The positive formalism: time and evolution

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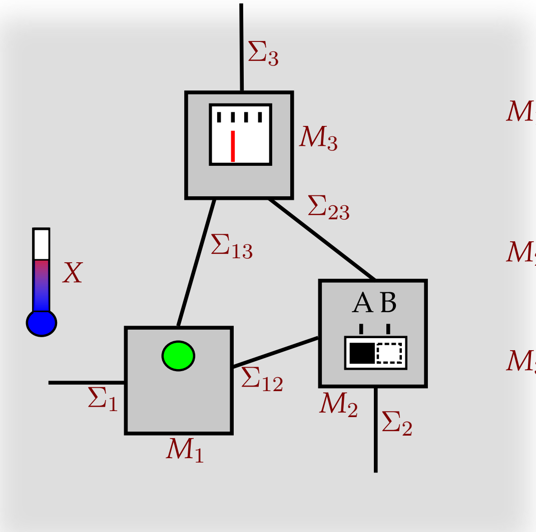
The positive formalism: An example



M_1: light

- $P(*)$ (apparatus)
- $P(r)$ (light red)
- $P(g)$ (light green)

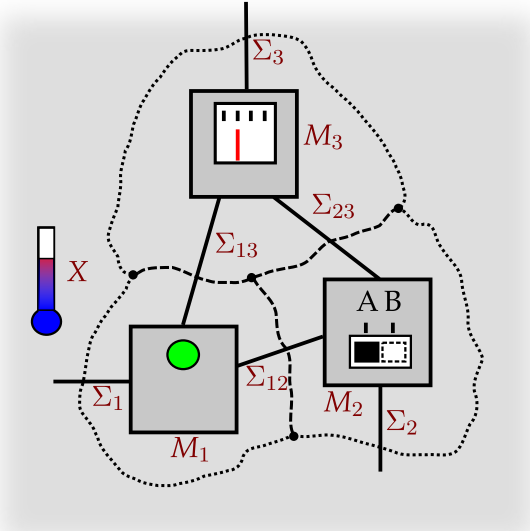
M_2: switch

- $Q(A)$ (position A)
- $Q(B)$ (position B)

M_3: meter

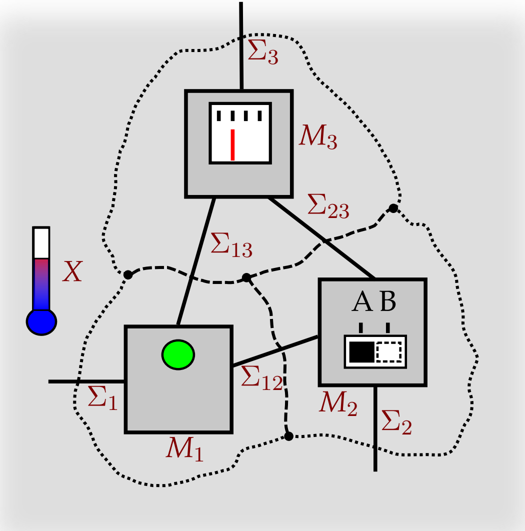
- $R[*]$ (apparatus)
- $R[a, b]$ (range $[a, b]$)
- R (reading)

Adding spacetime and locality



Spacetime locality provides a powerful organizing principle. Processes only interface with adjacent processes. This decreases considerably the inter-connectivity of the graph.

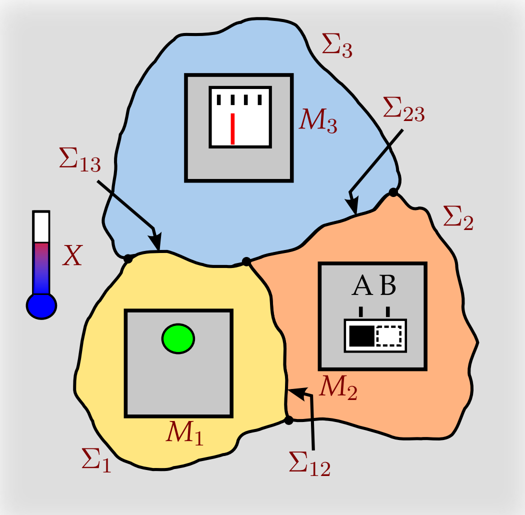
Adding spacetime and locality



Spacetime locality provides a powerful organizing principle. Processes only interface with adjacent processes. This decreases considerably the inter-connectivity of the graph.

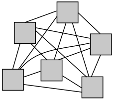
We associate a spacetime **region** to any process and a **hypersurface** to any interface. These form a **dual complex** to the graph of boxes and links.

