Local functorial quantization of field theory (II)

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Quantization (canonical)

concept	classical theory		quantum theory
states	phase space (L, ω)	$\longrightarrow$	Hilbert space $\mathcal{H}$
observables	functions on phase space $\mathcal{C}(L)$	$\longrightarrow$	operator algebra $\mathcal{B}(\mathcal{H})$
quantization condition	Poisson bracket $\{\cdot, \cdot\}$	$\longrightarrow$	commutator $[\cdot, \cdot]$

Quantization (local)

concept	classical theory		quantum theory
states: per hypersurface Σ	phase space $(L_\Sigma, \omega_\Sigma)$	$\longrightarrow$	Hilbert space $\mathcal{H}_\Sigma$
observables: per region M	functions $K_M \to \mathbb{C}$ on configuration space C_M	$\longrightarrow$	space of observable maps $\mathcal{O}_M \subseteq \mathcal{H}^\star_{\partial M}$
quantization condition	product of functions $C_M \times C_N \to C_{M\cup N}$	$\longrightarrow$	composition of observable maps $\mathcal{O}_M \times \mathcal{O}_N \to \mathcal{O}_{M\cup N}$

Schrödinger-Feynman quantization: hypersurfaces

Topological quantum field theory was originally inspired by the **Feynman path integral** and its composition properties. The Feynman path integral is defined in the **Schrödinger representation** where states are **wave functions** on **field configurations**.

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Topological quantum field theory was originally inspired by the **Feynman path integral** and its composition properties. The Feynman path integral is defined in the **Schrödinger representation** where states are **wave functions** on **field configurations**.

The state space $\mathcal{H}_\Sigma$ for the hypersurface Σ is the space of complex functions on K_Σ with inner product,

$$\langle \psi', \psi \rangle_\Sigma = \int_{K_\Sigma} \mathcal{D}\varphi \, \overline{\psi'(\varphi)} \psi(\varphi).$$

Here, $\mathcal{D}\varphi$ is a **translation invariant measure** on K_Σ.

Such a measure does not exist in most cases. As usual, it is fruitful to proceed pretending that it does.

Schrödinger-Feynman quantization: regions

The **Feynman path integral** serves to define the **amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ in a spacetime region M,

$$\rho_M(\psi) = \int_{\phi \in K_M} \mathcal{D}\phi\, \psi(\phi|_{\partial M})\, e^{\mathrm{i} S_M(\phi)}.$$

Similarly, it defines the **observable map** $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$ for observable $O : K_M \to \mathbb{R}$ in region M,

$$\rho_M^O(\psi) = \int_{\phi \in K_M} \mathcal{D}\phi\, \psi(\phi|_{\partial M})\, O(\phi)\, e^{\mathrm{i} S_M(\phi)}.$$

Again, the measure $\mathcal{D}\phi$ does not actually exist in most cases.

These quantum data "automatically" satisfy the quantum axioms.
Problem: This is not well defined.

Universal results in free field theory

[RO 2010; 2011; 2012]

For **free field theory** the informal Schrödinger-Feynman prescription can be replaced by a rigorous and functorial quantization scheme:

- For state spaces, the Schrödinger representation is replaced by the **holomorphic representation**, a flavor of **geometric quantization**.
- Additional data is required: One **complex structure** $J_\Sigma : L_\Sigma \to L_\Sigma$ per hypersurface.
- There is a **gluing anomaly**, modifying the gluing axiom **(T5b)** to $\diamond\rho_M = \rho_{M_1} \cdot c$
- Standard results in flat and globally hyperbolic spacetime are recovered.

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Theorem

The quantum data satisfies the QFT axioms (possibly with some infinite gluing anomalies).

Geometric quantization: Prequantization

Geometric quantization is designed to output the structures of the **standard formulation** of quantum theory, i.e., a Hilbert space of states and an operator algebra of observables acting on it. Its main input is the space L of solutions of the classical theory in spacetime with its symplectic structure ω. It proceeds roughly in two steps:

1. We consider a hermitian line bundle B over L with a connection ∇ that has curvature 2-form ω. Define the **prequantum** Hilbert space H as the space of square-integrable sections with inner product

$$\langle s', s \rangle = \int (s'(\eta), s(\eta))_\eta \, \mathrm{d}\mu(\eta).$$

   Here the measure $\mathrm{d}\mu$ is given by the $2n$-form $\omega \wedge \cdots \wedge \omega$ if L has dimension $2n$. Classical observables, i.e., functions on L, act naturally as operators on H with the "correct" commutation relations.

