The field theory of free fermions – quantum theory

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Surprises in fermionic field theory

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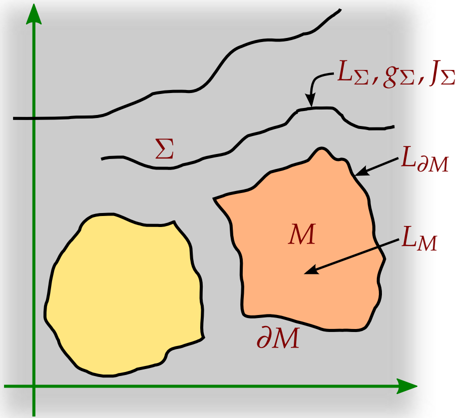
The **gluing anomaly** can be **renormalized**.

As in the bosonic case a gluing anomaly exists. But here it can be renormalized so that no **integrability condition** needs to be imposed.

Today, we consider the **quantum theory**.

Classical fermionic field theory (review)

Spacetime is modeled by a collection of **hypersurfaces** and **regions**.



To these geometric structures associate the classical data,

- per **hypersurface** Σ : a **real Krein space** (L_Σ, g_Σ),
- per **region** M : a **hypermaximal neutral subspace** $L_M \subseteq L_{\partial M}$.

In addition,

- per **hypersurface** Σ : a **complex structure** J_Σ .

Krein space

Recall that a **Krein space** V is a complete **indefinite inner product** space with an orthogonal decomposition

$$V = V^+ \oplus V^-.$$

V^+ is **positive definite** and V^- is **negative definite**. For $v \in V$ define the **signature**,

$$[v] := \begin{cases} 0 & \text{if } v \in V^+ \\ 1 & \text{if } v \in V^- \end{cases}.$$

All Krein spaces considered are **separable**. An **ON-basis** of V is the union of an ON-basis of V^+ with an ON-basis of V^-.

Quantum theory in the amplitude formalism

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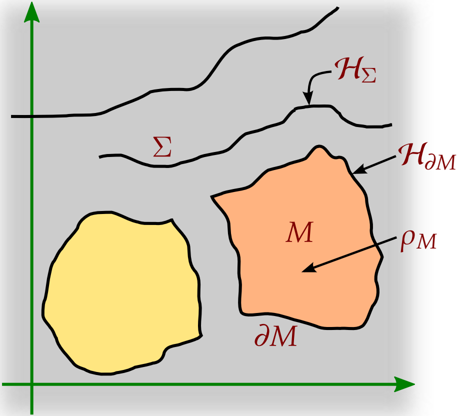
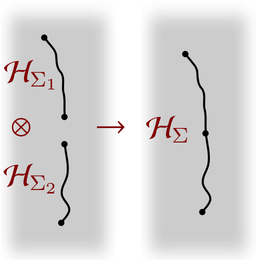


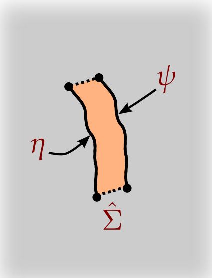
To these geometric structures associate the quantum data,

- per **hypersurface** Σ : an **f-graded Krein space** $\mathcal{H}_\Sigma$,
- per **region** M : a linear **f-graded amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$.

Compared to the purely bosonic case we have a $\mathbb{Z}_2$-grading called **f-grading** on all structures. Moreover, instead of **Hilbert spaces** we have **Krein spaces**.

Core axioms (I)





(T1b) per hypersurface Σ

A conjugate linear f-graded involutive isometry $\iota_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_{\overline{\Sigma}}$.

(T2) per hypersurface decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$

An isometry $\tau_{\Sigma_1,\Sigma_2;\Sigma} : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \to \mathcal{H}_\Sigma$ such that $\tau^{-1}_{\Sigma_2,\Sigma_1;\Sigma} \circ \tau_{\Sigma_1,\Sigma_2;\Sigma}$ is the f-graded transposition $\psi_1 \otimes \psi_2 \mapsto (-1)^{|\psi_1|+|\psi_2|} \psi_2 \otimes \psi_1$.

(T3x) per hypersurface Σ

The amplitude map gives rise to the **inner product** $\langle \iota_{\overline{\Sigma}}(\psi), \eta \rangle_\Sigma := \rho_{\hat{\Sigma}} \circ \tau(\psi \otimes \eta)$.

