# Klein-Gordon theory and the generalized S-matrix

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We consider a **real scalar field** theory in **Minkowski spacetime** with the action

$$S_M(\phi) = \frac{1}{2} \int \mathrm{d}^4x \, \left( (\partial_\mu \phi)\partial^\mu \phi - m^2\phi^2 \right).$$

The equations of motion are given by the **Klein-Gordon equation**:

$$(\square + m^2)\phi = 0.$$

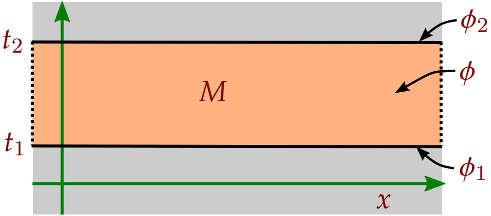
We take the **spaces of solutions** associated to **hypersurfaces** and **regions** to be vector spaces of real valued functions:

- $L_\Sigma := \{\phi : \hat{\Sigma} \to \mathbb{R}\}$ ($\hat{\Sigma}$ is a thickening of $\Sigma$)
- $L_M := \{\phi : M \to \mathbb{R}\}$

# Klein-Gordon Theory

## Standard geometry – spacelike hypersurplanes (I)

Consider **constant-time hypersurfaces** and **time-interval regions** as in the **standard formulation**.




Consider an **constant-time hypersurface** at time $t$. Expanding in **Fourier modes**, elements of $L_t$ are conveniently parametrized in terms of functions on **momentum space**,

$$\phi(t,x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left( \phi(k) e^{-\mathrm{i}(Et-kx)} + \overline{\phi(k)} e^{\mathrm{i}(Et-kx)} \right).$$

The Lagrangian gives rise to the **symplectic form**,

$$\omega_t(\phi_1,\phi_2) = \frac{1}{2}\int \mathrm{d}^3x\ (\phi_2(t,x)\partial_0\phi_1(t,x) - \phi_1(t,x)\partial_0\phi_2(t,x))$$
$$= \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left(\phi_2(k)\overline{\phi_1(k)} - \phi_1(k)\overline{\phi_2(k)}\right).$$

The standard **complex structure** is,

$$(J(\phi))(k) = -\mathrm{i}\phi(k).$$

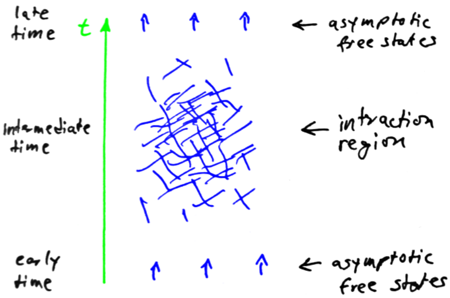
This yields the **complex inner product**,

$$\{\phi_1,\phi_2\}_t = 2\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\,\phi_1(k)\overline{\phi_2(k)}.$$

$L_t$ becomes the **one-particle Hilbert space**.
$\mathcal{H}_t$ is the space of **wave functions** or **Fock space** over $L_t$.

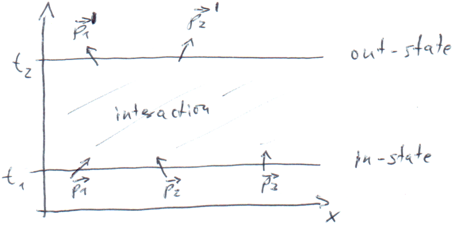
# S-matrix

Usually, interacting QFT is described via the S-matrix:

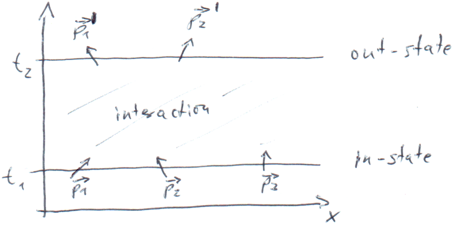



Assume interaction is relevant only after the initial time $t_1$ and before the final time $t_2$. The S-matrix is the asymptotic limit of the amplitude between free states at early and at late time:

$$\boxed{\langle\psi_2|S|\psi_1\rangle = \lim_{\substack{t_1\to-\infty \\ t_2\to+\infty}} \langle\psi_2|U_{\mathrm{int}}[t_1,t_2]|\psi_1\rangle}$$




Particles, i.e., elements of $L_t$, can be characterized by 3 **quantum numbers**: the components $p_i$ of the **3-momentum**. Moreover, each particle is part of either the **in-state** or the **out-state**.

