EXERCISES 2.2

1 Consider the system of linear algebraic equations

$$\begin{aligned} x_1 - 2x_2 + 3x_5 &= 7 \\ 2x_2 + x_4 + 5x_5 &= -6 \\ x_1 - x_3 + x_4 &= 0 \end{aligned}$$

Identify the coefficient matrix and the augmented matrix of the system.

2 Let

$$A = \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & -1 & 7 & 5 \\ 2 & 3 & -4 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 6 & 1 & 5 \\ 2 & 0 & -3 & 4 \\ 1 & -5 & 0 & 1 \end{pmatrix}$$

Compute $A + B$, $A - B$, $3A$, $-2B$, $5A - 7B$.

3 Let

$$C = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -2 \\ 0 & 5 \end{pmatrix}$$

Compute AC and BC, where A and B are defined in Exercise 2.

4 Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 3 \\ 1 & -2 & 4 \\ 5 & 0 & -7 \end{pmatrix}$$

Compute AB and BA. Is $AB = BA$?

5 Let A be an $m \times n$ matrix, and let 0 be an $m \times n$ zero matrix. Show that $A + 0 = 0 + A = A$ and that $A + (-1)A = 0$.

6 Let A be an $m \times n$ matrix, and let 0 be an $n \times p$ zero matrix. Find $A0$.

7 Let A be an mth order square matrix and I the mth order identity. Show that $AI = IA = A$.

8 Let

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Find a 2×2 matrix B such that $AB = I$. Compute BA. Can the same thing be done if

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

9 Let

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Compute E_1A, E_2A, E_3A where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

Generalize. *Hint:* E_1 is obtained from I by multiplying the second row by k, E_2 is obtained from I by interchanging rows 1 and 3, and E_3 is obtained from I by adding the second row to the first row.

10 Powers of square matrices are defined as follows: $A^1 = A$, $A^2 = AA$, $A^3 = AA^2$, etc. Prove that $A^2 - I = (A - I)(A + I) = (A + I)(A - I)$ and $A^3 - I = (A - I)(A^2 + A + I) = (A^2 + A + I)(A - I)$.

11 Let

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_1 &= b_{11}z_1 + b_{12}z_2 \\ x_2 &= b_{21}z_1 + b_{22}z_2 \\ x_3 &= b_{31}z_1 + b_{32}z_2 \end{aligned}$$

Find c_{11}, c_{12}, c_{21}, and c_{22}, where

$$\begin{aligned} y_1 &= c_{11}z_1 + c_{12}z_2 \\ y_2 &= c_{21}z_1 + c_{22}z_2 \end{aligned}$$

and verify the following matrix notation: $Y = AX$, $X = BZ$, $Y = CZ$ where $C = AB$.

2.3 ELIMINATION METHOD

In this section, we shall take up an elimination method which is general enough to find all solutions of a system of linear algebraic equations (if it has any) and which will, in fact, tell us whether a given system has solutions. The idea is quite simple. We try to eliminate the first variable from all but the first equation, the second variable from all but the first and second equations, the third variable from all but the first, second, and third equations, etc. This will not always be possible, but in the attempt we shall find out what is possible, and it will turn out that this is good enough to achieve our purpose.

Let us return to the example of Sec. 2.2:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ -2x_1 \qquad\quad - x_3 &= -3 \\ -x_1 + x_2 \qquad\quad &= 0 \\ -2x_2 + 4x_3 &= 2 \end{aligned}$$

$(-1,1,0,3,1)$ satisfy the homogeneous equations. In fact, the part of the general solution of the nonhomogeneous equations

$$a\begin{pmatrix}1\\2\\1\\0\\0\end{pmatrix}+b\begin{pmatrix}-1\\1\\0\\3\\1\end{pmatrix}$$

is the general solution of the homogeneous equations. This is the situation, in general, as indicated by the next theorem. This theorem shows that finding the general solution of the homogeneous system goes a long way toward solving the nonhomogeneous system.