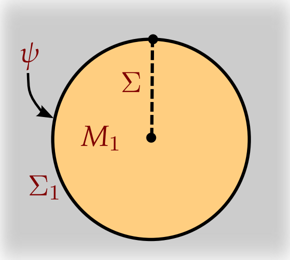
Core axioms (II)

(T5a) per disjoint composition of regions $M = M_1 \sqcup M_2$

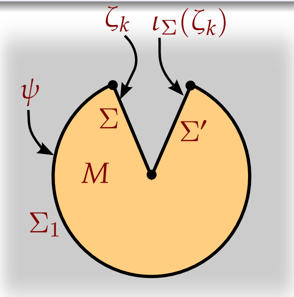
$\rho_M(\tau(\psi_1 \otimes \psi_2)) = \rho_{M_1}(\psi_1)\rho_{M_2}(\psi_2)$. We write $\rho_M = \rho_{M_1} \diamond \rho_{M_2}$.

(T5b) per self-composition of region M to M_1 along Σ

$\rho_{M_1}(\psi) \cdot c_{M,\Sigma} = \sum_k (-1)^{[\zeta_k]} \rho_M(\tau(\psi \otimes \zeta_k \otimes \iota_\Sigma(\zeta_k)))$.



$= \sum_k (-1)^{[\zeta_k]}$



$\{\zeta_k\}_{k \in I}$ ON-basis of $\mathcal{H}_\Sigma$. $c_{M,\Sigma}$ **gluing anomaly**.

Fock-Krein space (I)

We distinguish bosonic and fermionic case via

$$\kappa := 1 \quad \text{in the bosonic case,} \qquad \kappa := -1 \quad \text{in the fermionic case.}$$

Given a Krein space L, the **Fock-Krein space** $\mathcal{F}(L)$ over L is the completion of an $\mathbb{N}_0$-graded Krein space,

$$\mathcal{F}(L) = \widehat{\bigoplus_{n=0}^{\infty}} \mathcal{F}_n(L),$$

$$\mathcal{F}_n(L) := \{\psi : L \times \cdots \times L \to \mathbb{C}\ n\text{-lin. cont. } : \psi \circ \sigma = \kappa^{|\sigma|}\psi, \forall \sigma \in S^n\}.$$

There is a natural $\mathbb{Z}_2$-grading. In the bosonic case it is trivial, i.e., $|\psi| = 0$ for all $\psi \in \mathcal{F}(L)$. In the fermionic case it is,

$$|\psi| := \begin{cases} 0 & \text{if } \psi \in \mathcal{F}_n(L), n \text{ even} \\ 1 & \text{if } \psi \in \mathcal{F}_n(L), n \text{ odd.} \end{cases}$$

Fock-Krein space (II)

We write $\psi = \sum_{n=0}^{\infty} \psi_n$ for $\psi \in \mathcal{F}(L)$ decomposed into $\psi_n \in \mathcal{F}_n(L)$.

The **inner product** in Fock-Krein space is,

$$\langle \psi', \psi \rangle := \sum_{n=0}^{\infty} n!\, 2^n \sum_{k_1,\dots,k_n \in I} (-1)^{[\zeta_{k_1}]+\cdots+[\zeta_{k_n}]} \overline{\psi'_n(\zeta_{k_1},\dots,\zeta_{k_n})} \psi_n(\zeta_{k_1},\dots,\zeta_{k_n})$$

This makes $\mathcal{F}(L)$ into a **Krein space** as well.

Quantization: State spaces

For each **hypersurface** Σ we define the corresponding **state space** $\mathcal{H}_\Sigma$ to be the Fock-Krein space $\mathcal{F}(L_\Sigma)$.

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For all $n \in \mathbb{N}_0$ define $\iota_{\Sigma,n} : \mathcal{F}_n(L_\Sigma) \to \mathcal{F}_n(L_{\overline{\Sigma}})$ by,

$$(\iota_{\Sigma,n}(\psi))(\xi_1,\ldots,\xi_n) := \overline{\psi(\xi_n,\ldots,\xi_1)}.$$

Taking these maps together for all $n \in \mathbb{N}_0$ defines $\iota_\Sigma : \mathcal{F}(L_\Sigma) \to \mathcal{F}(L_{\overline{\Sigma}})$.

