Quantum abelian Yang-Mills fields on Riemannian manifolds

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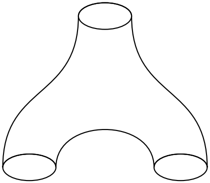
  4. *Special hypersurfaces:* boundary hypersurfaces:

$$\Sigma = \partial M,\ \partial\hat{\Sigma} = \Sigma \sqcup \overline{\Sigma}$$

$$\partial : Regions \to Hypersurfaces$$

$\mathcal{P}_{\partial M} \to \partial M$ induced by $\mathcal{P}_M \to M$

Gluing (Topology and geometry)



$$M, \quad \partial M = \Sigma_1 \sqcup \Sigma \sqcup \overline{\Sigma'}$$

$$M_1 = M/\sim \Sigma, \quad \overline{\Sigma'} \leftrightarrow \Sigma, \quad \partial M_1 = \Sigma_1$$

A principal bundle $\mathcal{P}_{M_1} \to M_1$ is obtained by gluing along Σ from $\mathcal{P}_M \to M$. Similarly for a Riemannian metric g_{M_1} obtained from g_M.

$$
\begin{tikzcd}
\text{Regions} \arrow[r, "\partial"] & \text{Hypersurfaces}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Spacetime system} \arrow[d, "\textit{classical}"] & \left(\left\{(M, \mathcal{P}_M, S_M)\right\}, \sqcup\right) \arrow[r, "\partial"] \arrow[d] & \left(\left\{(\Sigma, g_\Sigma, \partial/\partial n_\Sigma, \mathcal{P}_\Sigma)\right\}_\Sigma, \sqcup\right) \arrow[d] \\
\text{Lagr. embeddings} \arrow[d, "\textit{quantum}"] & \left(\{A_{\tilde{M}} \subseteq A_{M,\partial M}\}, \oplus\right) \arrow[r, mapsto] \arrow[d] & \left(\{\mathit{Lagr}(A_\Sigma, \omega_\Sigma)\}_\Sigma, \oplus\right) \arrow[d] \\
\text{Amplitude maps} & \left(\{\rho_M\}, \cdot\right) \arrow[r, mapsto] & \left(\{\mathit{Maps}(\mathcal{H}^\circ_\Sigma, \mathbb{C})\}_\Sigma, \otimes\right)
\end{tikzcd}
$$

$$
\mathit{Lagr}(A_\Sigma, \omega_\Sigma) = \{\tilde{A} \subseteq A' \mid \tilde{A} \subseteq A' \text{ lagr., } A' \subseteq A_\Sigma \text{ symplectic}\}
$$

Classical and quantum gluing rules

$$
\begin{tikzcd}
(M_1, \mathcal{P}_{M_1}, S_{M_1}) \arrow[d] \arrow[drr, dotted, |->, "\partial"] & & \\
A_{\tilde{M}_1} \subseteq A_{M_1, \partial M_1} \arrow[d] \arrow[drr, dotted, |->] & (M, \mathcal{P}_M, S_M) \arrow[ul, dashed, |->] \arrow[r, "\partial"] \arrow[d] & (\{\Sigma\}, \subseteq) \arrow[d] \\
\rho_{M_1} \arrow[drr, dotted, |->] & A_{\tilde{M}} \subseteq A_{M, \partial M} \arrow[ul, dashed, |->] \arrow[r, |->] \arrow[d] & \{Lagr(A_\Sigma, \omega_\Sigma)\}_\Sigma \arrow[d] \\
& \rho_M \arrow[ul, dashed, |->] \arrow[r, |->] & \{Maps(\mathcal{H}^\circ_\Sigma)\}_\Sigma
\end{tikzcd}
$$

the composition of the dotted maps $\begin{tikzcd} (\cdot)_M \arrow[r, dashed, |->] & (\cdot)_{M_1} \arrow[r, dotted, |->] & (\cdot)_{\Sigma_1} \arrow[r, hook] & (\cdot)_\Sigma \end{tikzcd}$
preserves the partial order induced by the inclusion of hypersurfaces $\Sigma \subseteq \Sigma_1$:

$$Lagr(A_{\Sigma_1}, \omega_{\Sigma_1}) \hookrightarrow Lagr(A_\Sigma, \omega_\Sigma)$$

$$Maps(\mathcal{H}_{\Sigma_1}, \mathbb{C}) \hookrightarrow Maps(\mathcal{H}_\Sigma, \mathbb{C})$$

