\tau_1, \tau_2] \times \mathbb{R}^3$.



The solution of the KG equation can be expressed in terms of φ_1 and φ_2.

$$\phi(\tau, \underline{x}) = \left(\frac{\Delta(\tau, \tau_2)}{\Delta(\tau_1, \tau_2)} \varphi_1 \right)(\underline{x}) + \left(\frac{\Delta(\tau_1, \tau)}{\Delta(\tau_1, \tau_2)} \varphi_2 \right)(\underline{x}),$$

where $\Delta(\tau_1, \tau_2) := X_a(\tau_1) X_b(\tau_2) - X_a(\tau_2) X_b(\tau_1)$.

All the deltas must be understood as operators acting on the boundary field configurations φ_1 and φ_2. We assume that these operators are invertible.

The free action in terms of the boundary field configuration is

$$S_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2) = \frac{1}{2}\int \mathrm{d}^3\underline{x}\,\begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix} W_{[\tau_1,\tau_2]}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},$$

where the $W_{[\tau_1,\tau_2]}$ is a 2×2 matrix with elements $W^{(i,j)}_{[\tau_1,\tau_2]}, (i,j=1,2)$, given by

$$W^{(1,1)}_{[\tau_1,\tau_2]} = -\sqrt{|g^{(3)}_{\tau_1} g^{\tau\tau}_{\tau_1}|}\,\frac{\Delta_1(\tau_1,\tau_2)}{\Delta(\tau_1,\tau_2)}, \quad W^{(1,2)}_{[\tau_1,\tau_2]} = -\sqrt{|g^{(3)}_{\tau_1} g^{\tau\tau}_{\tau_1}|}\,\frac{\Delta_2(\tau_1,\tau_1)}{\Delta(\tau_1,\tau_2)},$$

$$W^{(2,1)}_{[\tau_1,\tau_2]} = \sqrt{|g^{(3)}_{\tau_2} g^{\tau\tau}_{\tau_2}|}\,\frac{\Delta_1(\tau_2,\tau_2)}{\Delta(\tau_1,\tau_2)}, \quad W^{(2,2)}_{[\tau_1,\tau_2]} = \sqrt{|g^{(3)}_{\tau_2} g^{\tau\tau}_{\tau_2}|}\,\frac{\Delta_2(\tau_1,\tau_2)}{\Delta(\tau_1,\tau_2)},$$

where

$$\Delta_1(\tau_1,\tau_2) := \partial_\tau \Delta(\tau,\tau_2)\big|_{\tau=\tau_1} \qquad \Delta_2(\tau_1,\tau_2) := \partial_\tau \Delta(\tau_1,\tau)\big|_{\tau=\tau_2}.$$

A symmetry of the classical solution and the action

Both the classical solution (satisfying the boundary conditions) and the action are invariant under the transformation

$$X_a \to \beta X_a, \qquad X_b \to \beta^{-1} X_b,$$

with $\beta \in \mathbb{R} \setminus \{0\}$.

Region with one boundary hypersurface

Region M bounded one constant-τ hypersurface, $\Sigma_0 = \{(\tau, \underline{x}) : \tau = \tau_0\}$.

The classical solution satisfying the boundary condition $\phi\big|_{\tau_0} = \varphi$ is written as

$$\phi(\tau, \underline{x}) = \left(\frac{X(\tau)}{X(\tau_0)} \varphi \right)(\underline{x}),$$

where $X(\tau) = aX_a(\tau) + bX_b(\tau)$.

The free action is

$$S_{\tau_0,0}(\varphi) = \frac{1}{2} \int \mathrm{d}^3\underline{x}\, \varphi(\underline{x}) \sqrt{|g^{(3)}_{\tau_0} g^{\tau\tau}_{\tau_0}|} \frac{\partial_\tau X(\tau)\big|_{\tau_0}}{X(\tau_0)} \varphi(\underline{x})$$

Schrödinger-Feynman quantization: state space

According to the axioms of the GBF, to each hypersurface Σ (oriented manifold of dim $n-1$) is associated a state space $\mathcal{H}_\Sigma$.