Theorem 2.3.6 The general solution of the nonhomogeneous system of equations, $AX = B$, can be obtained by adding the general solution of the homogeneous system $AX = 0$ to any particular solution of the nonhomogeneous system.

PROOF Suppose Z is a particular solution of the nonhomogeneous system; then $AZ = B$. Suppose X is *any* other particular solution. Then $AX = B$, and

$$A(X - Z) = AX - AZ = B - B = 0$$

Therefore, $Y = X - Z$ is a solution of the homogeneous equations and so can be obtained from the general solution of the homogeneous equations by the appropriate choice of certain parameters. Hence, $X = Z + Y$, and since X was *any* particular solution, we can obtain the general solution of the nonhomogeneous system by adding the general solution of the homogeneous system to a particular solution of the nonhomogeneous system.

EXERCISES 2.3

1 Which of the following matrices are in reduced form?

$$(a)\ \begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad (b)\ \begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad (c)\ \begin{pmatrix}1&2&3&4\\0&0&1&0\\0&0&0&0\end{pmatrix}$$

$$(d)\ \begin{pmatrix}1&0&1&0&1\\0&1&0&1&0\\0&0&1&0&1\\0&1&0&1&0\end{pmatrix}\qquad (e)\ \begin{pmatrix}0&1&2&3&4\\0&0&1&2&3\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}\qquad (f)\ \begin{pmatrix}1&2&3\\0&1&2\\0&0&0\\0&0&0\end{pmatrix}$$

2 Using the three basic row operations only, change the following matrices to reduced form:

$$(a)\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & 0 & 1 & 2 & 3 \\ 0 & 1 & 5 & -2 & 4 \end{pmatrix}$$

$$(b)\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ -1 & 1 & 5 \\ -2 & 0 & 7 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

3 The following matrix is in reduced form:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Show that by using the basic row operations only, the matrix can be changed to the form

$$\begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Find a and b.

4 (a) If B can be obtained from A by multiplying a row of A by $k \neq 0$, can A be obtained from B by a basic row operation?

(b) If B can be obtained from A by interchanging two rows, can A be obtained from B by a basic row operation?

(c) If B can be obtained from A by adding one row to another, can A be obtained from B by basic row operations?

5 Let $A \to B$ stand for the property "B can be obtained from A by basic row operations." Prove that this is an equivalence relation. In other words, prove that:

(a) $A \to A$.

(b) If $A \to B$, then $B \to A$.

(c) If $A \to B$ and $B \to C$, then $A \to C$.

6 Find all possible solutions of the following systems of equations:

(a)
$$\begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 0 \\ -x_1 + 2x_3 + x_4 &= 0 \\ 2x_1 + x_2 - 2x_4 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 3x_3 &= 2 \\ x_2 - 5x_3 &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_3 - x_4 &= 0 \\ 2x_1 + x_2 - x_3 &= 5 \\ -x_1 + 2x_2 + x_3 + 2x_4 &= 3 \\ 3x_2 - 2x_3 + 5x_4 &= 1 \end{aligned}$$

(d)
$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 + x_5 &= 1 \\ 2x_2 - x_3 + 3x_4 &= 2 \\ 3x_1 + x_2 + 2x_3 - 2x_5 &= -1 \\ 4x_1 + x_2 + 2x_3 + 2x_4 - x_5 &= 2 \end{aligned}$$

(e)
$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 + 2x_5 &= -7 \\ x_2 + x_3 + 2x_4 - x_5 &= 5 \\ x_1 - x_2 + 2x_3 + 2x_4 + 2x_5 &= -1 \end{aligned}$$

7 For parts (d) and (e) of Exercise 6 find the general solutions of the corresponding homogeneous equations, and then find the general solution of the nonhomogeneous equations by adding the general solution of the homogeneous system to a particular solution of the nonhomogeneous system.

8 Let

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be the reduced coefficient matrix of a system of homogeneous equations. Find the general solution of the system. How many arbitrary parameters are there in the solution?

9 Let $AX = 0$ stand for a homogeneous system of linear algebraic equations. Show that if X_1 and X_2 are solutions, then $aX_1 + bX_2$ is a solution for any scalars a and b.

