2-dimensional quantum Yang-Mills theory with corners

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Motivation

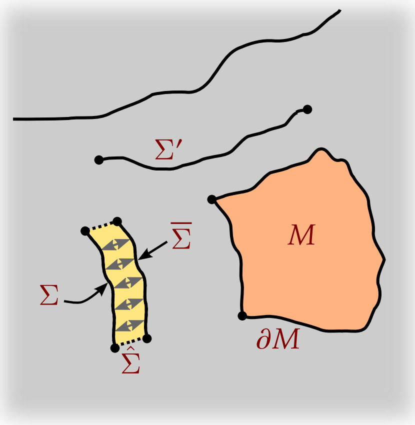
Quantum Yang-Mills theory in **2 dimensions** is an excellent toy example to study in the GBF because

- it is **non-linear**, different from linear examples,
- its is **solvable**, making quantization tractable,
- exhibits **non-abelian gauge symmetry**, such as QCD,
- allows for quantization with **corners**.

Overview

TQFT – manifolds

Fix dimension d. Manifolds are **oriented** and may carry additional structure: differentiable, metric, complex, etc.



region M

d-manifold with boundary.

hypersurface Σ

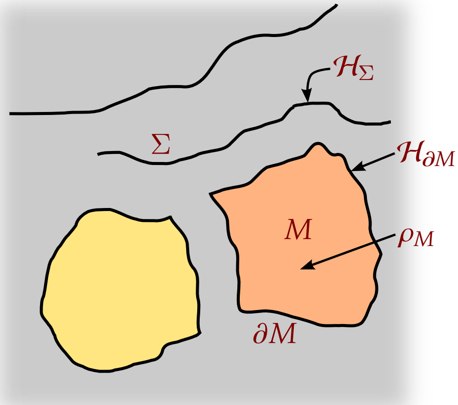
$d-1$-manifold with boundary, with germ of d-manifold.

slice region $\hat{\Sigma}$

$d-1$-manifold with boundary, with germ of d-manifold, interpreted as "infinitely thin" region.

TQFT – axioms I

Assignment of algebraic structures to geometric ones.



(T1) per hypersurface Σ

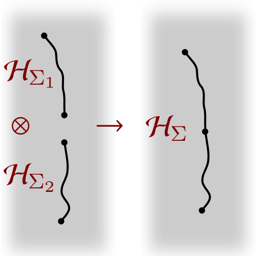
A complex vector space $\mathcal{H}_\Sigma$.
(state space)

$\mathcal{H}_\emptyset = \mathbb{C}$.

(T4) per region M

A linear map $\mathcal{H}_{\partial M} \to \mathbb{C}$.
(amplitude map)

TQFT – axioms II

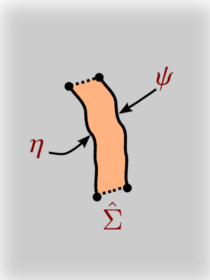


(T1b) per hypersurface Σ

A conjugate linear involution $\iota_\Sigma : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\overline{\Sigma}}$.

(T2) per hypersurface decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$

A partial isometry $\tau : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \rightarrow \mathcal{H}_\Sigma$.



(T3x) per hypersurface Σ

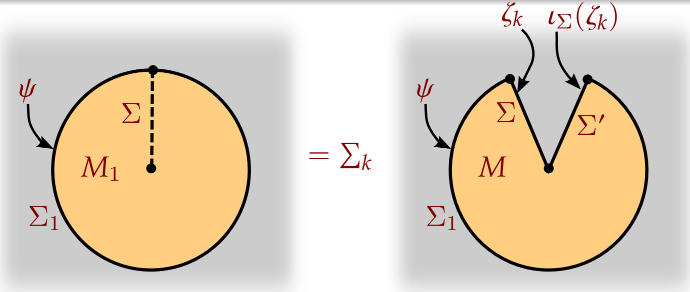
The amplitude map gives rise to a **positive-definite inner product** $\langle \iota_{\overline{\Sigma}}(\psi), \eta \rangle_\Sigma := \rho_{\hat{\Sigma}} \circ \tau(\psi \otimes \eta)$.

