The positive formalism: an operational approach to the foundations of physics

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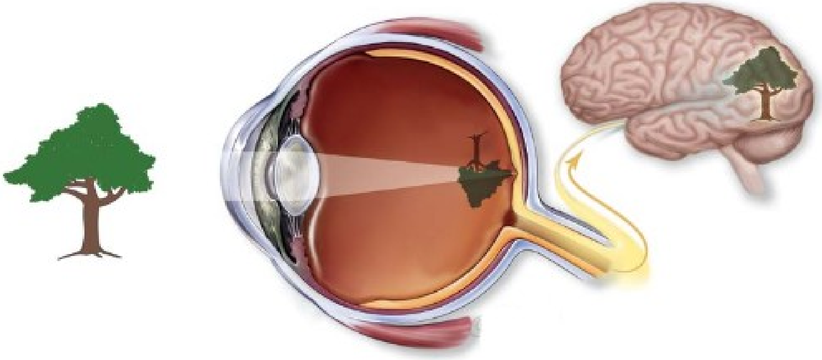
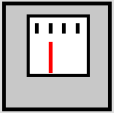
Vision and reconstruction



http://tpe.vision.aveugles.free.fr/vision.php

Processes and interfaces



Operational approach:
Fundamental notions

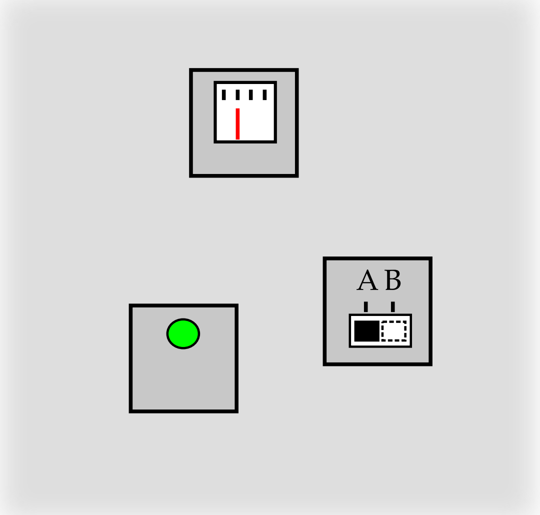
- experiment
- measurement
- observation
- preparation
- intervention

Subsume instance as:

- **process**

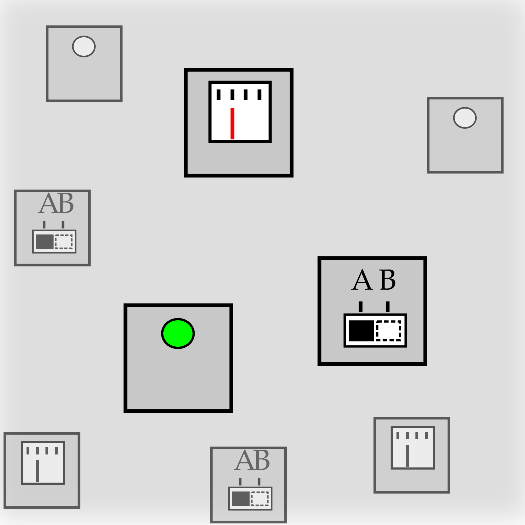
Processes have **outcomes**.
Represent processes as **boxes**.

Processes and interfaces



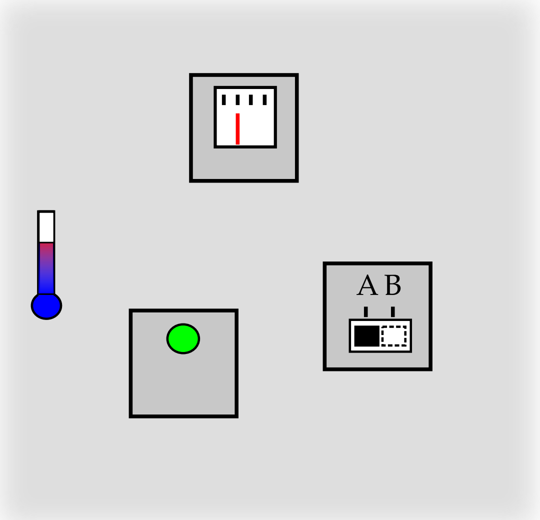
Processes are not isolated. Outcomes depend on other processes. We want to predict **correlations**.