- For the KG field in the Schrödinger representation $\mathcal{H}_\Sigma$ is the space of functions on field configurations K_Σ on Σ.
- Inner product,

$$\langle \psi_2 | \psi_1 \rangle = \int_{K_\Sigma} \mathcal{D}\varphi \, \psi_1(\varphi) \overline{\psi_2(\varphi)}.$$

Schrödinger-Feynman quantization: amplitude map

To each region M (oriented manifold of dim n) is associated an amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$

- Amplitude for a state $\psi \in \mathcal{H}_{\partial M}$,

$$\rho_M(\psi) = \int_{K_{\partial M}} \mathcal{D}\varphi \, \psi(\varphi) Z_M(\varphi),$$

  where Z_M is the field propagator given by the Feynman path integral,

$$Z_M(\varphi) = \int_{K_M, \phi|_{\partial M} = \varphi} \mathcal{D}\phi \, e^{\mathrm{i} S_M(\phi)}, \quad \forall \varphi \in K_{\partial M}.$$

  The integral is over the space K_M of space-time field configurations ϕ in the interior of M which agree with φ on the boundary ∂M.

Schrödinger-Feynman quantization: observable map

To each observable F defined in the region M is associated an observable map $\rho_M^F : \mathcal{H}_{\partial M} \to \mathbb{C}$

- A classical observable F in M is modelled as a function on K_M. The quantization of F is the linear map $\rho_M^F : \mathcal{H}_{\partial M} \to \mathbb{C}$ defined as

$$\rho_M^F(\psi) = \int_{K_{\partial M}} \mathcal{D}\varphi\, \psi(\varphi) Z_M^F(\varphi),$$

  where

$$Z_M^F(\varphi) = \int_{K_M, \phi|_{\partial M} = \varphi} \mathcal{D}\phi\, F(\phi) e^{\mathrm{i} S_M(\phi)}.$$

Field propagator in the two regions

The field propagator in the region M can be evaluated by shifting the integration variable by a classical solution matching the boundary field configurations $\phi|_{\partial M} = \varphi$,

$$Z_M(\varphi) = N_M \, e^{iS_M(\varphi)}.$$

where N_M is a normalisation factor.
In the two regions considered, the field propagator satisfies the composition rule

$$Z_{[\tau_1,\tau_3],0}(\varphi_1,\varphi_3) = \int \mathcal{D}\varphi_2 \, Z_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2) Z_{[\tau_2,\tau_3],0}(\varphi_2,\varphi_3)$$

$$Z_{[\tau_1],0}(\varphi_1) = \int \mathcal{D}\varphi_0 \, Z_{[\tau_0],0}(\varphi_0) Z_{[\tau_0,\tau_1],0}(\varphi_0,\varphi_1)$$

and the identity

$$\int \mathcal{D}\varphi_2 \, \overline{Z_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2)} Z_{[\tau_1,\tau_2],0}(\tilde{\varphi}_1,\varphi_2) = \delta\left(\varphi_1,\tilde{\varphi}_1\right),$$

where φ_1 and $\tilde{\varphi}_1$ are field configurations on Σ_1. The above identity can be interpreted in terms of the unitarity of the evolution implemented by the field propagator.

Vacuum state

The vacuum wave functional has the form of a Gaussian,

$$\psi_{\Sigma,0}(\varphi) = C_\Sigma \exp\left(-\frac{1}{2}\int \mathrm{d}^3\underline{x}\, \varphi(\underline{x}) \left(-\mathrm{i}\sqrt{|\mathrm{g}^3_\Sigma|}\,\frac{\partial_n(\Upsilon(\tau))}{\Upsilon(\tau)}\varphi\right)(\underline{x})\right),$$

where g^3_Σ is the determinant of the 3-metric on Σ, $\partial_n = \sqrt{|\mathrm{g}^{\tau\tau}_\Sigma|}\,\partial_\tau$ is the normal derivative to Σ and

$$\Upsilon(\tau) := c_a X_a(\tau) + c_b X_b(\tau),$$

where $c_{a,b}$ are complex numbers s.t. $\overline{c_a} c_b - \overline{c_b} c_a \neq 0$. This condition guarantees that the vacuum state is normalizable. Notice that this condition is left invariant by the β tranformation. However the vacuum state is not:

$$\psi^\beta_{\Sigma,0}(\varphi) = C_\Sigma \exp\left(-\frac{1}{2}\int \mathrm{d}^3\underline{x}\, \varphi(\underline{x}) \left(-\mathrm{i}\sqrt{|\mathrm{g}^3_\Sigma|}\,\frac{\partial_n(\Upsilon_\beta(\tau))}{\Upsilon_\beta(\tau)}\varphi\right)(\underline{x})\right),$$

with $\Upsilon_\beta(\tau) := c_a \beta X_a(\tau) + c_b \beta^{-1} X_b(\tau)$.