Classical and quantum spacetime reconstruction

$$
\begin{tikzcd}
 & & & (\{Lagr(A_\Sigma, \omega_\Sigma)\}_\Sigma, \subseteq) \\
{[M, \Sigma \subseteq \partial M]} \arrow[r] & {[M_1, \Sigma_1 \subseteq \partial M_1]} \arrow[r, "Finer"'] & \dots \arrow[r, "Coarser"'] & {[M_k, \Sigma_k \subseteq \partial M_k]} \arrow[u, "Classical"] \arrow[d, "Quantum"'] \\
 & & & (\{Maps(\mathcal{H}^\circ_\Sigma, \mathbb{C})\}_\Sigma, \subseteq)
\end{tikzcd}
$$

Classical gluing

For $\partial M = \Sigma_1 \sqcup \Sigma \sqcup \overline{\Sigma'}$, with $M_1 = M/\sim_\Sigma$

$$\begin{tikzcd}
A_{\tilde{M}_1} \arrow[rrr, "a_{M;\Sigma\overline{\Sigma'}}"] \arrow[dr] \arrow[dd, dashed] & & & A_{\tilde{M}} \arrow[dl] \arrow[dd, dashed] \\
& A_{M_1,\partial M_1} \arrow[dl, hook] & A_{M,\partial M} \arrow[l, two heads, "\alpha_1"'] \arrow[dr, hook] & \\
A_{\partial M_1} & & & A_{\partial M} \arrow[lll, hook, "i^*_{\partial M_1}"]
\end{tikzcd} .$$

Quantum gluing

$$
\begin{tikzcd}
\mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma} \otimes \mathcal{H}_{\overline{\Sigma}} \arrow[r, equal] \arrow[d] & \mathcal{H}_{\partial M} \arrow[r, "\rho_M"] \arrow[d] & \mathbb{C} \\
\mathcal{H}_{\Sigma_1} \arrow[r, equal] \arrow[u, hook, dashed, bend left] & \mathcal{H}_{\partial M_1} \arrow[ur, dashed, "\rho_{M_1}"'] &
\end{tikzcd}
$$

For $\{\xi_i\}_{i \in I}$ an orthonormal basis of $\mathcal{H}_\Sigma$

$$\rho_{M_1}(\psi) \cdot c(M; \Sigma, \overline{\Sigma'}) = \sum_{i \in I} \rho_M \left(\psi \otimes \xi_i \otimes \iota_\Sigma(\xi_i)\right)$$

In particular for $\Sigma_1 = \emptyset$, we have that $\partial M_1 = \emptyset$, therefore $\rho_{M_1} \in \mathbb{C}$.

Classical abelian YM fileds in regions

- $\mathcal{P}_M \to M$ a $U(1)$ principal bundle
- Action on connections $\eta \in \mathrm{Conn}(\mathcal{E}_M)$, with curvature F^η,

$$S_M(\eta) = \int_M F^\eta \wedge \star F^\eta$$

- For fixed $\eta_{\mathcal{P}} \in \mathrm{Conn}(\mathcal{P}_M)$ $\eta = \eta_{\mathcal{E}} + \varphi$, has Euler-Laagrange equations

$$d \star F^\eta = d \star d\varphi = 0$$

- Gauge symmetries

$$\tilde{\eta} = \eta + df, \qquad S_M(\tilde{\eta}) = S_M(\eta)$$

- *(Affine) Space of solutions modulo gauge* (Lorentz gauge fixing)

$$A_M = \{\eta = \eta_{\mathcal{E}} + d\varphi \in \mathrm{Conn}(\mathcal{P}_M) \mid d^\star F^\eta = 0, \, d \star \varphi = 0\}$$

modeled over a *linear* space

$$L_M \subseteq \{d^\star d\varphi = 0, \, d \star \varphi = 0\} \subseteq \Omega^1(M)$$