Processes and interfaces



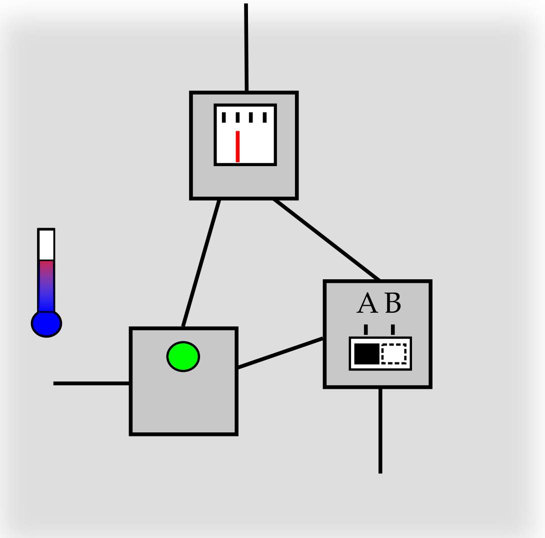
The outcome of a given set of processes depends generally on a large number of other processes.

Processes and interfaces



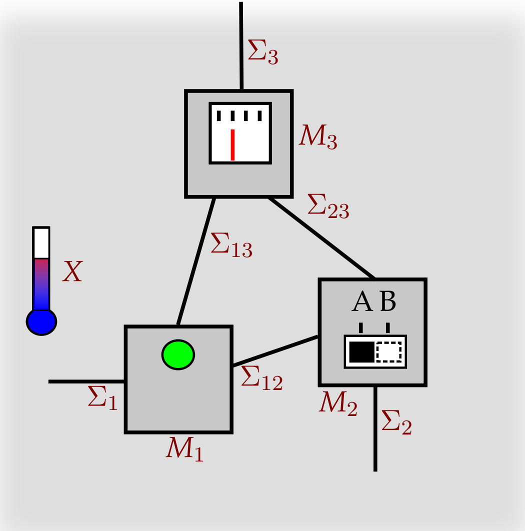
We treat these external processes collectively. We call this a **boundary condition**.

Processes and interfaces



We introduce the notion of **interface** to model interaction between processes. An interface encodes communication or information exchange between processes. we depict this as a **link**.

Processes and interfaces



Processes are of specific **types**. Interfaces are of specific **types**.

Types determine how processes and interfaces can be **connected**. Only matching types can be connected.

We indicate types with **labels**.

Processes and probes

Associated to each process of type M is a space $\mathcal{P}_M$ of **probes** with a subset of **primitive probes** $\mathcal{P}_M^+ \subseteq \mathcal{P}_M$.

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- specific apparatus settings
- the occurrence (or not) of a specific experimental outcome

A general **probe** may encode also

- measurement values

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There is always a **null-probe** $\boxslash \in \mathcal{P}_M^+$, representing the absence of any apparatus, observation or intervention.

Hierarchies of probes

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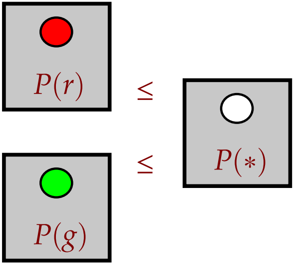
Consider an apparatus with one light that shows either `red` or `green`, encoded in three different probes:

- $P(r)$ for outcome `red`
- $P(g)$ for outcome `green`
- $P(*)$ for an unspecified outcome

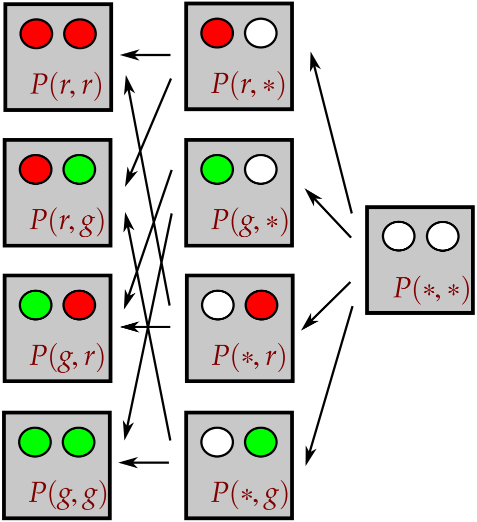
The unspecified state is more **general** than the others. Encode this in a **partial order** on $\mathcal{P}_M$, setting $P(r) \leq P(*)$ and $P(g) \leq P(*)$.