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How to fix the value of β?

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- Symmetries of spacetime
  - For Minkowski and Rindler, β is fixed to 1 (and c_a and c_b are also fixed)
  - For dS, all the β vacua are de Sitter invariant (Chernikov-Tagirov, Schomblond-Spindel, Allen, Mottola,...)

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- Wick rotation

  The vacuum state can be obtained as a limit of the Wick rotated field propagator defined in the slice region (Minkowski, Rindler, dS):

$$\lim_{\tau_2 \to \infty} Z_{[\tau_1, i\tau_2]}(\varphi_1, \varphi_2) \simeq \psi_{\Sigma,0}(\varphi_1).$$

  For dS this reduces to the Bunch-Davies vacuum.

Interacting theory

Consider the interaction of the scalar field with a real source field μ described by the action

$$S_{M,\mu}(\phi) = S_{M,0}(\phi) + D(\phi), \qquad D(\phi) = \int_M \mathrm{d}^4x \sqrt{-g(x)}\, \mu(x)\, \phi(x).$$

We assume that the field μ is confined in the interior of the region M, i.e. $\mu(x) = 0$ for $x \in \partial M$ and $x \notin M$.
The amplitude for a coherent state $\psi_\xi \in \mathcal{H}_{\partial M}$ factorizes as

$$\rho_{M,\mu}(\psi_\xi) = \rho_{M,0}(\psi_\xi)\, e^{\mathrm{i}D(\hat{\xi})}\, e^{\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\, \mathrm{d}^4x' \sqrt{g(x)g(x')}\, \mu(x)\, G_F(x,x')\, \mu(x')}$$

where the function $\hat{\xi}$ is complex solution of the KG equation and G_F is the Feynman propagator satisfying the inhomogeneous Klein-Gordon equation in both variables x and x'

$$\left(\Box + m^2\right) G_F(x,x') = (-g(x))^{-1/2}\, \delta^4(x - x').$$

- Slice region:

$$\hat{\xi}(x) = \frac{1}{W^{(1,2)}_{[\tau_1,\tau_2]}(\overline{c_a}\, c_b - \overline{c_b}\, c_a)\,\Delta(\tau_1,\tau_2)} \left(\left(\overline{\Upsilon_\beta(\tau)}\xi_1 \right)(\underline{x}) + \left(\Upsilon_\beta(\tau)\overline{\xi_2} \right)(\underline{x}) \right),$$

and

$$G_F(x,x') = \int \mathrm{d}^3\underline{k}\, \frac{1}{W^{(1,2)}_{[\tau_1,\tau_2]}\,\Delta(\tau_1,\tau_2)(\overline{c_a}\, c_b - \overline{c_b}\, c_a)} \left(\Upsilon_\beta(\tau_<)\varphi_{\underline{k}}\right)(\underline{x})\, \overline{\left(\Upsilon_\beta(\tau_>)\varphi_{\underline{k}}\right)(\underline{x}')}$$

- Region with one boundary hypersurface:

$$\hat{\xi}(x) = -\frac{1}{\mathcal{W}(\overline{c_a}\, c_b - \overline{c_b}\, c_a)} \left(\frac{X(\tau)}{a\, c_a - b\, c_b}\xi \right)(\underline{x})$$

and

$$G_F(x,x') = \int \mathrm{d}^3\underline{k}\, \frac{1}{\mathcal{W}(\overline{c_a}\, c_b - \overline{c_b}\, c_a)} \left(\frac{X(\tau_<)}{a\, c_a - b\, c_b}\varphi_{\underline{k}} \right)(\underline{x})\, \overline{\left(\Upsilon_\beta(\tau_>)\varphi_{\underline{k}}\right)(\underline{x}')}$$

General interaction

Consider the general interacting theory

$$S_{M,V}(\phi) = S_{M,0}(\phi) + \int_M \mathrm{d}^4x \, V(x, \phi(x)).$$

where V is an arbitrary potential that vanishes outside the region M. The corresponding field propagator is

$$Z_{M,V}(\varphi) = \exp\left(\mathrm{i} \int_M \mathrm{d}^4x \sqrt{-\mathrm{g}(x)} \, V\left(x, -\mathrm{i}\frac{\delta}{\delta\mu(x)}\right)\right) Z_{M,\mu}(\varphi)\Big|_{\mu=0},$$

and the amplitude for the general interacting theory is

$$\rho_{M,V}(\psi) = \exp\left(\mathrm{i} \int_M \mathrm{d}^4x \sqrt{-\mathrm{g}(x)} \, V\left(x, -\mathrm{i}\frac{\delta}{\delta\mu(x)}\right)\right) \rho_{M,\mu}(\psi)\Big|_{\mu=0}.$$

