The field theory of free fermions – classical theory

Robert Oeckl

Centro de Ciencias Matemáticas
Universidad Nacional Autónoma de México
Morelia, Mexico

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Outline

Overview

- So far in this seminar all talks have been essentially limited to the treatment of purely **bosonic** theories. Today we shall consider **fermionic** theories. We restrict ourselves to the simplest case of **free field theory**.
- In contrast to the bosonic case we can not directly use the powerful **holomorphic quantization approach** since there is no comparable notion of **coherent state**. Instead we shall use a **Fock space approach**. Bosonic and fermionic theories can then be treated in a unified way. Moreover, in the bosonic case, both approaches are equivalent.
- As in the bosonic case the basic ingredients in the fermionic case ca be motivated from **geometric quantization**.
- As with holomorphic quantization this leads to a rigorous and functorial quantization scheme.

Surprising results

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The **gluing anomaly** can be **renormalized**.

As in the bosonic case a gluing anomaly exists. But here it can be renormalized so that no **integrability condition** needs to be imposed.

Semiclassical theory

Today, we shall limit ourselves to **semiclassical theory**. Next time we shall consider the **quantum theory**.

Mini-review: Bosonic field theory (I)

Formulate field theory in terms of first order **Lagrangian density** $\Lambda(\varphi, \partial\varphi, x)$. Recall the **symplectic form**,

$$(\omega_\Sigma)_\phi(X,Y) = -\frac{1}{2}\int_\Sigma \Bigg((X^b Y^a - Y^b X^a)\, \partial_\mu \lrcorner \frac{\delta^2 \Lambda}{\delta \varphi^b \delta\, \partial_\mu \varphi^a}(\phi) \\ + (Y^a \partial_\nu X^b - X^a \partial_\nu Y^b)\, \partial_\mu \lrcorner \frac{\delta^2 \Lambda}{\delta\, \partial_\nu \varphi^b \delta\, \partial_\mu \varphi^a}(\phi) \Bigg).$$

In the case of linear field theory this is a bilinear form on the space L_Σ of germs of solutions on the hypersurface Σ. We suppose that ω_Σ is **non-degenerate**.

The symplectic form arises from the integral of a $(d-1)$-form on a hypersurface. Its sign thus depends on **orientation**: $\boxed{\omega_{\overline{\Sigma}} = -\omega_\Sigma}$.

Mini-review: Bosonic field theory (II)

The key additional ingredient for the **geometric quantization** on a hypersurface is the **complex structure** $J_\Sigma : L_\Sigma \to L_\Sigma$. Recall that this has to satisfy $J_\Sigma^2 = -\mathbf{1}$ and $\omega_\Sigma(J_\Sigma \cdot, J_\Sigma \cdot) = \omega_\Sigma(\cdot, \cdot)$.

The complex structure encodes a kind of global orientation. Its sign thus depends on **orientation**: $\boxed{J_{\overline{\Sigma}} = -J_\Sigma}$.

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The complex structure encodes a kind of global orientation. Its sign thus depends on **orientation**: $\boxed{J_{\overline{\Sigma}} = -J_\Sigma}$.

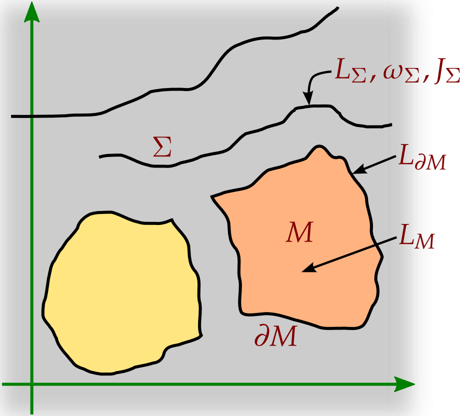
Let M be a region and L_M the space of solutions in M. Then we have a natural map $L_M \to L_{\partial M}$ by "forgetting" the solution in the interior of M. Recall the following key property for encoding the **classical dynamics**.