Boundary conditions of YM fields

- Dirichlet and Neumann conditions map

$$r_M : \varphi \in L_M \mapsto \left([\varphi^D], \varphi^N\right) \in \Omega^1(\partial M)^{\oplus 2}$$

- Gauge action $\varphi^D \mapsto \varphi^D + df$ with axial gauge fixing

$$d \star_{\partial M} \varphi^D = 0, \quad d \star_{\partial M} \varphi^N = 0$$

- *(Linear) space of boundary conditions of solutions modulo gauge*:

$$L_{\tilde{M}} = r_M(L_M)$$

$$A_{\tilde{M}} = r_M(\eta_{\mathcal{P}_M}) + L_{\tilde{M}}$$

- There is an affine fibration

$$a_M : A_M \to A_{\tilde{M}}$$

Boundary conditions on hypersurfaces

$$A_\Sigma = a_{\Sigma_\varepsilon}(\eta\varepsilon) + L_\Sigma$$
$$L_\Sigma \subseteq (\Omega^1(\Sigma)/d\Omega^0) \oplus \Omega^1(\Sigma)$$

Are *boundary conditions modulo gauge* in the bottom $\Sigma \times \{0\}$ of a (metric) cylinder

$$\Sigma_\varepsilon = \Sigma \times [0, \varepsilon]$$

Axial gauge fixing:

$$L_\Sigma \subseteq (\ker d^{\star \partial M}/\{exact\}) \oplus \ker d^{\star \partial M}$$

The rôle of relative topology

- If $H^1_{\mathrm{dR}}(M;\partial M)=0$, then:
  1. There is an isomorphism.
  $$a_M : A_M \leftrightarrow A_{\tilde{M}}$$
  2. Every pair $([\phi^D],\phi^N)\in L_{\partial M}$ can be realized as a boundary condition for a field $\varphi\in\Omega^1(M)$ inside the region (not necessarily a solution), so that
  $$r_M(\varphi)=([\phi^D],\phi^N)$$
- If $H^1_{\mathrm{dR}}(M;\partial M)\neq 0$, then:
  1. There is a projection but not one-to-one
  $$a_M : A_M \to A_{\tilde{M}}$$
  2. There exists proper subspaces of *topologically admissible boundary conditions modulo gauge*,
  $$L_{\tilde{M}} \subseteq L_{M,\partial M} \subsetneq L_{\partial M}, \qquad A_{\tilde{M}} \subseteq A_{M,\partial M} \subsetneq A_{\partial M}$$
  3. The harmonic projections of Dirichlet conditions $\{\varphi^D\}$ generate a *finite dimensional* isomorphic to $H^1_{\mathrm{dR}}(M;\partial M)$ contained in $H^1_{\mathrm{dR}}(\partial M)$.

Semiclassical axiom (Complex structure)

- Recall

$$H^1_{\mathrm{c}}(M\backslash\partial M) \simeq H^1_{\mathrm{dR}}(M;\partial M) \to H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(\partial M) \to \dots$$

- According to Belishev, Sharafutdinov, Shonkwiler, et. all.

$$H^1_{dR}(M,\partial M) \simeq (\ker \mathcal{N}_{YM})^{\perp} \subseteq \mathfrak{H}^1(\partial M) \simeq H^1_{\mathrm{dR}}(\partial M)$$

- Here $\mathcal{N}_{YM}([\varphi^D]) = \varphi^N$ The Dirichlet-Neumann operator, for the BVP

$$\begin{cases} \Delta\varphi = 0, & d^{\star}\varphi = 0, \quad \varphi \in \Omega^1(M) \\ i^*_{\partial M}\varphi = \phi, & \qquad\qquad\quad \phi \in \Omega^1(\partial M) \end{cases}$$