Geometric quantization: Polarization

2. This Hilbert space is too large. Choose in each complexified tangent space $(T_\phi L)^{\mathbb{C}}$ a Lagrangian subspace P_ϕ with respect to ω_ϕ. We then restrict H to those sections s of B such that

$$\nabla_{\overline{X}} s = 0,$$

if $X_\phi \in P_\phi$ for all $\phi \in L$. This is called a **polarization**. The subspace $\mathcal{H}$ of H obtained in this way is the Hilbert space of states. Not all observables are well defined on it as they might not leave the subspace $\mathcal{H} \subseteq H$ invariant.

Kähler polarization

We are interested in a **Kähler polarization**. Then P_ϕ is determined by a complex structure J_ϕ in $T_\phi L$ that is compatible with ω_ϕ. J_ϕ satisfies $J_\phi \circ J_\phi = -1$ and $\omega_\phi(J_\phi X, J_\phi Y) = \omega_\phi(X, Y)$. Then

$$P_\phi = \{X \in (T_\phi L)^{\mathbb{C}} : \mathrm{i}X = J_\phi X\}.$$

J_ϕ yields a real inner product on $T_\phi L$:

$$g_\phi(X_\phi, Y_\phi) := 2\omega_\phi(X_\phi, J_\phi Y_\phi).$$

We shall require g_ϕ to be positive definite. We also obtain a complex inner product on $T_\phi L$ viewed as a complex vector space:

$$\{X_\phi, Y_\phi\}_\phi := g_\phi(X_\phi, Y_\phi) + 2\mathrm{i}\omega_\phi(X_\phi, Y_\phi).$$

The Hilbert space $\mathcal{H}$ obtained from H through a Kähler polarization is also called the **holomorphic representation**.

Linear field theory

To be able to deal with the field theory case where L is generically infinite-dimensional we restrict ourselves to the simplest setting of linear field theory. That is, we take L to be a real vector space and the symplectic form ω to be invariant under translations in L. Not much is known beyond this setting.

Then, L can be naturally identified with its tangent space. Moreover, the symplectic form ω, the complex structure J, the real and complex inner products g, $\{\cdot,\cdot\}$ all become structures on the vector space L. The line bundle B becomes trivial and its section (the elements of H) can be identified with complex functions on L. For a Kähler polarization the elements of the subspace $\mathcal{H} \subseteq H$ are precisely the **holomorphic** functions on L. Moreover, the inner product formula simplifies,

$$\langle \psi', \psi \rangle = \int \overline{\psi'(\eta)} \psi(\eta) \exp\left(-\frac{1}{2} g(\eta, \eta)\right) \mathrm{d}\mu(\eta).$$

The measure

What is the measure $\mathrm{d}\mu$?

It turns out that on an infinite-dimensional vector space L no translation-invariant measure exists. Instead, we should look for a **Gaussian measure**

$$\mathrm{d}\nu \approx \exp\left(-\frac{1}{2}g(\eta,\eta)\right)\mathrm{d}\mu.$$

However, not even that exists on the Hilbert space L. The measure does exist if we extend L to a larger vector space $\hat{L}$. Concretely ν and $\hat{L}$ can be constructed as an inductive limit of finite-dimensional quotient spaces of L. It turns out that $\hat{L}$ can also be identified with the algebraic dual of the topological dual of L.

A priori, wave functions are thus really functions of $\hat{L}$ rather than on L. But, a function that is square-integrable on $\hat{L}$ and holomorphic is completely determined by its values on L. This allows us to "forget" about $\hat{L}$ to some extent.

Quantization: State spaces

For each hypersurface Σ the the complex structure J_Σ makes the space L_Σ into a complex Hilbert space with the inner product,

$$\{\phi', \phi\}_\Sigma := g_\Sigma(\phi', \phi) + 2\mathrm{i}\omega_\Sigma(\phi', \phi) \quad \text{where} \quad g_\Sigma(\phi', \phi) := 2\omega_\Sigma(\phi', J_\Sigma \phi).$$

