Local functorial quantization of field theory (I)

Robert Oeckl

Centro de Ciencias Matemáticas
Universidad Nacional Autónoma de México
Morelia, Mexico

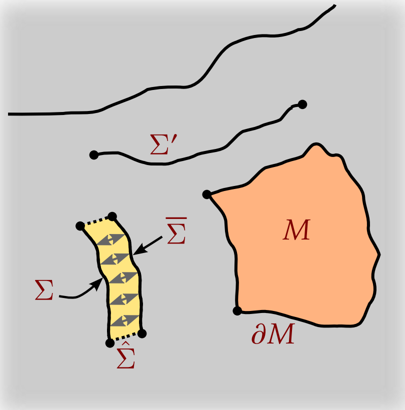
Seminar *General Boundary Formulation*
23 May 2018

Quantization (canonical)

concept	classical theory		quantum theory
states	phase space L	$\longrightarrow$	Hilbert space $\mathcal{H}$
observables	functions on phase space $C(L)$	$\longrightarrow$	operator algebra $\mathcal{B}(\mathcal{H})$
quantization condition	Poisson bracket $\{\cdot,\cdot\}$	$\longrightarrow$	commutator $[\cdot,\cdot]$

spacetime – manifolds

Fix dimension d. Manifolds are **oriented** and may carry additional structure: differentiable, metric, complex, etc.



region M

d-manifold with boundary.

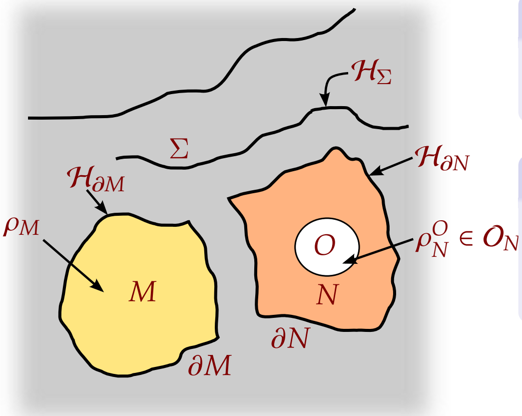
hypersurface Σ

$d-1$-manifold with boundary, with germ of d-manifold.

slice region $\hat{\Sigma}$

$d-1$-manifold with boundary, with germ of d-manifold, interpreted as "infinitely thin" region.

Assignment of algebraic structures to geometric ones.



(T1) per hypersurface Σ

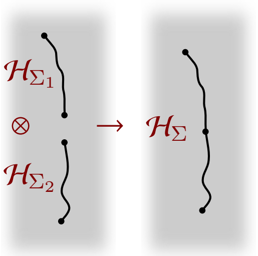
A complex Hilbert space $\mathcal{H}_\Sigma$.

(T4) per region M

A complex vector space $\mathcal{O}_M$ of linear maps $\mathcal{H}^\circ_{\partial M} \to \mathbb{C}$. A special **unit** $\rho_M \in \mathcal{O}_M$.

The choice of an element of $\mathcal{O}_M$ for a region M is indicated by a **label**.

QFT – axioms II

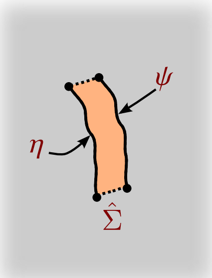


(T1b) per hypersurface Σ

A conjugate linear involution $\iota_\Sigma : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\overline{\Sigma}}$.

(T2) per hypersurface decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$

A linear isomorphism $\tau : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \rightarrow \mathcal{H}_\Sigma$.



(T3x) per hypersurface Σ

The unit gives rise to the **positive-definite inner product** $\langle \iota_{\overline{\Sigma}}(\psi), \eta \rangle_\Sigma = \rho_{\hat{\Sigma}} \circ \tau(\psi \otimes \eta)$.

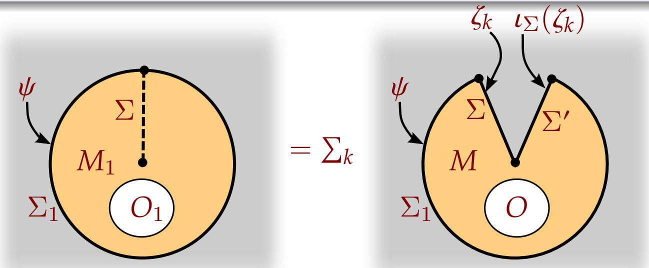
QFT – axioms III

(T5a) per disjoint composition of regions $M = M_1 \sqcup M_2$

$\diamond : \mathcal{O}_{M_1} \times \mathcal{O}_{M_2} \to \mathcal{O}_M$ given by
$(\rho_{M_1}^{O_1} \diamond \rho_{M_2}^{O_2})(\tau(\psi_1 \otimes \psi_2)) = \rho_{M_1}^{O_1}(\psi_1)\rho_{M_2}^{O_2}(\psi_2)$. Also $\rho_{M_1} \diamond \rho_{M_2} = \rho_M$.