(T5a) per disjoint composition of regions $M = M_1 \sqcup M_2$

$\rho_M(\tau(\psi_1 \otimes \psi_2)) = \rho_{M_1}(\psi_1)\rho_{M_2}(\psi_2)$. We write $\rho_M = \rho_{M_1} \diamond \rho_{M_2}$.

(T5b) per self-composition of region M to M_1 along Σ

$\rho_{M_1}(\psi) \cdot c_{M,\Sigma} = \sum_k \rho_M(\tau(\psi \otimes \zeta_k \otimes \iota_\Sigma(\zeta_k)))$. We write $\rho_{M_1} = \diamond \rho_M$.



$\{\zeta_k\}_{k \in I}$ ON-basis of $\mathcal{H}_\Sigma$. $c_{M,\Sigma}$ **gluing anomaly**.

Why corners?

If we are serious about describing physics **locally** we need to allow **compact** spacetime regions that may be considered arbitrarily **small**.

What is more, we need to study the **interaction** between small **neighboring regions**. In the GBF this means we need to be able to **glue** small regions together.

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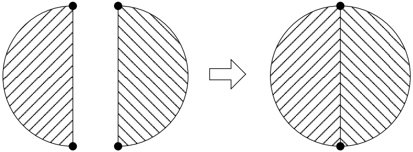
The generic **topology** of a small region is that of a **ball**.



We need to be able to glue two ball-shaped regions to a single ball-shaped region. This requires gluing along **parts** of boundaries.

This introduces **(virtual) corners** where boundaries are split. These are **boundaries of boundaries**.

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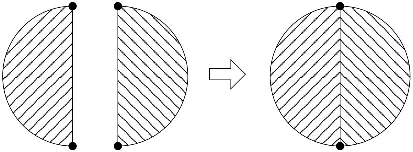

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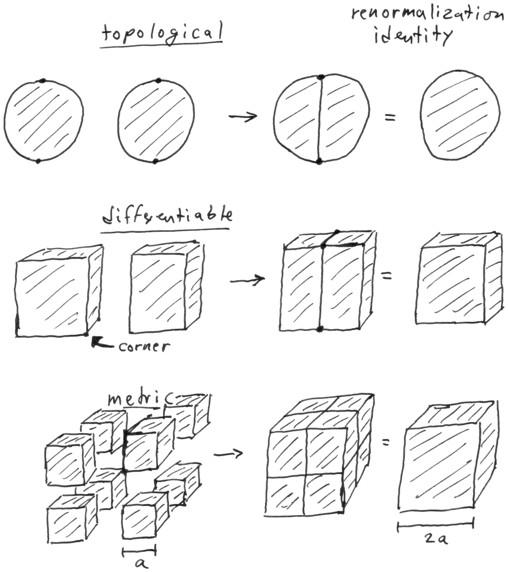
This is also crucial in **renormalization**.

Renormalization identities



topological

Relates regions of the same type. There is only one elementary region.

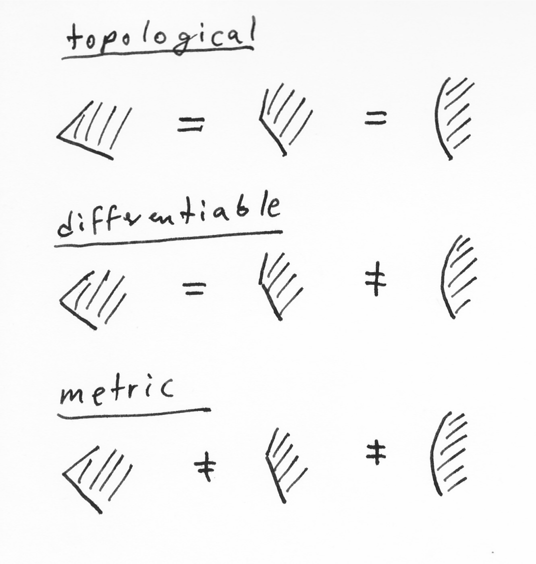
differentiable

Relates regions of the same type. Regions have **corners**.

metric

Relates regions of different sizes. Link to coupling constant renormalization.