- It yields a complex structure J, $J^2 = -Id$

$$J = \begin{pmatrix} 0 & -\mathcal{N}_{YM}^{-1} \\ \mathcal{N}_{YM} & 0 \end{pmatrix} : (\ker \mathcal{N}_{YM}/\{df\}) \oplus \operatorname{ran} \mathcal{N}_{YM} \circlearrowleft$$

- $\mathsf{L}_{M,\partial M} = \mathsf{L}_{\tilde{M}} \oplus J\mathsf{L}_{\tilde{M}}$
- Symplectic structure:

$$g_{\partial M}(\phi_1,\phi_2) = \int_{\partial M} \phi_1^D \wedge \star_{\partial M}\phi_2^D + \phi_1^N \wedge \star_{\partial M}\phi_2^N, \qquad \omega_{\partial M}(\cdot, J\cdot) = \frac{1}{2} g_{\partial M}(\cdot,\cdot)$$

Quantum abelian YM fields

- Linear Hilbert spaces:

$$\mathcal{H}^{\mathrm{L}}_{\Sigma} \subseteq \left\{ \chi : \hat{L}_{\Sigma} \to \mathbb{C} \; : \; \int_{\hat{L}_{\Sigma}} |\chi|^2 \nu_{\Sigma} \right\}$$

  holomorphic functions on a linear space

$$\mathit{supp}(\nu_{\Sigma}) \subseteq \hat{L}_{\Sigma} \subseteq \mathrm{hom}(L_{\Sigma}, \mathbb{C})$$

  ν_{Σ} gaussian with covariance $\frac{1}{2} g_{\Sigma}(\cdot, \cdot)$

- Holomorphic functions on the affine space $\hat{A}_{\Sigma} = a_M(\eta_{\varepsilon}) + \hat{L}_{\Sigma}$

$$\mathcal{H}_{\Sigma} \xrightarrow{\sim} \mathcal{H}^{\mathrm{L}}_{\Sigma}$$

$$\psi(\varphi) = \chi^{\eta\varepsilon}(\varphi) \cdot \alpha^{\eta\varepsilon}_{\Sigma}(\varphi) \longmapsto \chi^{\eta\varepsilon}(\varphi)$$

Amplitude map

- Amplitude map on space of linear functions: $\rho_M^{\mathrm{L}} : \mathcal{H}_{\partial M}^{\mathrm{L}\circ} \to \mathbb{C}$

$$\rho_M^{\mathrm{L}}(\chi) = \int_{\hat{L}_{\tilde{M}}} \chi(\phi) d\nu_M(\phi)$$

ν_M, covariance $\frac{1}{4}g_\Sigma(\cdot,\cdot)-$gaussian measure

$$supp(\nu_M) \subseteq \hat{L}_{\tilde{M}} \subseteq \hat{L}_{\partial M}$$

- $\chi : \hat{L}_{\partial M} \to \mathbb{C}$ is $\nu_{\partial M}$ measurable, not necessarily $\nu_{\tilde{M}}-$measurable:

$$\mathcal{H}_{\partial M}^{\mathrm{L}\circ} \subsetneq \mathcal{H}_{\partial M}^{\mathrm{L}}$$

- Amplitude map for *affine holomorphic wave functions*: $\rho_M : \mathcal{H}_{\partial M}^{\mathrm{L}\circ} \to \mathbb{C}$

$$\rho_M(\psi) = \exp\left(\mathrm{i}S_M(\eta_\varepsilon)\right) \int_{\hat{L}_{\tilde{M}}} \chi^{\eta_\varepsilon}(\phi) d\nu_M(\phi)$$

Example of amplitude map

- Let M_1 be a closed, $\partial M_1 = \emptyset$, Riemann surface of genus $\mathrm{g} \geq 2$ then

$$A_{\tilde{M}_1} = A_{\partial M_1} = A_{\partial M_1, M_1} = \{\eta\varepsilon\}$$