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A decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$ induces a direct sum of Krein spaces $L_\Sigma = L_{\Sigma_1} \oplus L_{\Sigma_2}$. This induces an isomorphism of Fock-Krein spaces

$$\tau_{\Sigma_1,\Sigma_2;\Sigma} : \mathcal{F}(L_{\Sigma_1}) \otimes \mathcal{F}(L_{\Sigma_2}) \to \mathcal{F}(L_\Sigma).$$

This also yields the f-graded transposition,

$$\mathcal{F}(L_{\Sigma_1}) \otimes \mathcal{F}(L_{\Sigma_2}) \to \mathcal{F}(L_{\Sigma_2}) \otimes \mathcal{F}(L_{\Sigma_1}) \;:\; \psi_1 \otimes \psi_2 \mapsto (-1)^{|\psi_1|+|\psi_2|} \psi_2 \otimes \psi_1.$$

Quantization: Amplitudes

Given a region M we recall the real orthogonal decomposition $L_{\partial M} = L_M \oplus J_{\partial M} L_M$ giving rise to the map $u_M : L_{\partial M} \to L_{\partial M}$,

$$u_M(\xi + J_{\partial M}\eta) = \xi - J_{\partial M}\eta, \quad \forall \xi, \eta \in L_M.$$

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The **amplitude** is defined as,

$$\rho_M(\psi) := \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \kappa^n \sum_{k_1,\ldots,k_n \in I} (-1)^{[\zeta_{k_1}]+\cdots+[\zeta_{k_n}]} \psi_{2n}(\zeta_{k_1}, \ldots, \zeta_{k_n}, u_M\zeta_{k_n}, \ldots, u_M\zeta_{k_1})$$

The amplitude vanishes for states with odd particle number.

Main result

This **quantization scheme** yields the data of a quantum theory in terms of the amplitude formalism.

Theorem

With an additional integrability assumption, the core axioms as well as the vacuum axioms are satisfied.

Theorem

In the bosonic case this quantization scheme is equivalent to **holomorphic quantization**. (See talk this afternoon.)

The quantization scheme may be viewed (in various ways) as a **functor** from semiclassical field theories to generalized quantum field theories.

The integrability assumptions amounts to requiring the finiteness of the **gluing anomaly factor**. Without it, **gluing axiom (T5b)** may be violated.

Algebraic time

Recall that $u_M : L_{\partial M} \to L_{\partial M}$ plays the role of a **generalized evolution map** in the classical theory and gives rise in the fermionic case to an **algebraic notion of time** via its restriction

$$\tilde{u}_M : L^+_{\partial M} \to L^-_{\partial M}.$$

We saw that in the Dirac field theory, this coincides with the **geometric notion of time** for the time-interval geometry.

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In the quantum theory we have a decomposition

$$\mathcal{H}_{\partial M} = \mathcal{F}(L^+_{\partial M} \oplus L^-_{\partial M}) = \mathcal{F}(L^+_{\partial M}) \otimes \mathcal{F}(L^-_{\partial M}).$$

The quantum analog of u_M is $U_M : \mathcal{H}_{\partial M} \to \mathcal{H}_{\partial M}$ given by

$$(U_M\psi)(\xi_1, \ldots, \xi_n) := \overline{\psi(u_M\xi_n, \ldots u_M\xi_1)}.$$

This induces an **f-graded** isometric isomorphism, representing the **quantum version** of the algebraic time evolution,

$$\tilde{U}_M : \mathcal{F}(L^+_{\partial M}) \to \mathcal{F}(L^-_{\partial M}).$$

Renormalizing the gluing anomaly (I)

Recall the main **gluing identity** of the **gluing axiom** **(T5b)**:

$$\rho_{M_1}(\psi) \cdot c = \sum_{k \in I} (-1)^{[\zeta_k]} \rho_M(\psi \otimes \zeta_k \otimes \iota_\Sigma(\zeta_k))$$

If all state spaces are **finite-dimensional** the sum on the right hand side is finite. The axiom is then satisfied without any additional **integrability condition** with finite **gluing anomaly factor** c (Theorem). This can only happen in the fermionic case. There, if L_Σ is finite-dimensional so is the Fock space $\mathcal{F}(L_\Sigma)$.

Consider now the set $\{L_{\Sigma,\alpha}\}_{\alpha \in A}$ of all **finite-dimensional** subspaces of L_Σ. This is an **injective system** with the inclusion. Moreover, it induces an **projective system** $\{\mathcal{F}(L_{\Sigma,\alpha})\}_{\alpha \in A}$ of the corresponding Fock-Krein spaces. Define P_α as the orthogonal projector $\mathcal{F}(L_\Sigma) \to \mathcal{F}(L_{\Sigma,\alpha})$.