Adding spacetime and locality

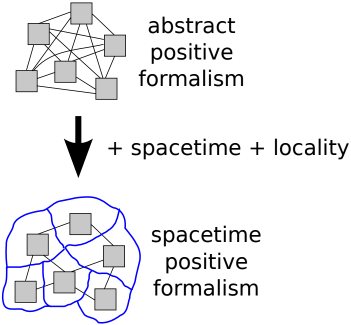


We may forget about the graph and identify **process types** with **regions** and **interface types** with **hypersurfaces**.

This framework is called the **local positive formalism**.



abstract
positive
formalism


abstract
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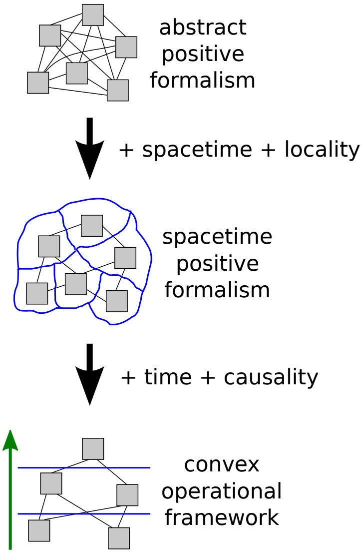
abstract positive formalism
+ spacetime + locality
spacetime positive formalism
+ time + causality
convex operational framework

Time-evolution

Specialize to a global factorizing spacetime $\mathbb{R} \times \Sigma$ and restrict the spacetime system to **equal-time hyperplanes** Σ_t and **time-interval regions** $[t_1, t_2] = [t_1, t_2] \times \Sigma$.

Write $\mathcal{B}_t := \mathcal{B}_{\Sigma_t}$ and call this the (generalized) **state space** at time t.

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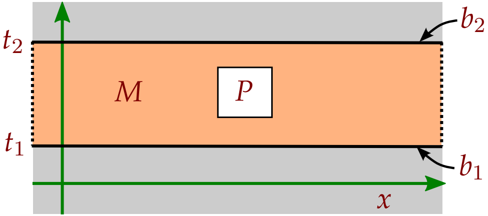

Consider probe $P \in \mathcal{P}_{[t_1,t_2]}$. Define the **probe map** $\tilde{P} : \mathcal{B}_{t_1} \to \mathcal{B}_{t_2}$ via

$$(\!| b_2, \tilde{P}(b_1) |\!)_{t_2} = [\![P, b_1 \otimes b_2]\!]_{[t_1,t_2]}, \qquad \forall b_1 \in \mathcal{B}_{t_1}, b_2 \in \mathcal{B}_{t_2}.$$

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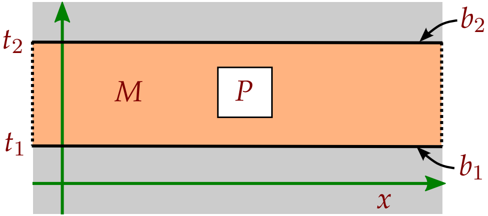

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That is, $\tilde{P}(b) = \sum_{k \in I} [\![P, b \otimes \xi_k]\!]_{[t_1,t_2]} \xi_k$.

Primitive probe maps and positivity

Probe maps for **primitive probes** are **positive**. They map proper states to proper states, $\mathcal{B}_{t_1}^+ \to \mathcal{B}_{t_2}^+$. They even have the stronger property of **boundary positivity**,

$$\sum_i \llparenthesis c_i, \tilde{P}(b_i) \rrparenthesis_{t_2} \geq 0 \qquad \text{if} \quad \sum_i b_i \otimes c_i \in \mathcal{B}_{\partial[t_1,t_2]}^+ \supseteq \mathcal{B}_{t_1}^+ \otimes \mathcal{B}_{t_2}^+$$

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In classical theory, positivity and boundary positivity are equivalent.

In quantum theory, boundary positivity is **complete positivity**.

Time-evolution maps

The probe map associated to the **null-probe** is the **time-evolution map** $T_{[t_1,t_2]} : \mathcal{B}_{t_1} \to \mathcal{B}_{t_2}$,

$$(\!| b_2, T_{[t_1,t_2]}(b_1) |\!)_{t_2} = [\![\boxslash, b_1 \otimes b_2]\!]_{[t_1,t_2]}$$

The time-evolution maps compose for $t_1 \le t_2 \le t_3$ as,

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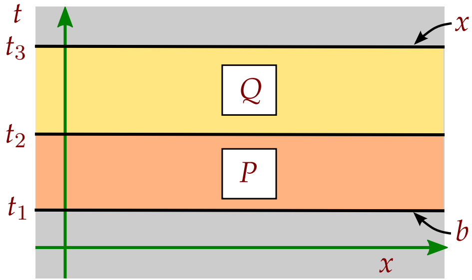
Usually, time-evolution preserves the state space. Thus, $\mathcal{B} = \mathcal{B}_t$. Probe maps become **operators** on $\mathcal{B}$. Assume this from now on.