Corners



topological

Corners are homeomorphic to smooth hypersurfaces.

differentiable

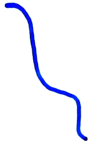
Corners of different angles are diffeomorphic, but distinct from smooth hypersurfaces.

metric

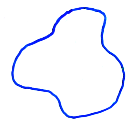
Corners of different angles are all distinct.

Two dimensions (topological): hypersurfaces

There are two types of elementary hypersurfaces (T1):



an open string with state space $\mathcal{H}_{\mathrm{O}}$



a closed string with state space $\mathcal{H}_{\mathrm{C}}$

We also assign $\iota_{\mathrm{O}} : \mathcal{H}_{\mathrm{O}} \to \mathcal{H}_{\bar{\mathrm{O}}}$ and $\iota_{\mathrm{C}} : \mathcal{H}_{\mathrm{C}} \to \mathcal{H}_{\bar{\mathrm{C}}}$ (T1b).

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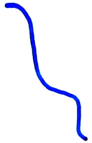

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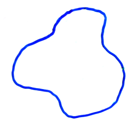

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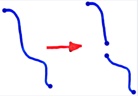
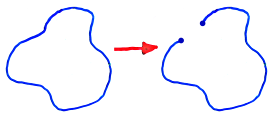
There are two types of elementary decompositions (T2):

open string to two open strings



$\tau_\mathrm{OO} : \mathcal{H}_\mathrm{O} \otimes \mathcal{H}_\mathrm{O} \to \mathcal{H}_\mathrm{O}$

closed string to an open string



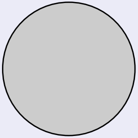
$\tau_\mathrm{OC} : \mathcal{H}_\mathrm{O} \to \mathcal{H}_\mathrm{C}$

Remarks: τ_OO must be associative, $\tau_\mathrm{OC} \circ \tau_\mathrm{OO}$ must be commutative.

Two dimensions (topological): regions

Any connected region is a **Riemann surface** with holes. It is characterized by two non-negative integers, the **genus** g and the number of **holes** n.

There is only one type of elementary region, the disc D with amplitude $\rho_D : \mathcal{H}_C \to \mathbb{C}$ (T4).



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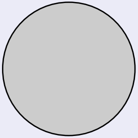
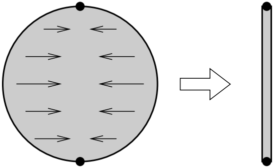


The slice region associated with an open string can be thought of as a squeezed disc:



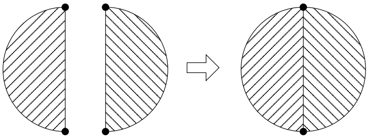
This gives rise to a bilinear pairing $\mathcal{H}_O \otimes \mathcal{H}_O \to \mathbb{C}$ defined by

$$(\cdot,\cdot)_O = \hat{\rho}_D := \rho_D \circ \tau_{OC} \circ \tau_{OO}$$

By axiom (T3x) this is related to the inner product on $\mathcal{H}_O$ via $\langle\cdot,\cdot\rangle_O = (\iota_O(\cdot),\cdot)_O$.

Two dimensions (topological): gluing

Gluing two discs to a new disc imposes consistency conditions on the disc amplitude via axiom (T5).