10 Referring to Exercise 3, suppose a reduced system is

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 1 \\ x_2 - x_3 + 2x_5 &= 0 \\ x_4 + x_5 &= 2 \\ x_5 &= -1 \end{aligned}$$

Solve the system by changing (by row operations only) the coefficient matrix to the form given in Exercise 3.

11 The following are reduced coefficient matrices of systems of linear algebraic equations. Which has a unique solution?

$$(a)\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.4 DETERMINANTS

Consider a system of two equations in two unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

(j,i)th element of $\tilde{B}$. Also $\sum_{k=1}^{n} a_{ik}b_{kj}$ is the (i,j)th element of AB. The (i,j)th element of $\widetilde{AB}$ is

$$\sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} \tilde{a}_{kj}\tilde{b}_{ik} = \sum_{k=1}^{n} \tilde{b}_{ik}\tilde{a}_{kj}$$

which is the (i,j)th element of $\tilde{B}\tilde{A}$. This completes the proof.

EXERCISES 2.4

1 Show that there are $n! = n(n-1)(n-2)\cdots 2\cdot 1$ different permutations of the integers $1, 2, 3, \ldots, n$.

2 Given a permutation P_1 of n integers. Obtain P_2 from P_1 by one inversion. Obtain P_3 from P_1 by one inversion. Show that P_3 can be obtained from P_2 by an even number of inversions. Use this to show that evenness or oddness of a permutation is independent of the particular set of inversions used to put it in normal order.

3 Show that a permutation is even or odd according to whether it takes respectively an even or odd number of inversions to obtain it from the normal order.

4 Write out all permutations of 1, 2, 3, 4, and classify them according to whether they are even or odd.

5 Write out the complete expansion of the determinant of a general 4×4 matrix. There should be 24 terms.

6 Evaluate the following determinants

$$(a)\ \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \qquad (b)\ \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \qquad (c)\ \begin{vmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 3 & 1 \\ -1 & 2 & -1 & 3 \\ -2 & 1 & 2 & -3 \end{vmatrix}$$

7 Evaluate the following determinant by showing that it is equal to an upper-triangular determinant (one in which all elements below the principal diagonal are zero).

$$\begin{vmatrix} 1 & 2 & -1 & 3 & -2 \\ 2 & 0 & 4 & -5 & 1 \\ -3 & 1 & 6 & 0 & -7 \\ 0 & 3 & 1 & -5 & 2 \\ -2 & 6 & 3 & -1 & 2 \end{vmatrix}$$

8 Consider the three basic row operations of Sec. 2.3. Show that if a square matrix has a zero determinant, after any number of basic row operations the resulting matrix will have a zero determinant. Also show that if a square matrix has a nonzero determinant, after any number of basic row operations the resulting matrix will have a nonzero determinant.

9 Prove that a square matrix has a zero determinant if and only if it can be reduced to upper-triangular form with at least one zero element on the principal diagonal.

10 Restate Theorems 2.3.3 and 2.3.5 in terms of the vanishing or nonvanishing of the determinant of the coefficient matrix.

11 Determine whether or not the following system of equations has a unique solution by evaluating the determinant of the coefficient matrix:

$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 &= -2 \\ x_1 - x_2 - x_3 + x_4 &= 1 \\ x_1 - 4x_2 - 2x_3 + 2x_4 &= 6 \\ 4x_1 + x_2 - 3x_3 + 3x_4 &= -1 \end{aligned}$$

12 Determine the values of λ for which the following system of equations has a nontrivial solution:

$$\begin{aligned} 9x_1 - 3x_2 \qquad &= \lambda x_1 \\ -3x_1 + 12x_2 - 3x_3 &= \lambda x_2 \\ - 3x_2 + 9x_3 &= \lambda x_3 \end{aligned}$$

13 Determine the values of λ for which the following system of equations has a nontrivial solution:

$$\begin{aligned} x_1 + x_2 &= \lambda x_1 \\ -x_1 + x_2 &= \lambda x_2 \end{aligned}$$

For what values of λ is there a *real* nontrivial solution?