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Denote $\psi(p_1, \ldots, p_n)$ the $n$-particle state with momenta $p_1, \ldots, p_n$ in $\mathcal{H}$.

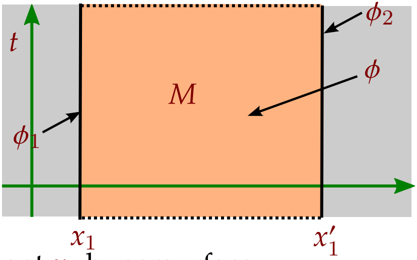
The **probability** to find outgoing particles with momenta $p_1', \ldots, p_m'$ given incoming particles $p_1, \ldots, p_n$ is,

$$|\langle \psi(p_1', \ldots, p_m'), U\psi(p_1, \ldots, p_n)\rangle|^2$$

# Klein-Gordon Theory

## Timelike Hyperplanes (I)

Consider hypersurfaces with constant $x_1$ coordinate and corresponding **space-interval regions**.




Parametrize solution near constant $x_1$ hypersurface,

$$\phi(t, x_1, \tilde{x}) = \int_{E^2 > \tilde{k}^2 + m^2} \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1} \left( \phi(E, \tilde{k}) e^{-\mathrm{i}(Et - \tilde{k}\tilde{x} - k_1 x_1)} + \overline{\phi(E, \tilde{k})}\, e^{\mathrm{i}(Et - \tilde{k}\tilde{x} - k_1 x_1)} \right)$$

where $\tilde{x} := (x_2, x_3)$, $\tilde{k} := (k_2, k_3)$, $k_1 := \sqrt{|E^2 - \tilde{k}^2 - m^2|}$.
Note that the sign of $E$ can be negative.

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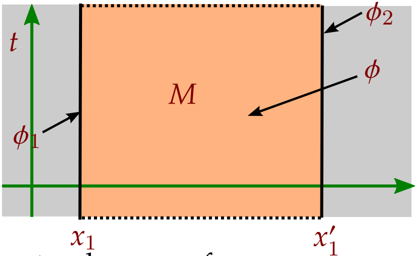


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These are the **propagating waves**: $E^2 > \tilde{k}^2 + m^2$, oscillate in space

# Klein-Gordon Theory

## Timelike Hyperplanes (II)

There are also **evanescent waves**: $E^2 < \tilde{k}^2 + m^2$, exponential in space

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with $\phi_\pm(E, \tilde{k}) = \overline{\phi_\pm(-E, -\tilde{k})}$.

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The space of solutions decomposes as $L_{x_1} = L^{\mathrm{p}}_{x_1} \oplus L^{\mathrm{e}}_{x_1}$.
The space of states is a tensor product $\mathcal{H}_{x_1} = \mathcal{H}^{\mathrm{p}}_{x_1} \otimes \mathcal{H}^{\mathrm{e}}_{x_1}$.

The construction of $\mathcal{H}_{x_1}^{\mathrm{p}}$ based on $L_{x_1}^{\mathrm{p}}$ parallels the spacelike case.

The Lagrangian gives rise to the **symplectic form**,

$$\omega_{x_1}(\phi_1,\phi_2) = -\frac{1}{2}\int \mathrm{d}^3x\, \left(\phi_2(t,x)\partial_{x_1}\phi_1(t,x) - \phi_1(t,x)\partial_{x_1}\phi_2(t,x)\right)$$
$$= \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1}\left(\phi_2(E,\tilde{k})\overline{\phi_1(E,\tilde{k})} - \phi_1(E,\tilde{k})\overline{\phi_2(E,\tilde{k})}\right).$$

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# Klein-Gordon Theory

## Timelike Hyperplanes (IV)

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A **vacuum** can be defined in $\mathcal{H}^{\mathrm{e}}_{x_1}$ [D. Colosi, RO 2007], but the existence and properties of particle states remain unclear.

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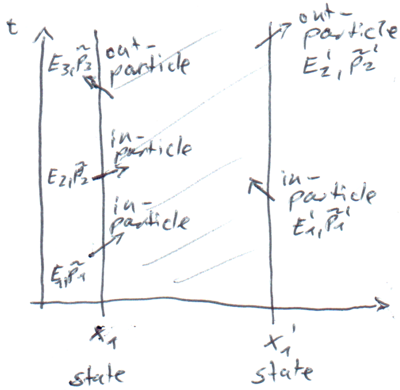
…but recently, progress is being made [D. Colosi, RO], stay tuned!