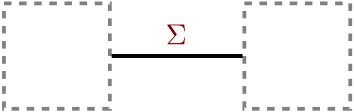
Hierarchies of probes

Hierarchies may become more complex when the apparatus allows for more distinct readings.

For example:
$P(r,r) \leq P(r,*) \leq P(*,*)$.

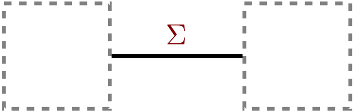


Interfaces and boundary conditions



We associate to each interface Σ a space of **boundary conditions** $\mathcal{B}_\Sigma^+$. This parametrizes possible signals/information exchange between adjacent processes.

Interfaces and boundary conditions



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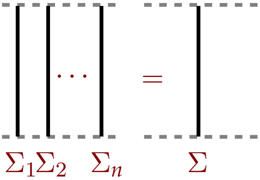
Interfaces between the same pair of processes can be combined arbitrarily. Write: $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$.



Induces a map between spaces of boundary conditions:

$\mathcal{B}_{\Sigma_1}^+ \times \mathcal{B}_{\Sigma_2}^+ \times \cdots \times \mathcal{B}_{\Sigma_n}^+ \to \mathcal{B}_\Sigma^+$

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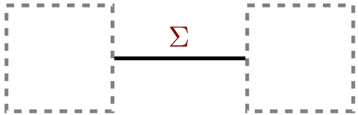

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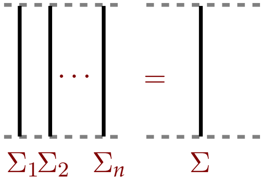

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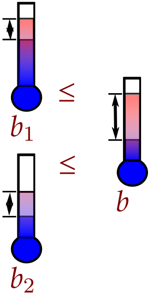
We denote the joint interface for a process of type M by ∂M.

Hierarchies of boundary conditions

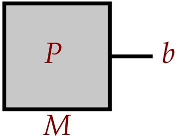
Boundary conditions also form **hierarchies of generality**. This gives rise to a **partial order** on $\mathcal{B}_\Sigma^+$. Here:

$$b_1 \leq b$$
$$b_2 \leq b$$



Values

Consider a process of type M.

$[\![P, b]\!]_M =$ 

To a **probe** $P \in \mathcal{P}_M$ and a **boundary condition** $b \in \mathcal{B}^+_{\partial M}$ we associate a **value** $[\![P, b]\!]_M$. We shall take this to be a **real number**. Formally, there is a **pairing** $[\![\cdot, \cdot]\!]_M : \mathcal{P}_M \times \mathcal{B}^+_{\partial M} \to \mathbb{R}$.

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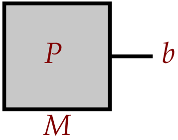
$[\![P,b]\!]_M =$ 

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$[\![P,b]\!]_M \in \mathbb{R}^+$ quantifies **compatibility** between the apparatus or outcome represented by the **primitive probe** $P \in \mathcal{P}^+_M$ and the **boundary condition** $b \in \mathcal{B}^+_{\partial M}$.

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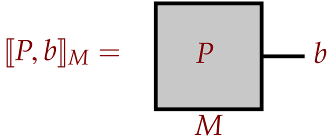

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Pairing and partial order structures are compatible:

$$P \le Q \quad \Longleftrightarrow \quad [\![P,b]\!]_M \le [\![Q,b]\!]_M \quad \forall b \in \mathcal{B}^+_{\partial M}$$

$$b \le c \quad \Longleftrightarrow \quad [\![P,b]\!]_M \le [\![P,c]\!]_M \quad \forall P \in \mathcal{P}^+_M$$

From values to measurements

A measurement is encoded by at least two **probes**:

- One **non-selective probe** Q encodes the measurement apparatus.
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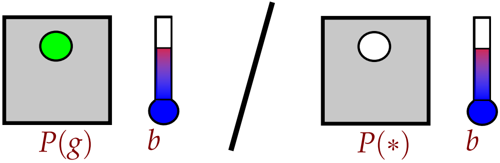
Measurement probabilities

In M consider the probe $P(*) \in \mathcal{P}_M^+$ encoding a measurement device and $P(g)$ encoding in addition a selected outcome.

Given $b \in \mathcal{B}_{\partial M}^+$ the **probability** Π for an affirmative outcome is:

$$\Pi = \frac{[\![P(g), b]\!]_M}{[\![P(*), b]\!]_M}$$



Since $0 \leq P(g) \leq P(*)$ we have $0 \leq \Pi \leq 1$ (if $[\![P(*), b]\!]_M \neq 0$).