14 Let A and B be $m \times n$ matrices. Show that

(*a*) $\tilde{\tilde{A}} = A$ (*b*) $\widetilde{A+B} = \tilde{A} + \tilde{B}$

(*c*) $\widetilde{A-B} = \tilde{A} - \tilde{B}$ (*d*) $\widetilde{aA} = a\tilde{A}$

2.5 INVERSE OF A MATRIX

One of the most important concepts in the matrix theory is the notion of inverse of a square matrix.

Definition 2.5.1 If A is an $n \times n$ matrix, then A has an inverse if there exists an $n \times n$ matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$, where I is the nth order identity.

Theorem 2.5.1 If a square matrix has an inverse, it is unique.

PROOF Suppose $B \neq A^{-1}$ was also an inverse for A. Then $BA = AB = I$. But then

$$A^{-1} = A^{-1}I = A^{-1}(AB) = (A^{-1}A)B = IB = B$$

EXERCISES 2.5

1 Which of the following matrices have inverses?

$$(a)\ \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\qquad (b)\ \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\qquad (c)\ \begin{pmatrix}1 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & -1\end{pmatrix}$$

$$(d)\ \begin{pmatrix}1 & 2 & 3\\ 0 & -1 & 0\\ 1 & 0 & -1\end{pmatrix}\qquad (e)\ \begin{pmatrix}1 & 3 & -1 & 2\\ 2 & 1 & 3 & 1\\ -1 & 2 & -4 & 1\\ -2 & 1 & 2 & -3\end{pmatrix}$$

$$(f)\ \begin{pmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 2 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\end{pmatrix}$$

2 Find inverses of matrices in Exercise 1 which are nonsingular.

3 Find the inverse of the diagonal matrix†

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{pmatrix}$$

When does a diagonal matrix have an inverse? State a general rule for finding the inverse of a diagonal matrix.

4 Show that an upper-triangular matrix with nonzero elements on the principal diagonal has an inverse which is upper-triangular.

5 Solve the following system of equations using Cramer's rule.

$$\begin{aligned} x_1 + x_2 - x_3 &= 7\\ -x_1 + 2x_2 + x_3 &= -3\\ 2x_1 - x_2 + 3x_3 &= 5 \end{aligned}$$

6 Solve the matrix equation $AB = C$ for B if

$$A = \begin{pmatrix}1 & 1 & -1\\ -1 & 2 & 1\\ 2 & -1 & 3\end{pmatrix}\quad \text{and} \quad C = \begin{pmatrix}2 & 0 & 1\\ 3 & -1 & 4\\ 1 & 5 & -1\end{pmatrix}$$

7 Let $AX = B$ be a system of n equations in n unknowns with $|A| = 0$. Show that there are no solutions unless all n determinants of matrices formed from A by inserting B in the n columns of A are zero. If solutions exist, are they unique? *Hint:* Multiply both sides of $AX = B$ on the left by $\tilde{C}$, the transpose of the matrix of cofactors of A.

† A diagonal matrix is a square matrix with zero elements off the principal diagonal.

8 Let

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Find A^{-1}. Is A orthogonal?

9 If A and B are nonsingular, show that AB is nonsingular. Show that

$$(AB)^{-1} = B^{-1}A^{-1}$$

10 Let A be nonsingular. Show that $\widetilde{A^{-1}} = (\tilde{A})^{-1}$.

11 Let A be nonsingular. Show that $(A^{-1})^{-1} = A$.

12 If $AB = 0$, $B \neq 0$, is A nonsingular?

13 If $A^2 = 0$, is A nonsingular?

14 If A is orthogonal, what are the possible values of $|A|$?

15 If A and B are orthogonal, is AB orthogonal? Is A^{-1} orthogonal?

16 If A and B are unitary, is AB unitary? Is A^{-1} unitary?

17 Define $A^0 = I$ and $A^{-n} = (A^{-1})^n$, n a positive integer. Let A be nonsingular. Prove the general exponential formulas $(A^p)^q = A^{pq}$ and $A^pA^q = A^{p+q}$, where p and q are integers.