(T5b) per self-composition of region M to M_1 along Σ

$\diamond : \mathcal{O}_M \to \mathcal{O}_{M_1}$ given by
$\rho_{M_1}^{O_1}(\psi) = (\diamond\rho_M^O)(\psi) := \sum_k \rho_M^O(\tau(\psi \otimes \zeta_k \otimes \iota_\Sigma(\zeta_k)))$. $\diamond\rho_M = \rho_{M_1}$.



$\{\zeta_k\}_{k\in I}$ orthonormal basis of $\mathcal{H}_\Sigma$.

Plan

continuum classical FT $\longrightarrow$ axiomatic classical FT $\longrightarrow$ axiomatic QFT

Lagrangian field theory (I)

Formulate field theory in terms of first order Lagrangian density $\Lambda(\varphi, \partial\varphi, x)$. For a spacetime region M the **action** of a field ϕ is

$$S_M(\phi) := \int_M \Lambda(\phi(\cdot), \partial\phi(\cdot), \cdot).$$

Classical solutions in M are extremal points of this action. These are obtained by setting to zero the first variation of the action,

$$(\mathrm{d}S_M)_\phi(X) = \int_M X^a \left(\frac{\delta\Lambda}{\delta\varphi^a} - \partial_\mu \frac{\delta\Lambda}{\delta\, \partial_\mu\varphi^a}\right)(\phi) + \int_{\partial M} X^a \partial_\mu \lrcorner \frac{\delta\Lambda}{\delta\partial_\mu\varphi^a}(\phi)$$

under the condition that the infinitesimal field X vanishes on ∂M. This yields the **Euler-Lagrange equations**,

$$\left(\frac{\delta\Lambda}{\delta\varphi^a} - \partial_\mu \frac{\delta\Lambda}{\delta\, \partial_\mu\varphi^a}\right)(\phi) = 0.$$

Lagrangian field theory (II)

The boundary term can be defined for an arbitrary hypersurface Σ.

$$(\theta_\Sigma)_\phi(X) = -\int_\Sigma X^a \partial_\mu \lrcorner \frac{\delta \Lambda}{\delta \partial_\mu \varphi^a}(\phi)$$

This 1-form is called the **symplectic potential**. Its exterior derivative is the **symplectic 2-form**,

$$(\omega_\Sigma)_\phi(X,Y) = (\mathrm{d}\theta_\Sigma)_\phi(X,Y) = -\frac{1}{2}\int_\Sigma \Bigg((X^b Y^a - Y^b X^a)\, \partial_\mu \lrcorner \frac{\delta^2 \Lambda}{\delta \varphi^b \delta\, \partial_\mu \varphi^a}(\phi) + (Y^a \partial_\nu X^b - X^a \partial_\nu Y^b)\, \partial_\mu \lrcorner \frac{\delta^2 \Lambda}{\delta\, \partial_\nu \varphi^b \delta\, \partial_\mu \varphi^a}(\phi) \Bigg).$$

We denote the space of solutions in M by L_M and the space of germs of solutions on a hypersurface Σ by L_Σ.

Lagrangian field theory (III)

Let M be a region and $\phi \in L_{\partial M}$. Then ϕ may or may not be induced from a solution in M. If ϕ arises from a solution in M and X, Y arise from infinitesimal solutions in M, then,

$$(\omega_{\partial M})_\phi(X, Y) = (\mathrm{d}\theta_{\partial M})_\phi(X, Y) = -(\mathrm{d}\mathrm{d}S_M)_\phi(X, Y) = 0.$$

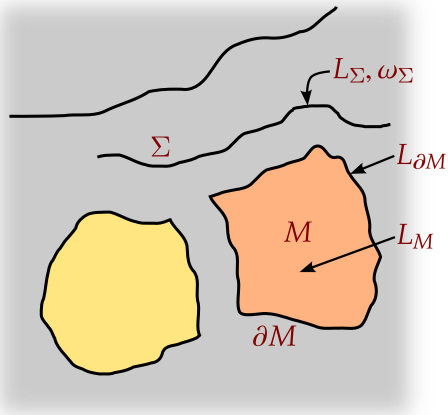
This means, L_M induces an **isotropic** submanifold of $L_{\partial M}$.