Outline

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Commutation relations, Wightman function, Feynman propagator in standard QFT

Commutators in Minkowski-QFT: $(\hbar = c = 1)$

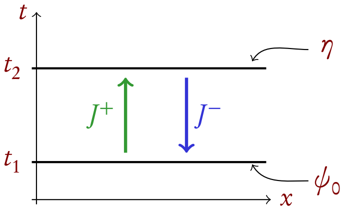
$$[\phi(x,t),\phi(x',t)] = [\partial_t\phi(x,t),\partial_t\phi(x',t)] = 0, \quad [\phi(x,t),\partial_t\phi(x',t)] = \mathrm{i}\delta(x-x')$$

Wightman function and Feynman propagator:

$$\langle 0|\phi(x)\phi(x')|0\rangle = \mathrm{i}D^{+}(x-x'),$$
$$\langle 0|T\phi(x)\phi(x')|0\rangle = \mathrm{i}G_F(x-x') = \mathrm{i}\left[\theta(t-t')D^{+}(x-x') + \theta(t'-t)D^{+}(x'-x)\right]$$

in-in formalism (CTP)

Free vacuum state ψ_0 evolves independently under two source J^+ and J^-; sum over all states at t_2. This is know as the close time path or in-in formalism.



$$Z[J^+,J^-] = \int \mathscr{D}\eta \, \langle\psi_0| e^{-i\int d^4x J^-(x)\phi(x)} |\eta\rangle \, \langle\eta| e^{i\int d^4x J^+(x)\phi(x)} |\psi_0\rangle$$

$$= \exp\left(\frac{-i}{2} \int d^4x \, d^4x' \begin{pmatrix} J^+(x) & J^-(x) \end{pmatrix} \begin{pmatrix} -G_F(x,x') & D^+(x,x') \\ D^+(x,x') & G_D(x,x') \end{pmatrix} \begin{pmatrix} J^+(x') \\ J^-(x') \end{pmatrix} \right)$$

G_F, G_D and D^+ can be obtain by functional derivatives of $Z[J^+,J^-]$

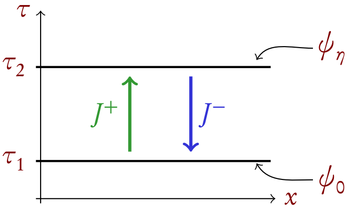
in-in formalism in the GBF

The interaction of the scalar field with the source $J^{\pm}$ is described by the action

$$S_M(\phi) = S_{M,0}(\phi) + \int_M \mathrm{d}^4x\, \sqrt{-g(x)}\, J^{\pm}(x)\, \phi(x).$$

We assume that $J^{\pm}$ are confined in the interior of the region M.

We consider the vacuum state ψ_0 at τ_1 and a coherent states ψ_η at τ_2 (coherent states satisfy a completeness relation: $K \int \mathrm{d}\eta\, \mathrm{d}\overline{\eta} |\psi_\eta\rangle\langle\psi_\eta| = \mathbb{I}$).



The functional $Z[J^+, J^-]$ in the slice region results to be

$$Z[J^+, J^-] = K \int \mathrm{d}\eta\, \mathrm{d}\overline{\eta}\; \rho^{J^+}_{[\tau_1,\tau_2]} \left(\psi_{\tau_1,0} \otimes \overline{\psi_{\tau_2,\eta}} \right) \rho^{J^-}_{[\tau_1,\tau_2]} \left(\overline{\psi_{\tau_1,0}} \otimes \psi_{\tau_2,\eta} \right)$$

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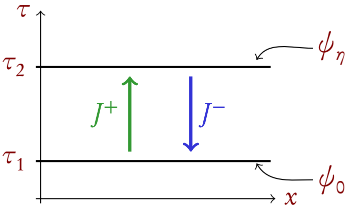

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$$= \exp\left(\frac{-i}{2} \int \mathrm{d}^4x\, \mathrm{d}^4x' \begin{pmatrix} J^+(x) & J^-(x) \end{pmatrix} \begin{pmatrix} -G_F(x,x') & D^+(x,x') \\ D^+(x,x') & G_D(x,x') \end{pmatrix} \begin{pmatrix} J^+(x') \\ J^-(x') \end{pmatrix}\right)$$