18 If C is the matrix of cofactors of the elements of A, what is the value of $|C|$?

*2.6 EXISTENCE AND UNIQUENESS THEOREMS

The two main questions concerning systems of linear algebraic equations (other than methods of finding explicit solutions) are (1) whether solutions exist (*existence*) and (2) if a solution exists, is it unique (*uniqueness*)? We have dealt with these questions to some extent in Sec. 2.3. In this section, we shall give a more systematic discussion of these two questions, but first we must introduce a new concept about matrices which can be defined in terms of determinants.

Every matrix, whether square or not, has square matrices in it which can be obtained by deleting whole rows and/or whole columns. There are of course only a finite number of such square matrices. Suppose we compute the determinants of all these matrices. We define the *rank* of the original matrix in terms of these determinants.

Definition 2.6.1 The rank of a matrix A is the order of the largest order nonsingular matrix which can be obtained from A by deleting whole rows and/or whole columns. We denote this number by rank (A); rank $(0) = 0$.

$$
\to \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 2 & 1 & -3 & 1 \\ 0 & 3 & -5 & 3 & -4 \\ 0 & 3 & -5 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 2 & 1 & -3 & 1 \\ 0 & 3 & -5 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

It is apparent that the rank of the coefficient matrix equals the rank of the augmented matrix, which is 3. Therefore, the system has a solution, but it is not unique because $3 < 4$, the number of unknowns. The general solution of the system will contain one arbitrary parameter.

EXERCISES 2.6

1 Prove that the rank of the augmented matrix of a system of linear algebraic equations cannot be less than the rank of the coefficient matrix.

2 Consider the three basic column operations on matrices: (1) multiplication of a column by $k \neq 0$, (2) interchange of two columns, (3) addition of one column to another. Prove that these column operations cannot change the rank of a matrix.

3 State Theorem 2.3.3 in terms of the rank of the coefficient matrix. Prove your version.

4 State Theorem 2.3.5 in terms of the rank of the coefficient matrix. Prove your version.

5 Determine whether the following systems of equations have solutions; if so, are they unique?

(*a*)
$$
\begin{aligned}
x_1 + x_2 - x_3 &= 1 \\
2x_1 + x_2 + 3x_3 &= 2 \\
x_2 - 5x_3 &= 1
\end{aligned}
$$

(*b*)
$$
\begin{aligned}
x_1 + + 2x_3 - x_4 &= 0 \\
2x_1 + x_2 - x_3 &= 5 \\
-x_1 + 2x_2 + x_3 + 2x_4 &= 3 \\
3x_2 - 2x_3 + 5x_4 &= 1
\end{aligned}
$$

(*c*)
$$
\begin{aligned}
2x_1 - x_2 + x_3 &= 1 \\
x_1 + 2x_2 - 3x_3 &= 0 \\
-x_1 + 3x_2 - x_3 &= 2 \\
x_1 - x_2 - 2x_3 &= -3
\end{aligned}
$$

(*d*)
$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 - x_4 + x_5 &= 5 \\
-x_1 + 3x_2 + 4x_3 + x_4 &= 2 \\
2x_1 + x_3 - 2x_4 + 2x_5 &= -1 \\
x_2 + 5x_3 - 2x_4 - x_5 &= 0
\end{aligned}
$$

6 Consider the system of equations $AX = B$, where A is $m \times m$. Suppose that rank $(A) = m - 1$. Prove that the system has a solution if and only if the m matrices formed from A by replacing the columns by B are all singular.

EXERCISES 3.2

1 Let $\mathbf{u} = (1,-2,1)$ and $\mathbf{v} = (3,1,-4)$. Compute $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u}$, $-\frac{1}{2}\mathbf{v}$, and $-\mathbf{u} + 2\mathbf{v}$. Make sketches of arrows representing each of these vectors.

2 Let $\mathbf{u} = (1,-2,1)$. Compute $|\mathbf{u}|$ and θ_1, θ_2, θ_3, the minimum nonnegative angles from the positive coordinate axes to the arrow of the vector.