It is natural to require that the symplectic form is **non-degenerate**. We are then led to the converse statement: If given X we have $(\omega_{\partial M})_\phi(X, Y) = 0$ for all induced Y, then X itself must be induced. This means, L_M induces a **coisotropic** submanifold of $L_{\partial M}$.

L_M induces a **Lagrangian** submanifold of $L_{\partial M}$.

[Kijowski, Tulczyjew 1979]

Axiomatic classical field theory



per hypersurface Σ :

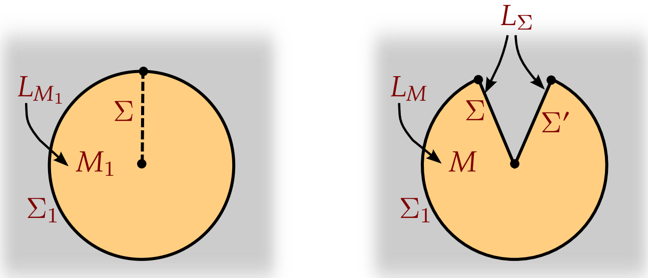
The **space of germs of solutions** near Σ. This is a **symplectic manifold** $(L_\Sigma, \omega_\Sigma)$.

per region M :

The **space of solutions** in M. Forgetting the interior yields a map $L_M \to L_{\partial M}$. Under this map L_M is a **Lagrangian submanifold** $L_M \subseteq L_{\partial M}$.

Axiomatic classical field theory

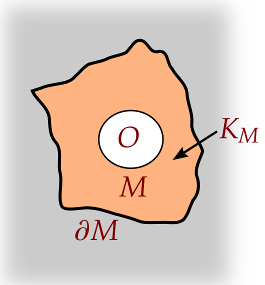
There are additional axioms related to gluing etc. like this one:



$$L_{M_1} \hookrightarrow L_M \rightrightarrows L_\Sigma$$

Observables

In relativistic field theory observables need to be defined on configuration space.



per region M :

The **configuration space** K_M in M. Have $L_M \subseteq K_M$. Also a space C_M of **classical observables** given by maps $K_M \to \mathbb{R}$.

per region M with label O:

Assign the **classical observable** $O \in C_M$. If there is no label assign the observable 1 with constant value 1.

Quantization (local)

concept	classical theory		quantum theory
states: per hypersurface Σ	phase space L_Σ	$\longrightarrow$	Hilbert space $\mathcal{H}_\Sigma$
observables: per region M	functions on configuration space C_M	$\longrightarrow$	space of observable maps $\mathcal{O}_M \subseteq \mathcal{H}^\star_{\partial M}$
quantization condition	product of functions $C_M \times C_N \to C_{M\cup N}$	$\longrightarrow$	composition of observable maps $\mathcal{O}_M \times \mathcal{O}_N \to \mathcal{O}_{M\cup N}$

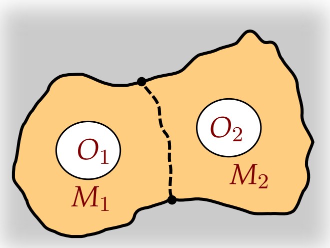
Quantization – requirements

For any region M we would like a linear **quantization map** $\mathcal{Q}_M : \mathcal{C}_M \to \mathcal{O}_M$. Moreover, $\mathcal{Q}_M(1) = \rho_M$.

What else? (No commutation relations!)

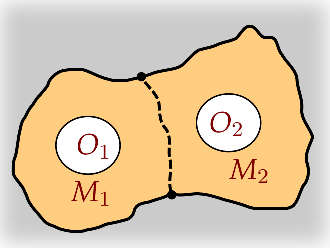
Observables: Composition correspondence

Consider **classical observables** O_1 and O_2 localized in spacetime regions M_1 and M_2, encoded in maps $O_i : K_{M_i} \to \mathbb{R}$. The natural **composition** of these observables given by the **product of functions** $O_1 \cdot O_2$ in the joint spacetime region $M_1 \cup M_2$ yielding a map $O_1 \cdot O_2 : K_{M_1 \cup M_2} \to \mathbb{R}$.



Observables: Composition correspondence

Consider **classical observables** O_1 and O_2 localized in spacetime regions M_1 and M_2, encoded in maps $O_i : K_{M_i} \to \mathbb{R}$. The natural **composition** of these observables given by the **product of functions** $O_1 \cdot O_2$ in the joint spacetime region $M_1 \cup M_2$ yielding a map $O_1 \cdot O_2 : K_{M_1 \cup M_2} \to \mathbb{R}$.