L_M induces a **Lagrangian subspace** of $L_{\partial M}$:

- $\omega_{\partial M}(\phi, \phi') = 0$ for all $\phi, \phi' \in L_M$.
- If $\phi \notin L_M$ then there is $\phi' \in L_M$ such that $\omega_{\partial M}(\phi, \phi') \neq 0$.

Mini-review: Bosonic field theory (III)

Spacetime is modeled by a collection of **hypersurfaces** and **regions**.



To these geometric structures associate the classical data,

- per **hypersurface** Σ :
  a symplectic vector space $(L_\Sigma, \omega_\Sigma)$,
- per **region** M :
  a **Lagrangian subspace** $L_M \subseteq L_{\partial M}$.

In addition,

- per **hypersurface** Σ :
  a **complex structure** J_Σ .

Fermionic field theory (I)

Starting with a **Lagrangian density** Λ we obtain a **symplectic form** $\tilde{\omega}_\Sigma$ associated to any hypersurface Σ as in the bosonic case.

A fermionic field is generally a section of a **complex vector bundle** (associated with the spin bundle). The associated complex structure can be used to produce a **symmetric bilinear form** g_Σ from $\tilde{\omega}_\Sigma$. This (and not $\tilde{\omega}_\Sigma$) is the "correct" object to encode fermionic field theory:

$$g_\Sigma(X, Y) = 2\tilde{\omega}_\Sigma(X, \mathrm{i}Y)$$

(g_Σ can be also be derived directly by already taking into account the "anti-commuting" nature of the fermionic field at the classical level.)

The symmetric form g_Σ arises from the integral of a $(d-1)$-form on a hypersurface. Its sign thus depends on **orientation**: $\boxed{g_{\overline{\Sigma}} = -g_\Sigma}$.

Fermionic field theory (II)

As in the bosonic case the additional ingredient for the **geometric quantization** on a hypersurface is the **complex structure** $J_\Sigma : L_\Sigma \to L_\Sigma$. This has to satisfy $J_\Sigma^2 = -\mathbf{1}$ and $g_\Sigma(J_\Sigma \cdot, J_\Sigma \cdot) = g_\Sigma(\cdot, \cdot)$.

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L_M induces a **hypermaximal neutral subspace** of $L_{\partial M}$:

- $g_{\partial M}(\phi, \phi') = 0$ for all $\phi, \phi' \in L_M$.
- If $\phi \notin L_M$ then there is $\phi' \in L_M$ such that $g_{\partial M}(\phi, \phi') \neq 0$.

There is a **compatibility condition** between $J_{\partial M}$ and L_M.

Fermionic field theory (III)

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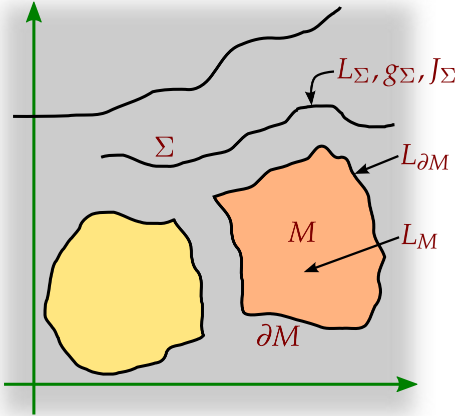

To these geometric structures associate the classical data,

- per **hypersurface** Σ :
  a **real Krein space** (L_Σ, g_Σ),
- per **region** M :
  a **hypermaximal neutral subspace** $L_M \subseteq L_{\partial M}$.

In addition,

- per **hypersurface** Σ :
  a **complex structure** J_Σ .

