The convex operational framework: classical and quantum theory

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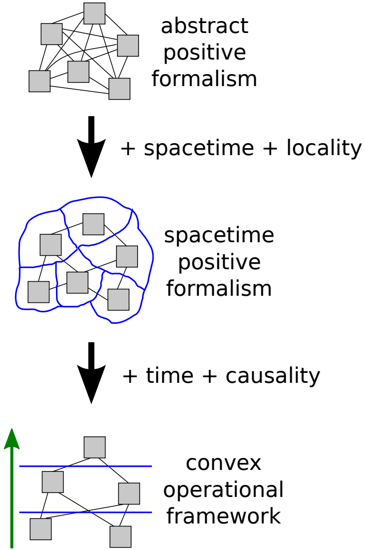
abstract positive formalism
+ spacetime + locality
spacetime positive formalism
+ time + causality
convex operational framework

The convex operational framework

We adopt a **time-evolution** picture of physics.

- To a **system** associate a space $\mathcal{B}$ of (generalized) **states**.
- The system evolves through **transformations** $P : \mathcal{B} \to \mathcal{B}$. These may involve **measurements**, **observations**, **interventions**.

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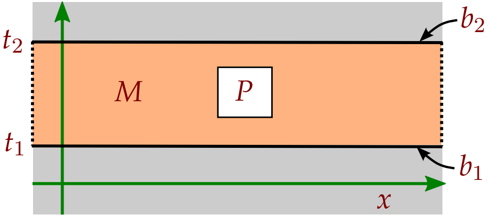


Here, $b_2 = P(b_1)$.

The state space

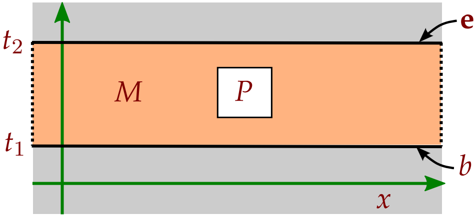
- The (generalized) state space $\mathcal{B}$ is a **partially ordered vector space** with **positive cone** $\mathcal{B}^+ \subseteq \mathcal{B}$.
- $\mathcal{B}$ carries a positive-definite, sharply positive **inner product** $(\!|\cdot, \cdot|\!)$.
- There is a special **state of maximal uncertainty** $\mathbf{e} \in \mathcal{B}^+$. It encodes complete lack of knowledge. It is an **order unit**.
- For $b \in \mathcal{B}^+$, $\|b\| = (\!|b, \mathbf{e}|\!)$ defines a **norm**. If $b \neq 0$ then $\|b\| > 0$. Denote the normalized elements of $\mathcal{B}^+$ by $\mathcal{B}^1$. These are called **proper states**. We denote $\mathcal{B}^{\leq 1} \subseteq \mathcal{B}^+$ the positive elements with norm less or equal to 1.

Transformations

- The (generalized) space of **transformations** $\mathcal{P}$ is a **partially ordered vector space** with positive cone $\mathcal{P}^+ \subseteq \mathcal{P}$.
- A (generalized) **transformation** P is a map $P : \mathcal{B} \to \mathcal{B}$. If $P \in \mathcal{P}^+$, then $P(\mathcal{B}^+) \subseteq \mathcal{B}^+$.
- If moreover, $P(\mathcal{B}^{\leq 1}) \subseteq \mathcal{B}^{\leq 1}$ we say that P is a **proper transformation**.
- If even $P(\mathcal{B}^1) \subseteq \mathcal{B}^1$ we say that P is **non-selective**. Otherwise it is called **selective**.
- For a time-interval $[t_1, t_2]$ there is a special non-selective transformation $T_{[t_1,t_2]}$ that encodes the **free time-evolution** of the system. This also preserves the inner product.

Measurement (without post-selection)

To encode a measurement with n possible outcomes we need n **selective transformations** $P_1, \ldots, P_n$ such that the sum $P = P_1 + \cdots + P_n$ is a **non-selective transformation**.