# Klein-Gordon Theory

## Timelike Hypersurfaces – scattering




Particles can be characterized by 3 **quantum numbers**: the momenta $k_2, k_3$ and the energy $E$. Recall that $E$ may be negative. This yields the **same degrees of freedom** as in the spacelike case.

But, in contrast to the spacelike case there is no notion of **in-state** or **out-state**. Rather each particle in a multi-particle state might individually be either **in-going** or **out-going**. This is what the **sign of the energy** $E$ encodes.

The boundary solution space decomposes as $L_{\partial M} = L_{x_1} \oplus L_{x'_1}$.
The boundary Hilbert space decomposes as $\mathcal{H}_{\partial M} = \mathcal{H}_{x_1} \otimes \mathcal{H}_{x'_1}$.

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For each hypersurface there is also a decomposition into **in-going** and **out-going** particle modes. For the boundary as a whole, $L_{\partial M} = L_{\text{in}} \oplus L_{\text{out}}$ and $\mathcal{H}_{\partial M} = \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$.

# Klein-Gordon Theory

## Timelike Hypersurfaces – probabilities

The boundary solution space decomposes as $L_{\partial M} = L_{x_1} \oplus L_{x_1'}$.
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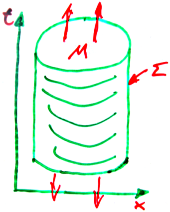
Denote the density matrix in $\mathcal{B}_{\partial M}$ for $n$ in-going particles with quantum numbers $(E_1, \tilde{k}_1) \ldots, (E_n, \tilde{k}_n)$ and $m$ out-going particles $(E_1', \tilde{k}_1') \ldots, (E_m', \tilde{k}_m')$ by $\sigma((E_1, \tilde{k}_1) \ldots; (E_1', \tilde{k}_1') \ldots)$, with same in-going but indeterminate out-going particles by, $\sigma((E_1, \tilde{k}_1) \ldots; *)$.

The probability for observing the out-going particles $(E_1', \tilde{k}_1') \ldots$ given the in-going particles are $(E_1, \tilde{k}_1) \ldots$ is,

$$\frac{A_M(\sigma((E_1, \tilde{k}_1) \ldots, (E_n, \tilde{k}_n); (E_1', \tilde{k}_1') \ldots, (E_m', \tilde{k}_m')))}{A_M(\sigma((E_1, \tilde{k}_1) \ldots, (E_n, \tilde{k}_n); *))}.$$

Consider a **hypercylinder** given by a **sphere** of radius $R$ in space, extended over all of time.




Parametrize **propagating** solutions ($E^2 > m^2$) near constant $R$ hypersurface, ($l = 0, 1, \dots,$ $m = -l, -l+1, \dots, l$)

$$\phi(t, r, \Omega) = \int_{|E|>m} \mathrm{d}E \, \frac{p}{4\pi} \sum_{l,m} \Big( \phi_{l,m}(E) h_l(pr) e^{-\mathrm{i}Et} Y_l^m(\Omega) + \overline{\phi_{l,m}(E)} \, \overline{h_l(pr)} e^{\mathrm{i}Et} Y_l^{-m}(\Omega) \Big) .$$

Here $Y_l^m$ denote the **spherical harmonics** and $p := \sqrt{|E^2 - m^2|}$. Also, $h_l = j_l + \mathrm{i} n_l$, where $j_l$ and $n_l$ are the **spherical Bessel functions** of the **first** and **second kind** respectively.

In the massive case $m > 0$, there are also **evanescent** solutions for $E^2 < m^2$, with exponential behaviour in space.

$$\phi(t,r,\Omega) = \int_{-m}^{m} \mathrm{d}E\, \frac{p}{4\pi} e^{-\mathrm{i}Et} \sum_{l,m} Y_l^m(\Omega) \left(\phi^{\mathrm{x}}_{l,m}(E) k_l(pr) + \phi^{\mathrm{i}}_{l,m}(E) \tilde{k}_l(pr)\right). \tag{1}$$

Here $Y_l^m$ denote the **spherical harmonics** and $p := \sqrt{|E^2 - m^2|}$. Also, $k_l(z) = -\mathrm{i}^l \pi h_l(\mathrm{i}z)/2$ and $\tilde{k}_l(z) = k_l(-z)$ are **modified spherical Bessel functions** that are real on $\mathbb{R}$.