Composition and quantization should commute

$$\rho_{M_1 \cup M_2}^{O_1 \cdot O_2} = \rho_{M_1}^{O_1} \diamond \rho_{M_2}^{O_2}$$

Schrödinger-Feynman quantization: hypersurfaces

Topological quantum field theory was originally inspired by the **Feynman path integral** and its composition properties. The Feynman path integral is defined in the **Schrödinger representation** where states are **wave functions** on **field configurations**.

Schrödinger-Feynman quantization: hypersurfaces

Topological quantum field theory was originally inspired by the **Feynman path integral** and its composition properties. The Feynman path integral is defined in the **Schrödinger representation** where states are **wave functions** on **field configurations**.

The state space $\mathcal{H}_\Sigma$ for the hypersurface Σ is the space of complex functions on K_Σ with inner product,

$$\langle \psi', \psi \rangle_\Sigma = \int_{K_\Sigma} \mathcal{D}\varphi \, \overline{\psi'(\varphi)} \psi(\varphi).$$

Here, $\mathcal{D}\varphi$ is a **translation invariant measure** on K_Σ.

Such a measure does not exist in most cases. As usual, it is fruitful to proceed pretending that it does.

Schrödinger-Feynman quantization: regions

The **Feynman path integral** serves to define the **amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ in a spacetime region M,

$$\rho_M(\psi) = \int_{\phi \in K_M} \mathcal{D}\phi\, \psi(\phi|_{\partial M})\, e^{\mathrm{i}S_M(\phi)}.$$

Similarly, it defines the **observable map** $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$ for observable $O : K_M \to \mathbb{R}$ in region M,

$$\rho_M^O(\psi) = \int_{\phi \in K_M} \mathcal{D}\phi\, \psi(\phi|_{\partial M})\, O(\phi)\, e^{\mathrm{i}S_M(\phi)}.$$

Again, the measure $\mathcal{D}\phi$ does not actually exist in most cases.

These quantum data "automatically" satisfy the quantum axioms.
Problem: This is not well defined.

Universal results in free field theory

[RO 2010; 2011; 2012]

For **free field theory** the informal Schrödinger-Feynman prescription can be replaced by a rigorous and functorial quantization scheme:

Universal results in free field theory

[RO 2010; 2011; 2012]

For **free field theory** the informal Schrödinger-Feynman prescription can be replaced by a rigorous and functorial quantization scheme:

- For state spaces, the Schrödinger representation is replaced by the **holomorphic representation**, a flavor of **geometric quantization**.

Universal results in free field theory

[RO 2010; 2011; 2012]

For **free field theory** the informal Schrödinger-Feynman prescription can be replaced by a rigorous and functorial quantization scheme:

- For state spaces, the Schrödinger representation is replaced by the **holomorphic representation**, a flavor of **geometric quantization**.
- Additional data is required: One **complex structure** $J_\Sigma : L_\Sigma \to L_\Sigma$ per hypersurface.

Universal results in free field theory

[RO 2010; 2011; 2012]

For **free field theory** the informal Schrödinger-Feynman prescription can be replaced by a rigorous and functorial quantization scheme:

- For state spaces, the Schrödinger representation is replaced by the **holomorphic representation**, a flavor of **geometric quantization**.
- Additional data is required: One **complex structure** $J_\Sigma : L_\Sigma \to L_\Sigma$ per hypersurface.
- There is a **gluing anomaly**, modifying the gluing axiom **(T5b)** to $\diamond \rho_M = \rho_{M_1} \cdot c$
- Standard results in flat and globally hyperbolic spacetime are recovered.

Universal results in free field theory

[RO 2010; 2011; 2012]

For **free field theory** the informal Schrödinger-Feynman prescription can be replaced by a rigorous and functorial quantization scheme:

- For state spaces, the Schrödinger representation is replaced by the **holomorphic representation**, a flavor of **geometric quantization**.
- Additional data is required: One **complex structure** $J_\Sigma : L_\Sigma \to L_\Sigma$ per hypersurface.
- There is a **gluing anomaly**, modifying the gluing axiom **(T5b)** to $\diamond \rho_M = \rho_{M_1} \cdot c$
- Standard results in flat and globally hyperbolic spacetime are recovered.

Theorem

The quantum data satisfies the QFT axioms (possibly with some infinite gluing anomalies).