The Hilbert space of states $\mathcal{H}_\Sigma$ is then the space of **holomorphic functions** on L_Σ with the inner product,

$$\langle \psi', \psi \rangle_\Sigma := \int_{\hat{L}_\Sigma} \overline{\psi'(\phi)} \psi(\phi) \, \mathrm{d}\nu_\Sigma(\phi).$$

where ν_Σ is the **Gaussian measure**,

$$\mathrm{d}\nu_\Sigma \approx \exp\left(-\frac{1}{2} g_\Sigma(\eta, \eta)\right) \mathrm{d}\mu.$$

Coherent States

The Hilbert spaces $\mathcal{H}_\Sigma$ are reproducing kernel Hilbert spaces and contain **coherent states** of the form

$$K_\xi(\phi) = \exp\left(\frac{1}{2}\{\xi,\phi\}_\Sigma\right)$$

associated to classical solutions $\xi \in L_\Sigma$. They have the **reproducing property**,

$$\langle K_\xi, \psi\rangle_\Sigma = \psi(\xi),$$

and satisfy the completeness relation

$$\langle \psi', \psi\rangle_\Sigma = \int_{\hat{L}_\Sigma} \langle \psi', K_\xi\rangle_\Sigma \langle K_\xi, \psi\rangle_\Sigma \, \mathrm{d}\nu_\Sigma(\xi).$$

They can be thought of as representing quantum states that approximate specific classical solutions.

Quantization: Amplitudes

For each region M we define the linear amplitude map $\rho_M : \mathcal{H}^\circ_{\partial M} \to \mathbb{C}$ by

$$\rho_M(\psi) := \int_{\hat{L}_M} \psi(\phi)\, \mathrm{d}\nu_M(\phi).$$

Here $\hat{L}_M$ is an extension of L_M and ν_M is a Gaussian measure on $\hat{L}_M$, depending on $g_{\partial M}$ that heuristically takes the form

$$\mathrm{d}\nu_M \approx \exp\left(-\frac{1}{4} g_{\partial M}(\eta, \eta)\right) \mathrm{d}\mu$$

with μ a (fictitious) translation-invariant measure.

It can be shown that this prescription is here equivalent to the Feynman path integral prescription.

Universal amplitude formula

The amplitude can be written down in closed form. M a region. [RO 2010]

- $L_{\partial M} = L_M \oplus J_{\partial M} L_M$ is a **real orthogonal decomposition** into classically continuable (L_M) and non-continuable ($J_{\partial M} L_M$) solutions.
- Let $\xi \in L_{\partial M}$ be a solution on the boundary of M. Decompose $\xi = \xi^{\mathrm{c}} + \xi^{\mathrm{n}}$ into $\xi^{\mathrm{c}} \in L_M$ and $\xi^{\mathrm{n}} \in J_{\partial M} L_M$.

The amplitude for the associated normalized coherent state $\tilde{K}_\xi$ is:

$$\rho_M(\tilde{K}_\xi) = \exp\left(\mathrm{i}\,\omega_{\partial M}(\xi^{\mathrm{n}}, \xi^{\mathrm{c}}) - \frac{1}{2} g_{\partial M}(\xi^{\mathrm{n}}, \xi^{\mathrm{n}})\right)$$

This has a simple and compelling physical interpretation.

Observables: Coherent factorization (I)

[RO 2011]

- Consider a region M with observable $F : K_M \to \mathbb{R}$.
- Let $\xi \in L_{\partial M}$ with $\xi = \xi^{\mathrm{c}} + \xi^{\mathrm{n}}$ and $\xi^{\mathrm{c}} \in L_M$, $\xi^{\mathrm{n}} \in J_{\partial M} L_M$. Define $\hat{\xi} \in L_M^{\mathbb{C}}$ by $\hat{\xi} := \xi^{\mathrm{c}} + \mathrm{i} J_{\partial M} \xi^{\mathrm{n}}$.

Coherent factorization for normal ordered quantization

$$\rho_M^{:F:}\left(K_\xi\right) = \rho_M\left(K_\xi\right) F(\hat{\xi})$$

Observables: Coherent factorization (II)

[RO 2012]

- Let $F = \exp(\mathrm{i}D)$ where $D : K_M \to \mathbb{R}$ is **linear**. Call F a **Weyl observable**.
- The action $S_M + D$ defines modified equations of motion. There is a unique solution η_D that is polarized on the boundary.