Let $\{\zeta\}_{i\in I}$ be an ON-basis of $\mathcal{H}_\mathrm{O}$. Then:

$$\hat{\rho}_D(\psi\otimes\eta) = \sum_i \hat{\rho}_D(\psi\otimes\zeta_i)\hat{\rho}_D(\iota_\mathrm{O}(\zeta_i)\otimes\eta)$$

This is **the renormalization identity** for topological manifolds.

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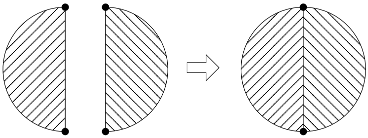

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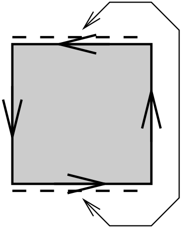
Any connected region can be obtained by gluing a disc with a suitably subdivided boundary to itself. Consider the cylinder:



$$\rho_{\mathrm{cyl}}\circ(\tau_{\mathrm{OC}}\otimes\tau_{\mathrm{OC}})(\psi\otimes\eta)$$
$$=\sum_i \rho_D\circ\tau_{\mathrm{OC}}\circ\tau\circ(\tau_{\mathrm{OO}}\otimes\tau_{\mathrm{OO}})(\psi\otimes\zeta_i\otimes\eta\otimes\iota_{\mathrm{O}}(\zeta_i))$$

Review of Schrödinger-Feynman quantization

Data of classical field theory

- A configuration space K_Σ per hypersurface Σ.
- A configuration space K_M per region M.
- An action $S_M : K_M \to \mathbb{R}$ per region M.

Review of Schrödinger-Feynman quantization

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Data of quantum field theory

- (T1) The state space $\mathcal{H}_\Sigma = \mathcal{C}(K_\Sigma)$ is the space of square integrable functions on K_Σ with $\langle\psi,\eta\rangle = \int_{K_\Sigma} \mathcal{D}\varphi\, \overline{\psi(\varphi)}\eta(\varphi)$.
- A field propagator $Z_M : K_{\partial M} \to \mathbb{C}$ per region M given by

$$Z_M(\varphi) = \int_{K_M,\phi|_{\partial M}=\varphi} \mathcal{D}\phi\, e^{\mathrm{i}S_M(\phi)}.$$

- (T4) The amplitude $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ per region M is

$$\rho_M(\psi) = \int_{K_\Sigma} \mathcal{D}\varphi\, \psi(\varphi) Z_M(\varphi).$$

Classical Yang-Mills theory

The **gauge group** G is a compact connected and simply connected Lie group. Consider trivial principal G-bundles over hypersurfaces and regions. Regions carry a metric.

- K_Σ is the space of **connection** 1-forms A on the hypersurface Σ.
- K_M is the space of **connection** 1-forms A on the region M.
- The **action** on a region M is:

$$S_M(A) = -\frac{1}{\gamma^2}\int_M \mathrm{tr}(F \wedge \star F)$$

  - F is the **curvature** 2-form of A
  - γ is the **coupling constant**

Gauge symmetry: Two connection 1-forms related by a **gauge transformation** (change of trivialization) are physically equivalent. A gauge transformation in M may be parametrized as a map $M \to G$.

Quantizing Yang-Mills theory

We restrict to **two dimensions**. Consider a **disc-shaped** region M with **boundary** $\Sigma \approx S^1$. Let $A_\Sigma \in K_\Sigma$.

$$Z_M(A_\Sigma) = \int_{K_M, A|_\Sigma = A_\Sigma} \mathcal{D}A\, e^{\mathrm{i} S_M(A)}.$$

- Due to **gauge invariance**, $Z_M(A_\Sigma)$ only depends on the **holonomy** $g \in G$ of A_Σ. Furthermore, it really only depends on the **conjugacy class** of g.
- In two dimensions $S_M(A)$ depends not on the full metric, but only on its **area form**. Moreover, by diffeomorphism invariance of $\mathcal{D}A$, $Z_M(A_\Sigma)$ depends only on the **total area** of M. What is more, the **differentiable structure** of M does not play any role.