The probability Π_k for the outcome k given an initial state $b \in \mathcal{B}^1$,

$$\Pi_k = (\!|\mathbf{e}, P_k(b)|\!) = \|P_k(b)\|.$$

The final state of the system after the measurement with outcome k is,

$$b' = \frac{P_k(b)}{\|P_k(b)\|}.$$

Expectation values

Suppose we associate with measurement outcome n the measured **value** $\lambda_n \in \mathbb{R}$. We form the generalized transformation $\hat{P} := \lambda_1 P_1 + \cdots + \lambda_n P_n$. Then, the **expectation value** of the measurement is given by,

$$\langle \hat{P} \rangle_b = \sum_{k=1}^{n} \lambda_k \Pi_k = \sum_{k=1}^{n} \lambda_k (\!| \mathbf{e}, P_k(b) |\!) = (\!| \mathbf{e}, \hat{P}(b) |\!).$$

Classical mechanics

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- The phase space carries a **symplectic structure** ω.
- A **Hamiltonian** function $H : L \to \mathbb{R}$ determines a **flow** X_H on phase space via,
$$\mathrm{d}H(Y) = 2\omega(Y, X_H).$$
This determines the **infinitesimal time-evolution** on L.
- Exponentiation yields finite **time-evolution** maps $v_{[t_1,t_2]} : L \to L$ given by $v_{[t_1,t_2]}(x) = \exp((t_2 - t_1)X_H)x$. These are **symplectomorphisms**.

Observables

- An **observable** O is a function $O : L \to \mathbb{R}$. It provides the value of a **measurement** on the system.
- Let $S \subseteq L$ and $\chi_S : L \to \mathbb{R}$ the characteristic function

$$\chi_S(x) = \begin{cases} 1 & \text{if} \quad x \in S \\ 0 & \text{if} \quad x \notin S \end{cases} \quad .$$

Then χ_S is the observable that tells us if we are in the subset S of the phase space.

Classical statistical mechanics

In classical **statistical** mechanics a **state** is a **statistical distribution** on phase space.

We introduce a reference **probability measure** μ on L and model states as functions. If L has dimension $2n$, then a suitable measure is $\omega^{\wedge n}$.

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We introduce a reference **probability measure** μ on L and model states as functions. If L has dimension $2n$, then a suitable measure is $\omega^{\wedge n}$.

Let $\mathcal{B}$ be the space of real valued functions on L (with suitable regularity properties). This is a partially ordered vector space. $\mathcal{B}^+$ is the set of **positive functions**, i.e., functions that take non-negative values everywhere.

The overall probability associated to a distribution $b : L \to \mathbb{R}_0^+$ is its integral,

$$\|b\| := \int |b(x)| \, \mathrm{d}\mu(x).$$

(This defines the L^1-norm on $\mathcal{B}$.)

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A proper state must yield total probability 1, so we require $\|b\| = 1$. This defines $\mathcal{B}^1 \subseteq \mathcal{B}^+$ (and $\mathcal{B}^{\leq 1}$).

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- $\mathbf{e} = \mathbf{1}$ the constant function with value 1 is an **order unit** (supposing functions are bounded). It encodes the **uniform statistical distribution**. It can be interpreted as the state of minimal knowledge, i.e., maximal uncertainty.
- $(\!| b, c |\!) = \int b(x)c(x)\mathrm{d}\mu(x)$ the L^2 **inner product** is a natural positive-definite, sharply positive inner product on $\mathcal{B}$.
- For $b \in \mathcal{B}^+$ we have $\|b\| = (\!| \mathbf{e}, b |\!)$.

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Time-evolution of distributions is, $(T_{[t_1,t_2]}(b))(x) = b(v^{-1}_{[t_1,t_2]}(x))$. This is a **non-selective transformation** that leaves the inner product invariant (if the measure is invariant under time-evolution).