Many systems are also **time-translation symmetric** meaning that $T_{[t_1,t_1+\Delta]} = T_{[t_2,t_2+\Delta]} = T_\Delta$. We then get a **one-parameter semigroup** of boundary positive operators,

$$T_{\Delta_1+\Delta_2} = T_{\Delta_1} \circ T_{\Delta_2}.$$

State "collapse" and Bayesian updating



Consider two consecutive measurements with initial state b and final state x. Suppose we have binary outcomes with probes,

- $Q_r + Q_g = Q_*$
- $P_r + P_g = P_*$

Predict probability for Q_r in the second measurement:

Outcome of P unknown: $$\Pi(Q_r) = \frac{(\!| x, \tilde{Q}_r \tilde{P}_* \, b |\!)}{(\!| x, \tilde{Q}_* \tilde{P}_* \, b |\!)}$$

Outcome P_r: $$\Pi(Q_r | P_r) = \frac{(\!| x, \tilde{Q}_r \tilde{P}_r \, b |\!)}{(\!| x, \tilde{Q}_* \tilde{P}_r \, b |\!)}$$

Outcome P_g: $$\Pi(Q_r | P_g) = \frac{(\!| x, \tilde{Q}_r \tilde{P}_g \, b |\!)}{(\!| x, \tilde{Q}_* \tilde{P}_g \, b |\!)}$$

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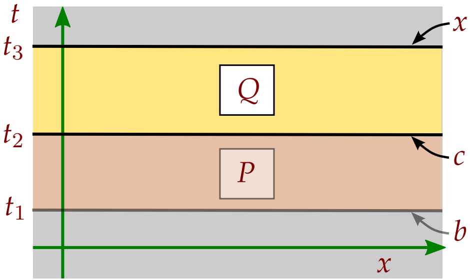

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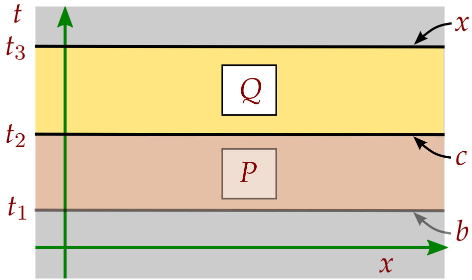

The outcome of P can be conveniently encoded in the state c. We may say:

> “The P measurement causes the state b to collapse to either $c = \tilde{P}_r b$ or $c = \tilde{P}_g b$.”

Predict probability for Q_r in the second measurement:

Outcome of P unknown: $\Pi(Q_r) = \frac{(\!|x, \tilde{Q}_r\, c|\!)}{(\!|x, \tilde{Q}_*\, c|\!)}$ with $c = \tilde{P}_* b$

Outcome P_r: $\Pi(Q_r|P_r) = \frac{(\!|x, \tilde{Q}_r\, c|\!)}{(\!|x, \tilde{Q}_*\, c|\!)}$ with $c = \tilde{P}_r b$

Outcome P_g: $\Pi(Q_r|P_g) = \frac{(\!|x, \tilde{Q}_r\, c|\!)}{(\!|x, \tilde{Q}_*\, c|\!)}$ with $c = \tilde{P}_g b$

The state of maximal uncertainty

Recall that the **boundary conditions** form a **hierarchy of generality**.

We assume that there exists a state $\mathbf{e} \in \mathcal{B}^+$ that is maximally general, call this the **state of maximal uncertainty**. This encodes a complete lack of knowledge.

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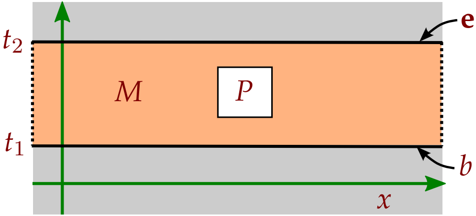
Most often in a measurement, we are only interested in the outcome given a fixed initial state b_1, but do not care about the state after the measurement.

This is encoded by setting the final state $b_2 = \mathbf{e}$.

Measurement without post-selection

Consider a binary measurement in $[t_1, t_2]$ encoded by a **non-selective probe** Q and a **selective probe** P.



The probability Π for an affirmative outcome given an initial state $b \in \mathcal{B}$, but disregarding the final fate of the system is thus,

$$\Pi = \frac{[\![P, b \otimes \mathbf{e}]\!]_{[t_1,t_2]}}{[\![Q, b \otimes \mathbf{e}]\!]_{[t_1,t_2]}} = \frac{(\!|\mathbf{e}, \tilde{P}(b)|\!)}{(\!|\mathbf{e}, \tilde{Q}(b)|\!)}.$$

One also says that this is a measurement **without post-selection**.

Main reference

R. O., *A local and operational framework for the foundations of physics*, arXiv:1610.09052.