Local functorial quantization of field theory (II)

Robert Oeckl

Centro de Ciencias Matemáticas
Universidad Nacional Autónoma de México
Morelia, Mexico

Seminar General Boundary Formulation
30 May 2018
Quantization (canonical)

<table>
<thead>
<tr>
<th>concept</th>
<th>classical theory</th>
<th>quantum theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>phase space ((L, \omega))</td>
<td>(\to) Hilbert space (\mathcal{H})</td>
</tr>
<tr>
<td>observables</td>
<td>functions on phase space (C(L))</td>
<td>(\to) operator algebra (\mathcal{B}(\mathcal{H}))</td>
</tr>
<tr>
<td>quantization condition</td>
<td>Poisson bracket {\cdot, \cdot}</td>
<td>(\to) commutator {\cdot, \cdot}</td>
</tr>
</tbody>
</table>
## Quantization (local)

<table>
<thead>
<tr>
<th>concept</th>
<th>classical theory</th>
<th>quantum theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>states: per hypersurface $\Sigma$</td>
<td>phase space $(L_\Sigma, \omega_\Sigma)$</td>
<td>$\rightarrow$ Hilbert space $\mathcal{H}_\Sigma$</td>
</tr>
<tr>
<td>observables: per region $M$</td>
<td>functions $K_M \rightarrow \mathbb{C}$ on configuration space $C_M$</td>
<td>$\rightarrow$ space of observable maps $O_M \subseteq \mathcal{H}_{\partial M}^*$</td>
</tr>
<tr>
<td>quantization condition</td>
<td>product of functions $C_M \times C_N \rightarrow C_{M\cup N}$</td>
<td>$\rightarrow$ composition of observable maps $O_M \times O_N \rightarrow O_{M\cup N}$</td>
</tr>
</tbody>
</table>
Topological quantum field theory was originally inspired by the **Feynman path integral** and its composition properties. The Feynman path integral is defined in the **Schrödinger representation** where states are **wave functions** on **field configurations**.
Topological quantum field theory was originally inspired by the **Feynman path integral** and its composition properties. The Feynman path integral is defined in the **Schrödinger representation** where states are **wave functions** on **field configurations**.

The state space $\mathcal{H}_\Sigma$ for the hypersurface $\Sigma$ is the space of complex functions on $K_\Sigma$ with inner product,

$$\langle \psi', \psi \rangle_\Sigma = \int_{K_\Sigma} \mathcal{D}\varphi \, \overline{\psi'(\varphi)} \psi(\varphi).$$

Here, $\mathcal{D}\varphi$ is a **translation invariant measure** on $K_\Sigma$.

Such a measure does not exist in most cases. As usual, it is fruitful to proceed pretending that it does.
Schrödinger-Feynman quantization: regions

The Feynman path integral serves to define the amplitude map \( \rho_M : \mathcal{H}_\partial M \to \mathbb{C} \) in a spacetime region \( M \),

\[
\rho_M(\psi) = \int_{\phi \in K_M} D\phi \, \psi(\phi|\partial M) \, e^{iS_M(\phi)}.
\]

Similarly, it defines the observable map \( \rho^O_M : \mathcal{H}_\partial M \to \mathbb{C} \) for observable \( O : K_M \to \mathbb{R} \) in region \( M \),

\[
\rho^O_M(\psi) = \int_{\phi \in K_M} D\phi \, \psi(\phi|\partial M) \, O(\phi) \, e^{iS_M(\phi)}.
\]

Again, the measure \( D\phi \) does not actually exist in most cases.

These quantum data “automatically” satisfy the quantum axioms. Problem: This is not well defined.
Universal results in free field theory

[RO 2010; 2011; 2012]

For **free field theory** the informal Schrödinger-Feynman prescription can be replaced by a rigorous and functorial quantization scheme:

- For state spaces, the Schrödinger representation is replaced by the **holomorphic representation**, a flavor of **geometric quantization**.
- Additional data is required: One **complex structure** $J_{\Sigma} : L_{\Sigma} \rightarrow L_{\Sigma}$ per hypersurface.
- There is a **gluing anomaly**, modifying the gluing axiom (T5b) to $\rho_M = \rho_{M_1} \cdot c$
- Standard results in flat and globally hyperbolic spacetime are recovered.