Results – State spaces

- A **region** is a pair (M, s) of a compact topological 2-manifold M and a non-negative real number s (the area).
- A **hypersurface** is a compact topological 1-manifold.
- The physical configuration space K'_{C} for the **closed string** is the space of **conjugacy classes** of G.
- The physical configuration space K'_{O} for the **open string** is the space of elements of G. (Gauge transformations must act identically at endpoints.)

The normalized invariant measure on G is the **Haar measure** $\mathrm{d}g$.

- $\mathcal{H}_{\mathrm{O}} = C(G)$ complex functions on G with **inner product** $\langle\psi,\eta\rangle = \int \mathrm{d}g\,\overline{\psi(g)}\eta(g)$.
- $\mathcal{H}_{\mathrm{C}} = C_{\mathrm{class}}(G) \subseteq C(G)$ is the space of **class functions**, i.e., functions invariant under conjugation.

Basis

- An orthogonal basis for $C(G)$ is given by matrix elements of simple representations. Thus, for any **simple representation** V choose a basis $\{v_i\}$. Denote the dual basis of V^* by $\{v_i^*\}$. Then set

$$t_{ij}^V(g) := (v_i^*, g \triangleright v_j).$$

- An orthogonal basis for the subspace $C_{\text{class}}(G)$ is given by characters, defined by

$$\chi^V(g) := \sum_i t_{ii}^V(g).$$

- The inner product on $C(G)$ is

$$\langle t_{ij}^V, t_{mn}^W \rangle = \delta_{V,W}\delta_{i,m}\delta_{j,n}\frac{1}{\dim V}, \quad \langle \chi^V, \chi^W \rangle = \delta_{V,W}.$$

Decomposition maps (I)

Decomposition maps:

- **Open** string to **two open** strings: $\tau_{\mathrm{OO}} : \mathcal{H}_{\mathrm{O}} \otimes \mathcal{H}_{\mathrm{O}} \to \mathcal{H}_{\mathrm{O}}$

$$
\begin{aligned}
(\tau_{\mathrm{OO}}(\psi \otimes \eta))\,(g) &= \int \mathrm{d}g_1 \mathrm{d}g_2\, \psi(g_1)\eta(g_2)\delta(g_1 g_2, g) \\
&= \int \mathrm{d}h\, \psi(gh)\eta(h^{-1}) = \int \mathrm{d}h\, \psi(h)\eta(h^{-1}g).
\end{aligned}
$$

- **Closed** string to **open** string: $\tau_{\mathrm{OC}} : \mathcal{H}_{\mathrm{O}} \to \mathcal{H}_{\mathrm{C}}$

$$
(\tau_{\mathrm{OC}}(\psi))\,(g) = \int_{h\in[g]} \mathrm{d}h \psi(h) = \int \mathrm{d}h\, \psi(hgh^{-1}).
$$

Integrals can be thought of as projections onto **gauge invariant states**. As required, τ_{OO} is associative and $\tau_{\mathrm{OC}} \circ \tau_{\mathrm{OO}}$ is commutative.

Decomposition maps (II)

In terms of matrix elements we get:

$$\tau_{\mathrm{OO}}(t_{ij}^V \otimes t_{mn}^W) = \delta_{V,W}\delta_{j,m}\frac{1}{\dim V}t_{in}^V.$$

$$\tau_{\mathrm{OC}}(t_{ij}^V) = \delta_{i,j}\frac{1}{\dim V}\chi^V.$$

Disc amplitude

Recall that the **propagator** for the disc D is a **class function** of the **holonomy**. Hence it can be expanded in **characters**:

$$Z_{(D,s)}(g) = \sum_V \dim V \, \alpha_V(s) \, \chi^V(g).$$

The sum runs over all **simple representations** V of G and $\alpha_V : \mathbb{R}_+ \to \mathbb{C}$ are functions depending on the area s. The amplitude $\rho_{(D,s)} : \mathcal{H}_C \to \mathbb{C}$ is thus:

$$\rho_{(D,s)}(\psi) = \sum_V \dim V \, \alpha_V(s) \int \mathrm{d}g \, \chi^V(g) \psi(g).$$