Comparison of structures per hypersurface

In the bosonic and fermionic case a complex inner product is induced:

$$g_\Sigma(\phi,\phi') = 2\omega_\Sigma(\phi, J_\Sigma\phi') \qquad \omega_\Sigma(\phi,\phi') = \frac{1}{2} g_\Sigma(J_\Sigma\phi,\phi')$$

$$\{\phi,\phi'\}_\Sigma := g_\Sigma(\phi,\phi') + 2\mathrm{i}\omega_\Sigma(\phi,\phi')$$

	bosonic theory	fermionic theory
basic structures	ω_Σ, J_Σ	g_Σ, J_Σ
derived structures	$g_\Sigma, \{\cdot,\cdot\}_\Sigma$	$\omega_\Sigma, \{\cdot,\cdot\}_\Sigma$
orientation change	$J_{\overline{\Sigma}} = -J_\Sigma$, $\omega_{\overline{\Sigma}} = -\omega_\Sigma$, $g_{\overline{\Sigma}} = g_\Sigma$, $\{\cdot,\cdot\}_{\overline{\Sigma}} = \overline{\{\cdot,\cdot\}_\Sigma}$	$J_{\overline{\Sigma}} = -J_\Sigma$, $\omega_{\overline{\Sigma}} = \omega_\Sigma$, $g_{\overline{\Sigma}} = -g_\Sigma$, $\{\cdot,\cdot\}_{\overline{\Sigma}} = -\overline{\{\cdot,\cdot\}_\Sigma}$

Krein spaces

- Orientation change: $g_{\overline{\Sigma}} = -g_{\Sigma}$.
- $L_M \subseteq L_{\partial M}$ **hypermaximal neutral subspace** implies: $g_{\partial M}(\phi, \phi') = 0$ if $\phi, \phi' \in L_M$.

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The spaces (L_Σ, g_Σ) and $(L_\Sigma, \{\cdot, \cdot\}_\Sigma)$ are real and complex **Krein spaces**. They decompose as an othogonal direct sum,

$$L_\Sigma = L_\Sigma^+ \oplus L_\Sigma^-.$$

L_Σ^+ is **positive definite** and L_Σ^- is **negative definite**.

Ecoding dynamics

The **dynamics** in a region M can be encoded equivalently:

- Through a **hypermaximal neutral subspace** $L_M \subseteq L_{\partial M}$
- Through a real linear map $u_M : L_{\partial M} \to L_{\partial M}$ that is
  - (a) involutive,
  - (b) is an anti-isometry,
  - (c) interchanges $L^+_{\partial M}$ and $L^-_{\partial M}$
  - (d) is the identity on L_M.

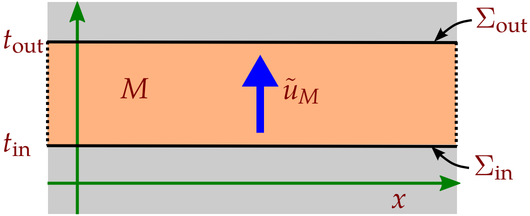
u_M also plays the role of a **real structure (complex conjugation)**. The **compatibility condition** for a **complex structure** $J_{\partial M}$ is that it has to anti-commute with u_M. Given such a complex structure there is a **real orthogonal decomposition** $L_{\partial M} = L_M \oplus J_{\partial M} L_M$. In terms of this,

$$u_M(\xi + J_{\partial M}\eta) = \xi - J_{\partial M}\eta, \quad \forall \xi, \eta \in L_M.$$

Dynamics and evolution

u_M also plays the role of a **generalized evolution** map.

Suppose we have a traditional notion of evolution in M: ∂M decomposes into an "initial part" Σ_{in} and a "final part" Σ_{out} with an evolution map $\tilde{u}_M : L_{\Sigma_{\text{in}}} \to L_{\Sigma_{\text{out}}}$. Then $\tilde{u}_M$ is identical to the restriction of u_M.



Time emerging

More generally, we may consider decompositions $L_{\partial M} = L_{\text{in}} \oplus L_{\text{out}}$ not necessarily arising geometrically so that u_M restricts to a bijection $\tilde{u}_M : L_{\text{in}} \to L_{\text{out}}$. We can think of these as generalized **algebraic evolutions**. In general, there are many such algebraic evolutions in a given region.