- G_F is the same Feynman propagator previously shown.
- the Wightman function is

$$D^{+}(x,x') = \mathrm{i}\int \mathrm{d}^3\underline{k}\,\frac{1}{d(\underline{k})(\overline{c_a}\,c_b - \overline{c_b}\,c_a)}\left(\Upsilon_\beta(\tau)\varphi_{\underline{k}}\right)(\underline{x})\,\overline{\left(\Upsilon_\beta(\tau')\varphi_{\underline{k}}\right)}(\underline{x}')$$

It reduces to the standard Wightman function in the standard setting.

Commutation relations

- In standard Minkowski-QFT, equal-time commutation relations can be obtained by a limiting procedure

$$
\begin{aligned}
&\langle 0|[\phi(x,t), \partial_t \phi(x',t)]|0\rangle = \mathrm{i}\delta(x-x') \\
&\qquad = \lim_{\epsilon \to 0} T\left[\phi(x,t+\epsilon)\partial_t\phi(x',t-\epsilon) - \phi(x,t-\epsilon)\partial_t\phi(x',t+\epsilon)\right]
\end{aligned}
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- In the GBF, we have

$$\langle 0|[\phi(x,\tau), \partial_\tau\phi(x',\tau)]|0\rangle$$
$$= \lim_{\epsilon\to 0}\left[\partial_{\tau-\epsilon}\frac{\delta}{\delta J(x,\tau+\epsilon)}\frac{\delta}{\delta J(x',\tau-\epsilon)} - \partial_{\tau-\epsilon}\frac{\delta}{\delta J(x,\tau+\epsilon)}\frac{\delta}{\delta J(x',\tau-\epsilon)}\right]\rho^J_{[\tau_1,\tau_2]}\left(\psi_0\otimes\overline{\psi_0}\right)$$

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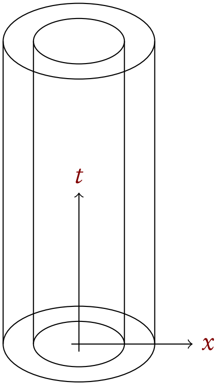
&= \mathrm{i}\lim_{\epsilon\to 0}\left[\partial_{\tau-\epsilon}G_F(x,\tau+\epsilon;x',\tau-\epsilon) - \partial_{\tau+\epsilon}G_F(x,\tau-\epsilon;x',\tau+\epsilon)\right] \\
&= \int \mathrm{d}^3\underline{k}\,\frac{\mathrm{i}}{d(\underline{k})(\overline{c_a}\,c_b - \overline{c_b}\,c_a)}\lim_{\epsilon\to 0}\left[(\partial_\tau\Upsilon(\tau-\epsilon))\,\overline{\Upsilon(\tau+\epsilon)} - \Upsilon(\tau-\epsilon)\left(\partial_\tau\overline{\Upsilon(\tau+\epsilon)}\right)\right]\varphi_{\underline{k}}(\underline{x}')\varphi_{\underline{k}}(\underline{x}) \\
&= \frac{\mathrm{i}}{\sqrt{|g^{(3)}_\tau g^{\tau\tau}_\tau|}}\,\delta(x-x')
\end{aligned}
$$

Non-standard commutation relations in Minkowski

Region bounded by two hyper cylinders: The boundary is the disjoint union of *timelike* hypersurfaces.
Propagation in the radial direction
Solution of the Klein-Gordon equation:



$$\phi(t,r,\Omega) = \int \mathrm{d}E \frac{p}{4\pi} \sum_{l,m} \left(a_{l,m}(E)\,\Upsilon_l(E,r)\, e^{-\mathrm{i}Et} Y_l^m(\Omega) + c.c. \right),$$

where Y_l^m are spherical harmonics and $p = \sqrt{|E^2 - m^2|}$ and

$$\Upsilon_l(E,r) = \begin{cases} j_l(pr) + \mathrm{i} n_l(pr) & E^2 > m^2 \\ \mathrm{i}^{-l} j_l(\mathrm{i}pr) - \mathrm{i}^l n_l(\mathrm{i}pr) & E^2 < m^2 \end{cases}$$

$$\left[\phi(t,r,\Omega), \partial_r \phi(t',r,\Omega')\right] = \frac{\mathrm{i}}{r^2} \delta(t-t') \delta^{(2)}(\Omega - \Omega')$$