---

Robert Oeckl  (CCM-UNAM)  
Local quantization  
2018-05-30 6 / 23
Universal results in free field theory

[RO 2010; 2011; 2012]
For free field theory the informal Schrödinger-Feynman prescription can be replaced by a rigorous and functorial quantization scheme:

- For state spaces, the Schrödinger representation is replaced by the holomorphic representation, a flavor of geometric quantization.
- Additional data is required: One complex structure $J_{\Sigma} : L_{\Sigma} \rightarrow L_{\Sigma}$ per hypersurface.
- There is a gluing anomaly, modifying the gluing axiom (T5b) to $\rho_{M} = \rho_{M_{1}} \cdot c$
- Standard results in flat and globally hyperbolic spacetime are recovered.

**Theorem**
The quantum data satisfies the QFT axioms (possibly with some infinite gluing anomalies).
Geometric quantization: Prequantization

Geometric quantization is designed to output the structures of the standard formulation of quantum theory, i.e., a Hilbert space of states and an operator algebra of observables acting on it. Its main input is the space $L$ of solutions of the classical theory in spacetime with its symplectic structure $\omega$. It proceeds roughly in two steps:

1. We consider a hermitian line bundle $B$ over $L$ with a connection $\nabla$ that has curvature 2-form $\omega$. Define the prequantum Hilbert space $H$ as the space of square-integrable sections with inner product

   $$\langle s', s \rangle = \int (s'(\eta), s(\eta))_{\eta} \, d\mu(\eta).$$

Here the measure $d\mu$ is given by the $2n$-form $\omega \wedge \cdots \wedge \omega$ if $L$ has dimension $2n$. Classical observables, i.e., functions on $L$, act naturally as operators on $H$ with the “correct” commutation relations.
This Hilbert space is too large. Choose in each complexified tangent space \((T_\phi L)_C\) a Lagrangian subspace \(P_\phi\) with respect to \(\omega_\phi\). We then restrict \(H\) to those sections \(s\) of \(B\) such that

\[
\nabla_X s = 0,
\]

if \(X_\phi \in P_\phi\) for all \(\phi \in L\). This is called a **polarization**. The subspace \(\mathcal{H}\) of \(H\) obtained in this way is the Hilbert space of states. Not all observables are well defined on it as they might not leave the subspace \(\mathcal{H} \subseteq H\) invariant.
Kähler polarization

We are interested in a Kähler polarization. Then $P_\phi$ is determined by a complex structure $J_\phi$ in $T_\phi L$ that is compatible with $\omega_\phi$. $J_\phi$ satisfies $J_\phi \circ J_\phi = -1$ and $\omega_\phi(J_\phi X, J_\phi Y) = \omega_\phi(X, Y)$. Then

$$P_\phi = \{X \in (T_\phi L)^C : iX = J_\phi X\}.$$ 

$J_\phi$ yields a real inner product on $T_\phi L$:

$$g_\phi(X_\phi, Y_\phi) := 2 \omega_\phi(X_\phi, J_\phi Y_\phi).$$

We shall require $g_\phi$ to be positive definite. We also obtain a complex inner product on $T_\phi L$ viewed as a complex vector space:

$$\{X_\phi, Y_\phi\}_\phi := g_\phi(X_\phi, Y_\phi) + 2i\omega_\phi(X_\phi, Y_\phi).$$

The Hilbert space $\mathcal{H}$ obtained from $H$ through a Kähler polarization is also called the holomorphic representation.
Linear field theory

To be able to deal with the field theory case where $L$ is generically infinite-dimensional we restrict ourselves to the simplest setting of linear field theory. That is, we take $L$ to be a real vector space and the symplectic form $\omega$ to be invariant under translations in $L$. Not much is known beyond this setting.

Then, $L$ can be naturally identified with its tangent space. Moreover, the symplectic form $\omega$, the complex structure $J$, the real and complex inner products $g, \{\cdot, \cdot\}$ all become structures on the vector space $L$. The line bundle $B$ becomes trivial and its section (the elements of $H$) can be identified with complex functions on $L$. For a Kähler polarization the elements of the subspace $\mathcal{H} \subseteq H$ are precisely the holomorphic functions on $L$. Moreover, the inner product formula simplifies,

$$\langle \psi', \psi \rangle = \int \overline{\psi'(\eta)} \psi(\eta) \exp \left( -\frac{1}{2} g(\eta, \eta) \right) d\mu(\eta).$$
The measure

What is the measure $d\mu$?

It turns out that on an infinite-dimensional vector space $L$ no translation-invariant measure exists. Instead, we should look for a Gaussian measure

$$d\nu \approx \exp\left(-\frac{1}{2}g(\eta,\eta)\right) d\mu.$$ 

However, not even that exists on the Hilbert space $L$. The measure does exist if we extend $L$ to a larger vector space $\hat{L}$. Concretely $\nu$ and $\hat{L}$ can be constructed as an inductive limit of finite-dimensional quotient spaces of $L$. It turns out that $\hat{L}$ can also be identified with the algebraic dual of the topological dual of $L$.