Renormalizing the gluing anomaly (II)

Consider a “reduced version” of the gluing identity,

$$\rho_{M_1}(\psi) \cdot c_\alpha = \sum_{k \in I} (-1)^{[\zeta_k]} \rho_M(\psi \otimes P_\alpha \zeta_k \otimes \iota_\Sigma(P_\alpha \zeta_k)). \quad (1)$$

This of course will not hold for arbitrary states ψ if we fix α.

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But, (*Theorem*) there exists a set $\{c_\alpha\}_{\alpha\in A}$ such that for any state ψ there is $\beta\in A$ such that for all $\gamma\geq\beta$ the identity (1) holds. This implies,

$$\varinjlim_{\alpha}\left(\rho_{M_1}(\psi)\cdot c_\alpha - \sum_{k\in I}(-1)^{[\zeta_k]}\rho_M(\psi\otimes P_\alpha\zeta_k\otimes\iota_\Sigma(P_\alpha\zeta_k))\right)=0.$$

This is the **renormalized gluing identity**. It is satisfied in the fermionic theory without any **integrability condition**.

Note: The limit $\varinjlim_{\alpha} c_\alpha$ does not exist in general!

The positive formalism for fermions

For a **hypersurface** Σ we consider the space $\mathcal{B}_\Sigma$ of "self-adjoint" operators on $\mathcal{H}_\Sigma$. Here σ **self-adjoint** means,

$$\langle \sigma\xi, \eta\rangle_\Sigma = (-1)^{[\xi]+[\eta]+|\xi|\cdot|\eta|}\langle \xi, \sigma\eta\rangle_\Sigma.$$

Recall that $\mathcal{H}_\Sigma$ is **f-graded**. We write, $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma,0} \oplus \mathcal{H}_{\Sigma,1}$. This induces a **bigrading** on $\mathcal{B}_\Sigma$. Write $\mathcal{B}_{\Sigma,00}$ for the subspace of operators on $\mathcal{H}_{\Sigma,0}$. Then self-adjointness for $\sigma \in \mathcal{B}_{\Sigma,00}$ is self-adjointness in the Hilbert space sense. This is also how we define **positivity**,

$$\sigma \geq 0 \quad \text{iff} \quad (-1)^{[\xi]}\langle \sigma\xi, \xi\rangle_\Sigma \geq 0 \qquad \forall \xi \in \mathcal{H}_{\Sigma,0}$$

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For a **region** M the **probability map** $A_M : \mathcal{B}_{\partial M} \to \mathbb{R}$ is given by,

$$A_M(\sigma) = [\![\boxslash, \sigma]\!]_M = \sum_{k\in I} \overline{\rho_M(\zeta_k)}\rho_M(\sigma\zeta_k)$$

ρ_M vanishes on the complement of $\mathcal{H}_{\partial M,0}$.

A_M vanishes on the complement of $\mathcal{B}_{\partial M,00}$.

Probabilities

What is the **probability** Π for measuring a **boundary condition** P on ∂M given a more general boundary condition Q?

$\mathsf{P}, \mathsf{Q} \in \mathcal{B}_{\partial M,00}$ by the **fermionic superselection rule** [Wick, Wightman, Wigner 1952] and $0 \le \mathsf{P} \le \mathsf{Q}$.

$$\Pi(\mathsf{P}|\mathsf{Q}) = \frac{A_M(\mathsf{P})}{A_M(\mathsf{Q})} = \frac{[\![\boxslash, \mathsf{P}]\!]_M}{[\![\boxslash, \mathsf{Q}]\!]_M}$$

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If P, Q are **projection operators** this reduces to the old probability rule [RO 2005]. Given subspaces $\mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{H}_{\partial M}$ let P be the projector onto $\mathcal{A}$ and Q the projector onto $\mathcal{S}$. Then,

$$\Pi(\mathsf{P}|\mathsf{Q}) = \frac{\sum_{k\in K} |\rho_M(\zeta_k)|^2}{\sum_{k\in J} |\rho_M(\zeta_k)|^2}$$

Here $\{\zeta_k\}_{k\in I}$ reduces on $K \subseteq J \subseteq I$ to ON-bases of $\mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{H}_{\partial M,0}$.

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