Measurement

$O : L \to \mathbb{R}$ an observable. Its expectation value in a state $b \in \mathcal{B}^1$ is,

$$\langle O \rangle_b = \int O(x) b(x) \, \mathrm{d}\mu(x).$$

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We introduce the (generalized) **transformation** $\tilde{O} : \mathcal{B} \to \mathcal{B}$ given by, $(\tilde{O}(b))(x) := O(x)b(x)$. Then,

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Suppose O takes values $\lambda_1 \ldots, \lambda_n$ on $S_1, \ldots, S_n$ with $L = S_1 \sqcup \ldots \sqcup S_n$. That is, $O = \sum_{k=1}^n \lambda_k \chi_{S_k}$. Note that $\tilde{\chi}_{S_k}$ is **selective** while $\sum_{k=1}^n \tilde{\chi}_{S_k} = \mathrm{id}$ is **non-selective**. The probability for measuring λ_k (outcome k) is,

$$\Pi_k = (\!| \mathbf{e}, \tilde{\chi}_{S_k}(b) |\!) = \| \tilde{\chi}_{S_k}(b) \|.$$

After the measurement the state is updated to

$$b' = \frac{\tilde{\chi}_{S_k}(b)}{\| \tilde{\chi}_{S_k}(b) \|}.$$

Quantum mechanics

- To a system we associate a complex **Hilbert space** $\mathcal{H}$.
- A self-adjoint **Hamiltonian operator** $H : \mathcal{H} \to \mathcal{H}$ determines **infinitesimal time-evolution** on $\mathcal{H}$ via the **Schrödinger equation**,

$$\frac{\partial}{\partial t}\psi = -\mathrm{i}H\psi, \qquad \text{for} \quad \psi \in \mathcal{H}.$$

- The operator describing time-evolution in the interval $[t_1, t_2]$ is $U_{[t_1,t_2]} = \exp(-\mathrm{i}(t_2 - t_1)H)$. It is **unitary**.

Observables and measurement

An **observable** is a self-adjoint operator $O : \mathcal{H} \to \mathcal{H}$.

Measuring an observable in a state ψ yields the expectation value,

$$\langle O \rangle_\psi = \langle \psi, O\psi \rangle_{\mathcal{H}}.$$

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Suppose $O = \sum_{k=1}^n \lambda_k P_k$, where P_k are orthogonal projection operators. The probability for obtaining the measurement result λ_k is,

$$\Pi_k = \langle \psi, P_k \psi \rangle_{\mathcal{H}} = \|P_k \psi\|^2_{\mathcal{H}}.$$

After the measurement, the state "collapses to"

$$\psi' = \frac{P_k \psi}{\|P_k \psi\|_{\mathcal{H}}}.$$

Quantum statistical mechanics

The (generalized) **state space** is the space $\mathcal{B}$ of **self-adjoint operators** on $\mathcal{H}$. This is a **partially ordered vector space** with positive cone $\mathcal{B}^+$ the **positive operators**.

To an element $\psi \in \mathcal{H}$ we associate in $\mathcal{B}^+$ the projector on the subspace spanned by ψ. Such an element of $\mathcal{B}^+$ is called a **pure state**.

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- $\mathbf{e} = \mathrm{id}_{\mathcal{H}}$ the **identity operator**. This is an **order unit** in $\mathcal{B}$. We can interpret $\mathbf{e}$ as the state of maximal uncertainty.
- $(\!| b, c |\!) = \mathrm{tr}(bc)$ the **Hilbert-Schmidt inner product** on $\mathcal{B}$ is positive-definite and sharply positive.
- For $b \in \mathcal{B}^+$ set $\|b\| = (\!| \mathbf{e}, b |\!) = \mathrm{tr}(b)$. This defines the **trace norm** on $\mathcal{B}$. The **proper states** are the elements of $\mathcal{B}^1 \subseteq \mathcal{B}^{\leq 1} \subseteq \mathcal{B}^+ \subseteq \mathcal{B}$.

Quantum statistical mechanics

The **positive transformations** $\mathcal{P}^+$ are all **completely positive maps** $\mathcal{B} \to \mathcal{B}$. The proper (norm decreasing) ones are also called **quantum operations**.

The time-evolution map $T_{[t_1,t_2]} : \mathcal{B} \to \mathcal{B}$ is given by $b \mapsto U_{[t_1,t_2]} b\, U^\dagger_{[t_1,t_2]}$. This is a non-selective transformation that leaves the inner product $(\!(\cdot, \cdot)\!)$ invariant.

Transformations and Measurements

A general measurement with n outcomes and its expectation value is given as seen for the convex operational framework.