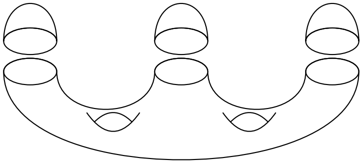
hence

$$\rho_{M_1}(\psi) = \exp(\mathrm{i} S_{M_1}(\eta_{\mathcal{E}_1})) = \exp\left(\mathrm{i}2\pi^2 \mathrm{area}(M_1) \cdot (c(\mathcal{E}_1))^2\right)$$

- Adding over all characteristic classes $c(\mathcal{E}_1) = \left[\frac{1}{2\pi} F^{\eta_{\mathcal{E}_1}}\right] \in H^2_{\mathrm{dR}}(M_1, \mathbb{Z})$ with Euclidean action (Wick rotation) $-S_M$ instead of $\mathrm{i}S_M$

$$\frac{1}{Z} \sum_{c(\mathcal{E}_1) \in \mathbb{Z}} \exp\left(-2\pi^2 \mathrm{area}(M_1) \cdot (c(\mathcal{E}_1))^2\right)$$

Example of gluing



- M' surface of genus $\mathrm{g} \geq 2$ with $m \geq 1$ boundary components

$$M = M' \sqcup M'', \qquad M'' = B_1 \sqcup \dots B_m$$

- Gluing along: $\Sigma = \partial M'$ and $\overline{\Sigma'} = \partial B_1 \sqcup \dots \partial B_m$:

$$M_1 = M' \cup M'' / \sim \Sigma$$

Example of gluing (continued)

$$\begin{tikzcd} A_{M''} = A_{\tilde{M}''} \arrow[r, hook, "a_M"] \arrow[d, equal] & A_{\partial M''} = A_{M'',\partial M''} \arrow[d, equal] \\ \mathbb{R}^m & \{(\eta_i^D, \eta_i^N)\} = \mathbb{R}^{2m} \end{tikzcd}$$

Geometric condition on each pair (η_i^D, η_i^N) for $A_{\tilde{M}''}$

$$\begin{tikzcd} A_{M'} \arrow[r, two heads, "a_{M'}"] \arrow[d, leftrightarrow] & A_{\tilde{M}'} \arrow[r, hook] \arrow[d, leftrightarrow] & A_{M'',\partial M''} \arrow[r, hook] \arrow[d, leftrightarrow] & A_{\partial M''} \arrow[d, leftrightarrow] \\ \mathbb{R}^{2g+m-1} & \mathbb{R}^{m-1} & \mathbb{R}^{2m-2} & \mathbb{R}^{2m} \end{tikzcd}$$

$d\varphi = 0$ condition on $A_{M'',\partial M''}$

- Gluing anomaly factor: $c(M; \Sigma, \overline{\Sigma'}) = 1$
- For $g = 1$, $c(M; \Sigma, \overline{\Sigma'})$ diverges as well as Z

Further exercises

- What is the precise relationship between the divergence:

$$c(M; \Sigma, \overline{\Sigma'}) \sim Z$$

- Reproduce other known calculations, E.Verlinde (1995):
  1. $M_1 = \mathbb{CP}^1 \times \mathbb{CP}^1$ ruled surface

$$\frac{1}{Z} \sum_{n_v^{\mathcal{E}}, n_h^{\mathcal{E}} \in \mathbb{Z}} \exp\left[-2\pi^2 \left((n_h^{\mathcal{E}})^2 \frac{r_v^2}{r_h^2} + (n_v^{\mathcal{E}})^2 \frac{r_h^2}{r_v^2}\right)\right],$$

  2. $M_1 = \mathbb{CP}^2$

$$\frac{1}{Z} \sum_{n^{\mathcal{E}} \in \mathbb{Z}} \exp\left(-2\pi^2 (n^{\mathcal{E}})^2\right).$$