Expectation values

Say we have an apparatus in M represented by a primitive probe $Q \in \mathcal{P}_M^+$. The measurement may have n different outcome represented by primitive probes $P_1, \ldots, P_n \in \mathcal{P}_M^+$. We associate with each outcome a pointer reading $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. The **expectation value** E for the pointer reading is,

$$E = \sum_{i=1}^{n} \lambda_i \frac{[\![P_i, b]\!]_M}{[\![Q, b]\!]_M} = \frac{[\![P, b]\!]_M}{[\![Q, b]\!]_M}$$

where we define,

$$P = \sum_{i=1}^{n} \lambda_i P_i .$$

$P \in \mathcal{P}_M$ is a general probe, not necessarily primitive.

Convexity

In a probabilistic setting it makes sense to combine different probes probabilistically, even when they correspond to different experimental situations. Say we have probes $P_1, \ldots, P_n$ and probabilities $p_1, \ldots, p_n$ such that $\sum_k p_k = 1$. Then we can consider $P := \sum_k p_k P_k$ as a probe.

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Since an arbitrary real multiple of a probe is a probe, this equips the space $\mathcal{P}_M$ of probes with the structure of a **real vector space**. The subset of primitive probes $\mathcal{P}_M^+ \subset \mathcal{P}_M$ is a **positive cone** making $\mathcal{P}_M$ into a **partially ordered vector space**.

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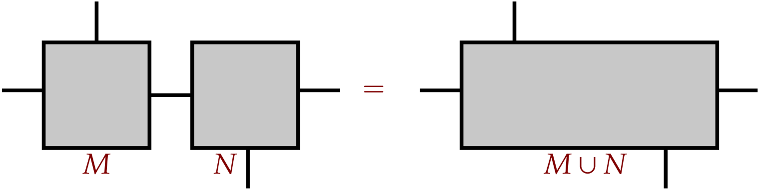
We extend the pairing, $[\![\cdot,\cdot]\!]_M : \mathcal{P}_M \times \mathcal{B}_{\partial M} \to \mathbb{R}$

Composition

A set of processes joined by interfaces may be considered itself a process. Say we have a process of type M and one of type N. We say the **composite** process has type $M \cup N$.



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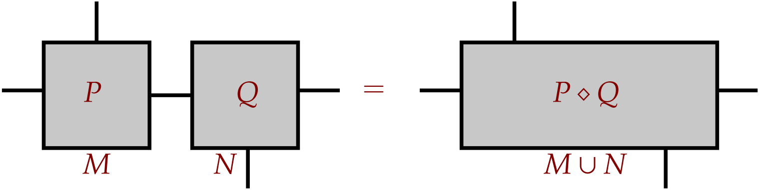
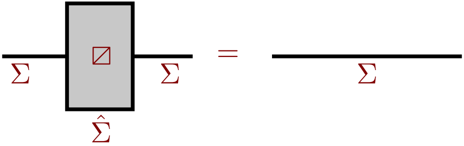


This induces a composition of associated probes $P \in \mathcal{P}_M$ with $Q \in \mathcal{P}_N$.
We write for the composite probe $P \diamond Q \in \mathcal{P}_{M \cup N}$.
This yields a **composition map** $\diamond : \mathcal{P}_M \times \mathcal{P}_N \to \mathcal{P}_{M \cup N}$.

Slice processes and inner product

For any type Σ of **interface** we postulate a type of **slice process** $\hat{\Sigma}$:

- $\partial\hat{\Sigma} = \Sigma \cup \Sigma$
- the **null probe** “passes signals through”



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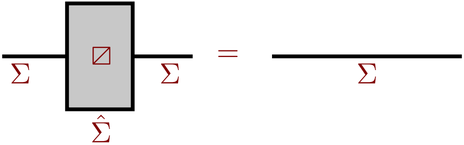
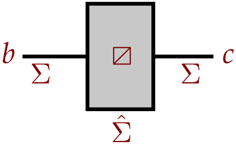


Putting **boundary conditions** on the two sides allows evaluation. This yields an **inner product** $\mathcal{B}_\Sigma \times \mathcal{B}_\Sigma \to \mathbb{R}$ on the space of boundary conditions.