Coherent factorization for Weyl observables

$$\rho_M^F(K_\xi) = \rho_M^{:F:}(K_\xi)\,\rho_M^F(K_0),$$
$$\rho_M^F(K_0) = \exp\left(\frac{\mathrm{i}}{2}D(\eta_D)\right)$$

General observables

More general observables, in particular polynomial ones, can be obtained through **derivatives** from Weyl observables. Let $D_1, \ldots D_n : K_M \to \mathbb{R}$ be linear. We are interested in quantizing the monomial observable $G := D_1 \cdots D_n$. We define the Weyl observable F, depending on real parameters $\lambda_1, \ldots, \lambda_n$,

$$F_{\lambda_1,\ldots,\lambda_n} := \exp\left(\mathrm{i}\sum_{k=1}^{n} \lambda_k D_k\right), \quad \text{so,}$$

$$G = (-\mathrm{i})^n \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_n} F_{\lambda_1,\ldots,\lambda_n}\Big|_{\lambda_1=0,\ldots,\lambda_n=0}.$$

Since quantization is linear, this implies,

$$\rho_M^G = (-\mathrm{i})^n \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_n} \rho_M^{F_{\lambda_1,\ldots,\lambda_n}}\Big|_{\lambda_1=0,\ldots,\lambda_n=0}.$$

QFT in Minkowski spacetime (I)

Specialize to **quantum field theory (QFT) in Minkowski spacetime**. Consider the linear observable determined by a **source field** μ with support in a spacetime region M.

$$D(\phi) := \int \mu(x)\phi(x)\, \mathrm{d}x.$$

As before, the modified action $S_M + D$ yields modified equations of motions. These are here given by an **inhomogeneous PDE** with source μ. Recall the complexified special solution η_P satisfying **polarized boundary conditions**. Here these boundary conditions (encoded in the complex structure) are the **Feynman boundary conditions**. Therefore,

$$\eta_P(x) = \int G_F(x, x')\mu(x')\, \mathrm{d}x',$$

where G_F is the Feynman propagator.

QFT in Minkowski spacetime (II)

For the Weyl observable $F := \exp(\mathrm{i}D)$ we obtain from coherent factorization,

$$\rho_M^F(K_\xi) = \rho_M(K_\xi) \exp\left(\mathrm{i}\int \mu(x)\hat{\xi}(x)\,\mathrm{d}x\right)\exp\left(\frac{\mathrm{i}}{2}\int \mu(x)G_F(x,x')\mu(x')\,\mathrm{d}x\mathrm{d}x'\right).$$

In the special case where we take M to be a time-interval region $[t_{\mathrm{in}}, t_{\mathrm{out}}] \times \mathbb{R}^3$ and send $t_{\mathrm{in}} \to -\infty$, $t_{\mathrm{out}} \to \infty$ we recover the well known "generating function" or "kernel" of the S-matrix.

So the coherent factorization property is a vast generalization of this.

Perturbation theory (I)

Non-linear field theories can be treated **perturbatively**. The corresponding methods from quantum field theory readily generalize. Thus, we consider the theory determined by a modified action $S_M + S_M^{\text{int}}$, where S_M^{int} is considered a perturbative correction. We may treat $G := \exp(\mathrm{i}S_M^{\text{int}})$ simply as an **observable** and apply previous considerations.

Perturbation theory (II)

More specifically, for a potential term

$$S_M^{\mathrm{int}}(\phi) = \int_M V(x, \phi(x))\,\mathrm{d}x,$$

we may take advantage of its spacetime integral form. Define as before a linear observable determined by a source μ in M,

$$D_\mu(\phi) := \int \mu(x)\phi(x)\,\mathrm{d}x \quad \text{and} \quad F_\mu := \exp(\mathrm{i}D_\mu).$$

The interacting theory is then formally determined by the amplitude map,

$$\rho_M^G(\psi) = \exp\left(\mathrm{i}\int V\left(x, -\mathrm{i}\frac{\delta}{\delta\mu(x)}\right)\mathrm{d}x\right)\rho_M^{F_\mu}(\psi)\Big|_{\mu=0}.$$

Some extensions and applications

Extensions

- affine field theory [RO 2011]
- free fermionic field theory [RO 2012]
- abelian YM in Riemannian manifolds [H. Díaz-Marin, RO 2017]

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Applications

- generalized S-matrix in Minkowski [D. Colosi, RO 2007; 2008]
- S-matrix for AdS [M. Dohse, RO 2015]
- Unruh effect [D. Colosi, D. Rätzel 2012]
- Casimir effect [D. Colosi in progress]
- QFT in various curved spacetimes [D. Colosi, M. Dohse, R. Banisch, F. Hellmann,…2009–]