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In the **fermionic case**, there exists a preferred algebraic evolution due to the Krein space structure. This is given by $L_{\partial M} = L^+_{\partial M} \oplus L^-_{\partial M}$. u_M thus gives rise to a bijection $\tilde{u}_M : L^+_{\partial M} \to L^-_{\partial M}$. What is more, these evolutions automatically match up correctly under gluing, forming an **algebraic notion of time**. This does not require a metric or any other further geometric structure.

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In the example of the **Dirac field** one may verify that this **algebraic notion of time** coincides with the usual **geometric notion of time**.

Example: The Dirac field

The **Dirac field** in Minkowski spacetime is a 4-dimensional complex vector field X. Its free **Lagrangian** is,

$$\mathcal{L}(X) = -\Im\left(X^\dagger \gamma^0 \gamma^\mu \partial_\mu X\right) - m X^\dagger \gamma^0 X.$$

Here, γ^μ are the usual γ-matrices of high energy physics.
The Lagrangian leads to the symplectic structure,

$$\tilde{\omega}_\Sigma(X, Y) = \int_\Sigma \Im\left(X^\dagger \gamma^0 \gamma^\mu Y\right) n_\mu \mathrm{d}^3 x.$$

This in turn leads to the symmetric bilinear form,

$$g_\Sigma(X, Y) = 2\tilde{\omega}_\Sigma(X, \mathrm{i}Y) = 2\int_\Sigma \Re\left(X^\dagger \gamma^0 \gamma^\mu Y\right) n_\mu \mathrm{d}^3 x.$$

Decomposing the inner product

Rewrite this as

$$g_\Sigma(X, Y) = 2\int_\Sigma \Re\left(X^\dagger P Y\right) \mathrm{d}^3x,$$

with $P(x) = \gamma^0\gamma^\mu n_\mu(x)$ an operator valued function. Since $P(x)$ is **self-adjoint** we can decompose it as,

$$P(x) = P^+(x) + P^-(x)$$

where $P^+(x)$ has only non-negative and $P^-(x)$ only non-positive eigenvalues. Restricting to eigenspaces of $P^+(x)$ or $P^-(x)$ at each point $x \in \Sigma$ leads to subspaces L_Σ^+ and L_Σ^- of the space L_Σ of fields on Σ. Moreover, g_Σ is then **positive definite** on L_Σ^+ and **negative definite** on L_Σ^-. If $P(x)$ is non-degenerate (almost) for all $x \in \Sigma$, then L_Σ is a **Krein space**,

$$L_\Sigma = L_\Sigma^+ \oplus L_\Sigma^-.$$

Spacelike hypersurfaces

Consider an **equal time hypersurface** Σ in Minkowski space. Its future pointing normal vector is,

$$n(x) = (1, 0, 0, 0). \tag{1}$$

This yields $P(x) = \gamma^0\gamma^0 = \mathbf{1}$. Thus, $P^+(x) = P(x)$ and $L_\Sigma^+ = L_\Sigma$. That is, g_Σ is purely positive definite and L_Σ is a real Hilbert space.

The normal vector to an arbitrary future oriented **spacelike hypersurface** Σ can be locally brought into the form (1) by a **Lorentz transformation**. Since by continuity arguments the rank of $P(x)$ cannot change, it must be positive as for (1). That is, $P^+(x) = P(x)$ and L_Σ is a **real Hilbert space**.

Restricting to spacelike hypersurfaces with future orientation yields only Hilbert spaces. This explains why Krein spaces do not appear in the standard approach.

The opposite orientation yields negative definite spaces.