3 A force in pounds is exerted on a body as designated by the vector $(3,1,-2)$. Find the magnitude and direction of this force. Another force of $(-4,5,3)$, also measured in pounds, is exerted on the same body. Find the combined effect (resultant) of the two forces acting together. Find the magnitude and direction of the resultant. *Hint*: The resultant is the vector sum of the two forces.

4 Let $\mathbf{u} = (2,0,-3)$ and $\mathbf{v} = (-1,4,5)$. Find the cosine of the angle between the two vectors. If $\cos\theta_1$, $\cos\theta_2$, $\cos\theta_3$ are the direction cosines of $\mathbf{u}$ and $\cos\phi_1$, $\cos\phi_2$, $\cos\phi_3$ are the direction cosines of $\mathbf{v}$, show that $\cos\theta = \cos\theta_1\cos\phi_1 + \cos\theta_2\cos\phi_2 + \cos\theta_3\cos\phi_3$, where θ is the angle between the two vectors.

5 Consider a nonzero vector $\mathbf{u}$ represented by the arrow from O to P. Consider the vector $\mathbf{v}$ represented by the arrow from O to Q. The projection of $\mathbf{v}$ on $\mathbf{u}$ is defined to be the vector O to N, where N is the foot of the perpendicular drawn from Q to the line of $\mathbf{u}$. Show that this projection is given by

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u}$$

Compute the projection of $(1,-2,1)$ on $(3,1,-4)$.

6 Consider a plane represented implicitly by $ax + by + cz = d$. Consider a vector $\mathbf{v}$ represented by an arrow from the point (x_0,y_0,z_0) in the plane to the point Q. The projection of $\mathbf{v}$ on the plane is defined to be the vector from (x_0,y_0,z_0) to N, the foot of the perpendicular drawn from Q to the plane. Let $\mathbf{u} = (a,b,c)$. Show that the projection of $\mathbf{v}$ on the plane is given by

$$\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u}$$

Compute the projection of $(2,3,-1)$ on the plane given by $x - 3y + 2z = 7$.

7 Show that the distance from the point (x_1,y_1,z_1) to the plane represented by $ax + by + cz = d$ is given by

$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Compute the distance from $(1,-2,3)$ to the plane represented by $2x - y + 3z = 7$.

8 Find an equation of the line through the point $(1,2,-3)$ in the direction of the vector $(-2,3,5)$.

9 Find an equation of the line through the two points $(3,-5,7)$ and $(-2,1,4)$.

10 Find an equation of the line through the point $(4,3,-5)$ and perpendicular to the plane given by $2x + 3y + 4z = 3$.

11 Find an equation of the line of intersection of the two planes given by $2x + 3y - 4z = 5$ and $-x + 7y + 5z = 2$.

12 Find an equation of the plane through the origin and parallel to the vectors $(1,-2,3)$ and $(5,0,7)$.

13 Find an implicit representation of the plane containing the three points $(1,-3,4)$, $(2,5,-1)$, $(0,7,-4)$.

14 Find an equation of the plane containing the two lines $(x,y,z) = (1,-3,4) + t(2,5,-1)$ and $(x,y,z) = (1,-3,4) + s(0,7,-4)$.

15 Find an equation of the plane containing the point $(1,2,3)$ and perpendicular to the line $(x,y,z) = (1,-3,4) + t(2,5,-1)$.

16 One way to compute a three-dimensional vector perpendicular to two given vectors is to use the *vector product*. Let $\mathbf{u} = (u_1,u_2,u_3)$ and $\mathbf{v} = (v_1,v_2,v_3)$ be two nonzero and nonparallel vectors. Show that

$$\mathbf{w} = \mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

is a vector perpendicular (orthogonal) to both $\mathbf{u}$ and $\mathbf{v}$. The vector product has no counterpart in other spaces.