A priori, wave functions are thus really functions of $\hat{L}$ rather than on $L$. But, a function that is square-integrable on $\hat{L}$ and holomorphic is completely determined by its values on $L$. This allows us to “forget” about $\hat{L}$ to some extent.
Quantization: State spaces

For each hypersurface $\Sigma$ the complex structure $J_\Sigma$ makes the space $L_\Sigma$ into a complex Hilbert space with the inner product,

$$\{\phi', \phi\}_\Sigma := g_\Sigma(\phi', \phi) + 2i\omega_\Sigma(\phi', \phi)$$

where $g_\Sigma(\phi', \phi) := 2\omega_\Sigma(\phi', J_\Sigma \phi)$.

The Hilbert space of states $\mathcal{H}_\Sigma$ is then the space of holomorphic functions on $L_\Sigma$ with the inner product,

$$\langle \psi', \psi \rangle_\Sigma := \int_{L_\Sigma} \overline{\psi'(\phi)} \psi(\phi) \, d\nu_\Sigma(\phi).$$

where $\nu_\Sigma$ is the Gaussian measure,

$$d\nu_\Sigma \approx \exp \left( -\frac{1}{2} g_\Sigma(\eta, \eta) \right) d\mu.$$
Coherent States

The Hilbert spaces $\mathcal{H}_\Sigma$ are reproducing kernel Hilbert spaces and contain coherent states of the form

$$K_\xi(\phi) = \exp \left( \frac{1}{2} \{ \xi, \phi \}_\Sigma \right)$$

associated to classical solutions $\xi \in L_\Sigma$. They have the reproducing property,

$$\langle K_\xi, \psi \rangle_\Sigma = \psi(\xi),$$

and satisfy the completeness relation

$$\langle \psi', \psi \rangle_\Sigma = \int_{\hat{L}_\Sigma} \langle \psi', K_\xi \rangle_\Sigma \langle K_\xi, \psi \rangle_\Sigma \, d\nu_\Sigma(\xi).$$

They can be thought of as representing quantum states that approximate specific classical solutions.
Quantization: Amplitudes

For each region $M$ we define the linear amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ by

$$\rho_M(\psi) := \int_{\hat{L}_M} \psi(\phi) \, d\nu_M(\phi).$$

Here $\hat{L}_M$ is an extension of $L_M$ and $\nu_M$ is a Gaussian measure on $\hat{L}_M$, depending on $g_{\partial M}$ that heuristically takes the form

$$d\nu_M \approx \exp \left(-\frac{1}{4} g_{\partial M}(\eta, \eta)\right) d\mu$$

with $\mu$ a (fictitious) translation-invariant measure.

It can be shown that this prescription is here equivalent to the Feynman path integral prescription.
Universal amplitude formula

The amplitude can be written down in closed form. $M$ a region.

[RO 2010]

- $L_{\partial M} = L_M \oplus J_{\partial M}L_M$ is a real orthogonal decomposition into classically continuable ($L_M$) and non-continuable ($J_{\partial M}L_M$) solutions.

- Let $\xi \in L_{\partial M}$ be a solution on the boundary of $M$. Decompose $\xi = \xi^c + \xi^n$ into $\xi^c \in L_M$ and $\xi^n \in J_{\partial M}L_M$.

The amplitude for the associated normalized coherent state $\tilde{K}_\xi$ is:

$$\rho_M(\tilde{K}_\xi) = \exp \left( i \omega_{\partial M}(\xi^n, \xi^c) - \frac{1}{2} g_{\partial M}(\xi^n, \xi^n) \right)$$

This has a simple and compelling physical interpretation.
Consider a region $M$ with observable $F : K_M \to \mathbb{R}$.

Let $\xi \in L_{\partial M}$ with $\xi = \xi^c + \xi^n$ and $\xi^c \in L_M$, $\xi^n \in J_{\partial M}L_M$. Define $\hat{\xi} \in L^c_M$ by $\hat{\xi} := \xi^c + iJ_{\partial M}\xi^n$.

Coherent factorization for normal ordered quantization

$$\rho^F_M (K_\xi) = \rho_M (K_\xi) F(\hat{\xi})$$
Let $F = \exp(iD)$ where $D : K_M \rightarrow \mathbb{R}$ is linear. Call $F$ a Weyl observable.