Timelike hypersurfaces

Consider a **timelike hyperplane** Σ in Minkowski space characterized by the normal vector,

$$n(x) = (0, 0, 0, 1). \tag{2}$$

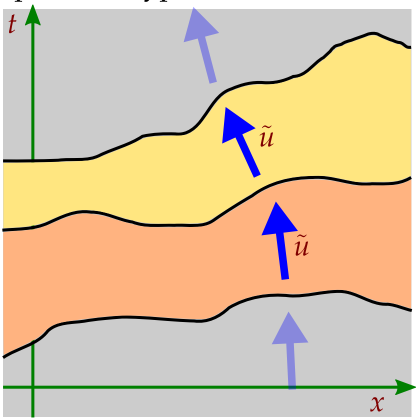
This yields (using the standard or the chiral representation) the operator

$$P(x) = -\gamma^0\gamma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $P^+(x)$ and $P^-(x)$ have both rank 2 and eigenvalues 1 and -1 respectively. L_Σ decomposes non-trivially with the positive and negative definite parts consisting of spinors of rank 2 at each point. Since Lorentz transformations cannot change the rank, an argument analogous to that of the spacelike case shows that this type of decomposition applies to **any timelike hypersurface**.

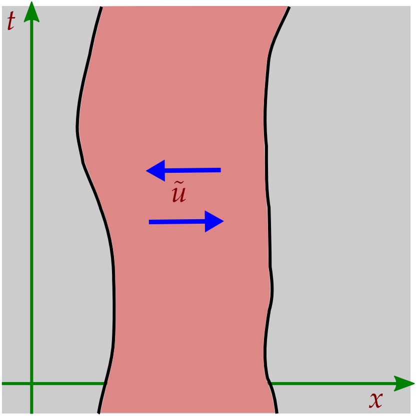
Algebraic time versus geometric time

spacelike hypersurfaces



The algebraic arrow of time coincides with the geometric one.

timelike hypersurfaces



The algebraic arrow of time does not have a definite direction in geometric terms.

Plane waves

Expand solutions of the Dirac equation in Minkowski space in terms of plane waves:

$$X(t,x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \sum_{s=1,2} \left(X_a^s(k) u^s(k) e^{-\mathrm{i}(Et-kx)} + \overline{X_b^s(k)} v^s(k) e^{\mathrm{i}(Et-kx)} \right) .$$

Here, u^s and v^s with $s \in \{1,2\}$ are the usual spinors in momentum space.

Real inner product on plane waves

Consider an equal-time hypersurface located at time t. We take the space L_t of solutions near this hypersurface to be the space of global solutions in terms of plane waves. The **positive definite** real inner product on L_t is,

$$g_t(X,Y) = 2\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} \sum_{s=1,2} \Re\left(\overline{X_a^s(k)}Y_a^s(k) + X_b^s(k)\overline{Y_b^s(k)}\right).$$

Consider now a constant x^3 hypersurface. (Set $z := x^3$.) Again we set L_z to be the global solution space, excluding thus evanescent waves. The **indefinite** real inner product on L_z is,

$$g_z(X,Y) = 2\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} \frac{k_3}{|k_3|} \sum_{s=1,2} \Re\left(\overline{X_a^s(k)}Y_a^s(k) + X_b^s(k)\overline{Y_b^s(k)}\right).$$

The subspaces L_z^+ and L_z^- are distinguished by the direction of the momentum component k_3 that is perpendicular to the hypersurface.

Complex structure

The **complex structure** encodes the distinction between "positive energy" and "negative energy" solutions. More generally we can think of it as distinguishing between propagation in the two opposed normal directions to the hypersurface. This leads to,

$$(J_t X)_a^s(k) = \mathrm{i} X_a^s(k), \quad (J_t X)_b^s(k) = \mathrm{i} X_b^s(k)$$
$$(J_z X)_a^s(k) = \mathrm{i}\frac{k_3}{|k_3|} X_a^s(k), \quad (J_z X)_b^s(k) = \mathrm{i}\frac{k_3}{|k_3|} X_b^s(k).$$

Remarkably the induced **symplectic form** is the same for both types of hypersurfaces,

$$\omega(X, Y) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \sum_{s=1,2} \Im\left(\overline{X_a^s(k)} Y_a^s(k) + \overline{X_b^s(k)} Y_b^s(k)\right).$$

References

Main reference:

R. O., *Free Fermi and Bose Fields in TQFT and GBF*, SIGMA **9** (2013) 028.
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