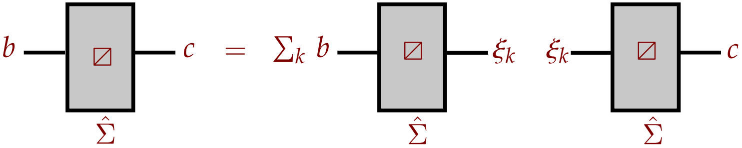
$$(\!|b, c|\!)_\Sigma := [\![\boxslash, b \otimes c]\!]_{\hat{\Sigma}}$$

This should be **symmetric** and **positive-definite**.

Composition of slices

Two processes of the same slice type compose to one process of this slice type. Null probes then compose to a null probe.

A decomposition of the identity in terms of a **basis** yields a notion of **composition** of slice probes. For the null probe,

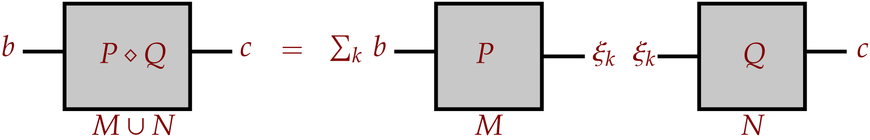


$$(\!|b, c|\!)_{\hat{\Sigma}} = \sum_{k \in I} (\!|b, \xi_k|\!)_{\hat{\Sigma}} \, (\!|\xi_k, c|\!)_{\hat{\Sigma}}$$

Here, $\{\xi_k\}_{k \in I}$ is an ON-basis of $\mathcal{B}_{\Sigma}$.

Composition rule for probes

As a generalization we obtain the **composition rule for probes**.



$$[\![P \diamond Q, b \otimes c]\!]_{M \cup N} = \sum_{k \in I} [\![P, b \otimes \xi_k]\!]_M \, [\![Q, \xi_k \otimes c]\!]_N$$

Here, $\{\xi_k\}_{k\in I}$ is an ON-basis of $\mathcal{B}_\Sigma$.

(Abstract) Positive Formalism

Obtain an **axiomatic framework** for encoding physical theories with:

- **types**
  - a collection of **process types**
  - a collections of **interface types**
  - a **boundary map** from process types to interface types $M \mapsto \partial M$
- objects
  - a partially ordered vector space of **probes** $\mathcal{P}_M$ per process type M
  - a partially ordered vector space of **generalized boundary conditions** $\mathcal{B}_\Sigma$ per interface type Σ
- **compositions**
  - **decomposition** of interface types $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$ with associated positive isomorphism $\mathcal{B}_{\Sigma_1} \otimes \cdots \otimes \mathcal{B}_{\Sigma_n} \to \mathcal{B}_\Sigma$
  - **composition** of process types M and N to $M \cup N$ and probes $\diamond : \mathcal{P}_M \times \mathcal{P}_N \to \mathcal{P}_{M \cup N}$
- **values**: positive pairings $[\![\cdot,\cdot]\!]_M : \mathcal{P}_M \times \mathcal{B}_{\partial M} \to \mathbb{R}$

Partially ordered vector space

A **real vector space** V with a **partial order** such that:

- $a \leq b \iff a + c \leq b + c \quad \forall a, b, c \in V$
- $a \leq b \iff \lambda a \leq \lambda b \quad \forall a, b \in V, \forall \lambda > 0$

Require **generating cone**, i.e., $V = V^+ - V^+$. Require **Archimedean order**, i.e., for any $v \in V$ have that if there exists $w \in V^+$ such that $v \leq \lambda w$ for all $\lambda > 0$ then $v \leq 0$.

Positive map

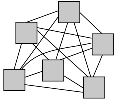
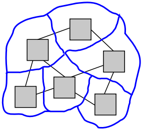
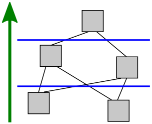
A linear map that maps positive elements to positive elements.

Sharply positive inner product

A symmetric bilinear form $V \times V \to \mathbb{R}$ such that if $a, b \in V^+$ then $(a, b) \geq 0$ and if for some $a \in V$ we have $(a, b) \geq 0$ for all $b \in V^+$ then $a \in V^+$.

classical
(lattices)

quantum
(anti-lattices)

classical		quantum
abstract classical statistical theory	 abstract positive formalism	abstract quantum theory
	+ spacetime + locality	
spacetime statistical field theory	 spacetime positive formalism	general boundary formulation / axiomatic QFT
	+ time + causality	
statistical mechanics	 convex operational framework	standard formulation of quantum theory

Main reference

R. O., *A local and operational framework for the foundations of physics*, arXiv:1610.09052.