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The space of solutions decomposes as $L_R = L_R^{\mathrm{p}} \oplus L_R^{\mathrm{e}}$.
The space of states is a tensor product $\mathcal{H}_R = \mathcal{H}_R^{\mathrm{p}} \otimes \mathcal{H}_R^{\mathrm{e}}$.

# Klein-Gordon Theory

## Timelike Hypercylinder (III)

We restrict considerations to **propagating solutions**.

The Lagrangian gives rise to the **symplectic form**,

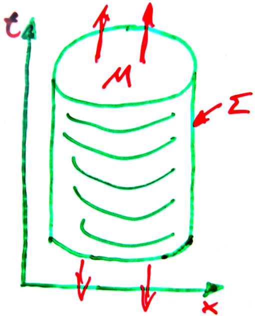
$$\omega_R(\phi,\xi) = \frac{R^2}{2}\int \mathrm{d}t\,\mathrm{d}\Omega\ \left(\xi(t,R,\Omega)\partial_r\phi(t,R,\Omega) - \phi(t,R,\Omega)\partial_r\xi(t,R,\Omega)\right)$$
$$= \int \mathrm{d}E\,\frac{\mathrm{i}p}{8\pi}\sum_{l,m}\left(\phi_{l,m}(E)\overline{\xi_{l,m}(E)} - \overline{\phi_{l,m}(E)}\xi_{l,m}(E)\right).$$

The standard **complex structure** is,

$$(J(\phi))_{l,m}(E) = \mathrm{i}\phi_{l,m}(E).$$

This yields the **complex inner product**,

$$\{\phi,\xi\}_R = \int \mathrm{d}E\,\frac{p}{2\pi}\sum_{l,m}\overline{\phi_{l,m}(E)}\xi_{l,m}(E).$$



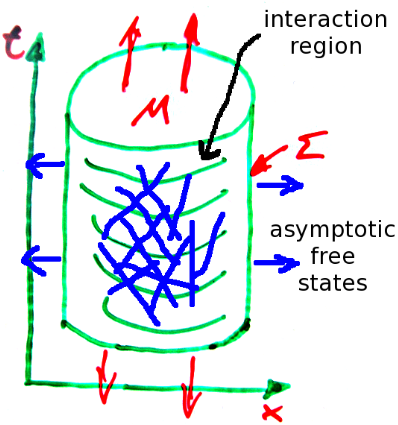

To go beyond standard transition amplitudes, consider an example with a connected boundary. [RO 2005]

- $M = \mathbb{R} \times B^3_R$.
- $\partial M = \Sigma_R = \mathbb{R} \times S^2_R$.

(Consider propagating waves only.)

- The state space $\mathcal{H}_{\Sigma_R}$ is again a **Fock space**.
- A particle can be characterized by three quantum numbers: **energy** $E$ and **angular momentum** $l, m$.
- The sign of the energy determines if a particle is **in-going** or **out-going**. The state space decomposes as $\mathcal{H}_{\Sigma_R} = \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$.
- This decomposition is neither geometrical nor temporal.

# Spatially asymptotic S-matrix (I)




Similarly, we can describe interacting QFT via a spatially asymptotic amplitude. Assume interaction is relevant only within a radius $R$ from the origin in space (but at all times). Consider then the asymptotic limit of the amplitude of a free state on the hypercylinder when the radius goes to infinity:

$$\boxed{S(\psi) = \lim_{R\to\infty} \rho_R(\psi)}$$

[D. Colosi, RO 2007–2008]

# Spatially asymptotic S-matrix (II)

Results:

- The **perturbative description of interactions** works as in the standard path integral and S-matrix picture. Technically, the interactions are introduced via **sources**. In the hypercylinder geometry, this involves **evanescent modes** in an essential way, even if they vanish asymptotically.
- The S-matrices are equivalent when the interaction is confined in space and time. This equivalence is realized through an **isomorphism of** the asymptotic state spaces.
- In the standard formulation, **crossing symmetry** is an emergent feature of the S-matrix. In the hypercylinder setting of the GBF crossing symmetry is manifest.