In terms of matrix elements:

$$\rho_{(D,s)}(\chi^V) = \dim V \, \alpha_V(s).$$

Consistency conditions

1. Applying axiom (T3x) to the **squashed disc** (with area $s = 0$) fixes (up to a choice of sign) axiom (T1b) for the **open string**

$$(\iota_O(\psi))(g) = \overline{\psi(g^{-1})}, \qquad \iota_O(t_{ij}^V) = t_{ji}^V.$$

2. At the same time this fixes $\alpha_V(0) = 1$, $\forall V$.
3. Consistency then yields for the **closed string**

$$(\iota_C(\psi))(g) = \overline{\psi(g^{-1})}, \qquad \iota_C(\chi^V) = \chi^V.$$

4. Applying axioms (T5a), (T5b), gluing a disc of **area** s and a disc of **area** t to a disc of **area** $s + t$ yields $\alpha_V(s)\alpha_V(t) = \alpha_V(s + t)$, $\forall V$.
5. 2 + 4 + continuity then yield for unknown β_V:

$$\alpha_V(s) = \exp(s\,\beta_V).$$

6. Unitarity requires the β_V to be **imaginary**.

General regions

Gluing a disc to itself after suitably decomposing its boundary yields the amplitude for all **Riemann surfaces with holes**. Let g be the genus, $n > 0$ the number of holes and s the area. Then,

$$\rho_{\mathrm{g,n,s}}(\chi^{V_1} \otimes \cdots \otimes \chi^{V_n}) = \delta_{V_1,\ldots,V_n} \exp(s\,\beta_{V_1})\,(\dim V_1)^{2-2g-n}.$$

For the case of a **closed Riemann surface** we get

$$\rho_{\mathrm{g,0,s}} = \sum_V \exp(s\,\beta_V)\,(\dim V)^{2-2g}.$$

This sum might be ill defined since generally there are infinitely many inequivalent simple representations.
To have a well defined theory we might have to exclude certain gluings and closed manifolds. Typically (depending on the coefficients β_V) this sum is ill defined for $g = 0$ (sphere) $g = 1$ (torus).

Recovers results from [Witten, …1990] obtained without corners.

The constants β_V

A more detailed analysis of the action and the propagator shows that β_V should take the form

$$\beta_V = \frac{\mathrm{i}}{4}\gamma^2 C_V,$$

where C_V is the value of the **quadratic Casimir** operator on the representation V.
For example:

- $G = U(1)$: simple reps. $k \in \mathbb{Z}$, $\dim V_k = 1$, $C_k = k^2$
- $G = SU(2)$: simple reps. $j \in \frac{1}{2}\mathbb{N}_0$, $\dim V_j = 2j + 1$, $C_V = 2j(j+1)$

Discussion [2006]

- Quantum Yang-Mills theory provides a successful realization of the axioms in two dimensions, including **infinite dimensional state spaces** and **corners**.
- The Yang-Mills example provides insight into the role of **gauge symmetries** in relation to the axioms.
- The axioms provide strong **consistency conditions**, severely constraining possible theories.

Outlook

- Apply axioms to more complicated theories in two dimensions, e.g. to **conformal field theory**.
- Are the axioms suitable in higher dimensions? Or should they be extended or modified? Apply to higher dimensional theories to find out.

Main reference

R. O., *2-dimensional quantum Yang-Mills theory with corners*, J. Phys. A **41** (2008) 135401, `arXiv:hep-th/0608218`.