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Any positive transformation $P \in \mathcal{P}^+$, i.e., completely positive map $\mathcal{B} \to \mathcal{B}$ can be decomposed in terms of operators $\{K_i\}_{i \in I}$, called **Kraus operators** so that,

$$P(b) = \sum_{i \in I} K_i b K_i^\dagger .$$

P is proper, i.e., norm decreasing iff $\sum_{i \in I} K_i^\dagger K_i \leq \mathrm{id}$ and norm preserving iff $\sum_{i \in I} K_i^\dagger K_i = \mathrm{id}$.

For the time-evolution map $T_{[t_1,t_2]}$ we have $I = \{1\}$ and $K_1 = U_{[t_1,t_2]}$.

Observable measurement as a special case

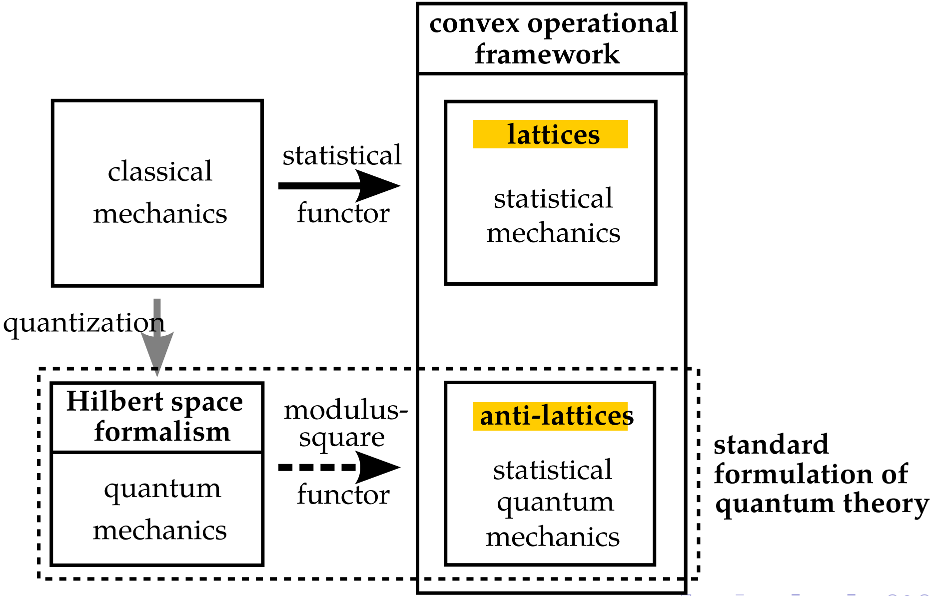
An observable O defines for each eigenvalue λ_k of $O = \sum_{k=1}^{n} \lambda_k P_k$ a **selective transformation** via $\tilde{P}_k(b) = P_k b P_k^\dagger$. Set $\tilde{O} := \sum_{i=1}^{n} \lambda_k \tilde{P}_k$. If $b = P_\psi$,

$$\langle O \rangle_{P_\psi} = (\!|\mathbf{e}, \tilde{O}(P_\psi)|\!) = \sum_{k=1}^{n} \lambda_k \mathrm{tr}(P_k P_\psi P_k^\dagger) = \sum_{k=1}^{n} \lambda_k \langle \psi, P_k \psi \rangle_{\mathcal{H}} = \langle \psi, O\psi \rangle_{\mathcal{H}}$$

$$\Pi_k = (\!|\mathbf{e}, \tilde{P}_k(P_\psi)|\!) = \|\tilde{P}_k(P_\psi)\| = \mathrm{tr}(P_k P_\psi) = \langle \psi, P_k \psi \rangle_{\mathcal{H}} = \|P_k \psi\|_{\mathcal{H}}^2.$$

$$b' = \frac{\tilde{P}_k(P_\psi)}{\|\tilde{P}_k(P_\psi)\|} = \frac{P_k P_\psi P_k^\dagger}{\|P_k \psi\|_{\mathcal{H}}^2} = P_{P_k \psi}.$$

Time-evolution frameworks



Main reference

R. O., *A local and operational framework for the foundations of physics*, `arXiv:1610.09052`.