The action $S_M + D$ defines modified equations of motion. There is a unique solution $\eta_D$ that is polarized on the boundary.

Coherent factorization for Weyl observables

$$\rho^F_M (K_\xi) = \rho^F_M (K_\xi) \rho^F_M (K_0),$$

$$\rho^F_M (K_0) = \exp \left( \frac{i}{2} D(\eta_D) \right)$$
General observables

More general observables, in particular polynomial ones, can be obtained through derivatives from Weyl observables. Let $D_1, \ldots, D_n : K_M \to \mathbb{R}$ be linear. We are interested in quantizing the monomial observable $G := D_1 \cdots D_n$. We define the Weyl observable $F$, depending on real parameters $\lambda_1, \ldots, \lambda_n$,

$$F_{\lambda_1, \ldots, \lambda_n} := \exp \left( i \sum_{k=1}^{n} \lambda_k D_k \right),$$

so,

$$G = (-i)^n \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_n} F_{\lambda_1, \ldots, \lambda_n} \bigg|_{\lambda_1=0, \ldots, \lambda_n=0}.$$

Since quantization is linear, this implies,

$$\rho^G_M = (-i)^n \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_n} \rho^F_M \bigg|_{\lambda_1=0, \ldots, \lambda_n=0}.$$
Specialize to quantum field theory (QFT) in Minkowski spacetime. Consider the linear observable determined by a source field \( \mu \) with support in a spacetime region \( M \).

\[
D(\phi) := \int \mu(x)\phi(x) \, dx.
\]

As before, the modified action \( S_M + D \) yields modified equations of motions. These are here given by an inhomogeneous PDE with source \( \mu \). Recall the complexified special solution \( \eta_P \) satisfying polarized boundary conditions. Here these boundary conditions (encoded in the complex structure) are the Feynman boundary conditions. Therefore,

\[
\eta_P(x) = \int G_F(x, x') \mu(x') \, dx',
\]

where \( G_F \) is the Feynman propagator.
For the Weyl observable $F := \exp(iD)$ we obtain from coherent factorization,

$$\rho_M^K = \rho_M^K 
\exp \left( i \int \mu(x) \hat{\xi}(x) \, dx \right) \exp \left( \frac{i}{2} \int \mu(x) G_F(x, x') \mu(x') \, dx \, dx' \right).$$

In the special case where we take $M$ to be a time-interval region $[t_{in}, t_{out}] \times \mathbb{R}^3$ and send $t_{in} \to -\infty$, $t_{out} \to \infty$ we recover the well known “generating function” or “kernel” of the S-matrix.

So the coherent factorization property is a vast generalization of this.
Perturbation theory (I)

Non-linear field theories can be treated perturbatively. The corresponding methods from quantum field theory readily generalize. Thus, we consider the theory determined by a modified action $S_M + S_M^{\text{int}}$, where $S_M^{\text{int}}$ is considered a perturbative correction. We may treat $G := \exp(i S_M^{\text{int}})$ simply as an observable and apply previous considerations.
Perturbation theory (II)

More specifically, for a potential term

\[ S_M^{\text{int}}(\phi) = \int_M V(x, \phi(x)) \, dx, \]

we may take advantage of its spacetime integral form. Define as before a linear observable determined by a source \( \mu \) in \( M \),

\[ D_\mu(\phi) := \int \mu(x)\phi(x) \, dx \quad \text{and} \quad F_\mu := \exp(iD_\mu). \]

The interacting theory is then formally determined by the amplitude map,

\[ \rho_M^G(\psi) = \exp \left( i \int V(x, -i \frac{\delta}{\delta\mu(x)}) \, dx \right) \rho_M^{F_\mu}(\psi) \bigg|_{\mu=0}. \]
Some extensions and applications

Extensions

- affine field theory [RO 2011]
- free fermionic field theory [RO 2012]
- abelian YM in Riemannian manifolds [H. Díaz-Marin, RO 2017]
Some extensions and applications

Extensions

- affine field theory [RO 2011]
- free fermionic field theory [RO 2012]
- abelian YM in Riemannian manifolds [H. Díaz-Marin, RO 2017]

Applications

- generalized S-matrix in Minkowski [D. Colosi, RO 2007; 2008]
- S-matrix for AdS [M. Dohse, RO 2015]
- Unruh effect [D. Colosi, D. Rätzel 2012]
- Casimir effect [D. Colosi in progress]
- QFT in various curved spacetimes [D. Colosi, M. Dohse, R. Banisch, F. Hellmann,…2009–]