17 Show that $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. *Hint*: Write $\mathbf{u} = |\mathbf{u}|(\cos\theta_1, \cos\theta_2, \cos\theta_3)$ and $\mathbf{v} = |\mathbf{v}|(\cos\phi_1, \cos\phi_2, \cos\phi_3)$.

18 Repeat Exercise 13, using the vector product to compute a vector perpendicular to the required plane.

19 Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be three vectors whose arrow representations from the origin form the three edges of a parallelepiped. Show that $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$ is equal to the volume of the parallelepiped.

20 Show that:

(*a*) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

(*b*) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$.

(*c*) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = -\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$.

21 Consider three planes given implicitly by

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The intersection of these three planes could be (1) empty, (2) a line, (3) a plane, or (4) a point. In terms of the solutions of the three equations in three unknowns give an algebraic condition for each of the cases.

22 A point moves along a curve in three-dimensional space with its displacement vector from the origin given by the vector-valued function of time $\mathbf{r}(t) = (x(t),y(t),z(t))$, where x, y, and z have first and second derivatives. The first derivative $\mathbf{r}'(t) = \mathbf{v}(t) = (x'(t),y'(t),z'(t))$ is called the *velocity* of the point, and the second derivative $\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{v}'(t) = (x''(t),y''(t),z''(t))$ is called the *acceleration*.

If $\mathbf{v}(t) \neq \mathbf{0}$, show that the velocity is tangent to the curve. The magnitude of the velocity is called the *speed*. If the speed is constant and positive, show that the acceleration is *normal* to the curve (perpendicular to the tangent vector).

23 If the speed $s(t) = |\mathbf{v}(t)|$ of a point is not zero, show that $\mathbf{a}(t) = s'(t)\mathbf{T} + s(t)|\mathbf{T}'|\mathbf{n}$, where $\mathbf{T}$ is a unit tangent vector and $\mathbf{n}$ is a unit normal vector.

3.3 AXIOMS OF A VECTOR SPACE

We have already seen several examples of algebraic systems which, at least in certain respects, behave similarly. We have in mind those properties of complex numbers, two-dimensional euclidean vectors, three-dimensional euclidean vectors, and matrices with respect to addition and multiplication by a scalar. We now take the modern mathematical point of view and define abstract systems with those properties we wish to study. These systems we shall call *vector spaces*. This approach will have the distinct advantage that any properties we derive from this definition will be true of all vector spaces, and we shall not have to study each system separately as it comes up. We begin with the axioms for a vector space.

Definition 3.3.1 Consider a system V of objects, called *vectors*, for which we have defined two operations, addition and multiplication by a scalar, either real or complex. Then V is a vector space if these operations satisfy the following properties:

A1 If $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u} + \mathbf{v}$ is in V.
A2 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
A3 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
A4 There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V.
A5 If $\mathbf{u}$ is in V, then there is a vector $-\mathbf{u}$ in V, called the negative of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
M1 If a is a scalar and $\mathbf{u}$ is in V, then $a\mathbf{u}$ is in V.
M2 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
M3 $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
M4 $(ab)\mathbf{u} = a(b\mathbf{u})$.
M5 $1\mathbf{u} = \mathbf{u}$.

If the set of scalars is the set of all real numbers, then we say that V is a *real vector space*. If the set of scalars is the set of all complex numbers, then we say that V is a *complex vector space*.

Multiplying the first equation by a and the second by b and adding, we have

$$\alpha(ax_1 + bx_2) + \beta(ay_1 + by_2) + \gamma(az_1 + bz_2) = 0$$

Therefore, $a\mathbf{u} + b\mathbf{v}$ is in the plane. We argue geometrically to show that these are all the subspaces. If there is a point $(x_0,y_0,z_0) \neq \mathbf{0}$ in the subspace, then the whole line (tx_0,ty_0,tz_0), $-\infty < t < \infty$, is in the subspace. If these are all the points in the subspace, then we have a line through the origin. If there are two points $\mathbf{u} = (x_1,y_1,z_1)$ and $\mathbf{v} = (x_2,y_2,z_2)$, such that $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ are noncollinear, then the subspace contains the plane through the origin given by $a\mathbf{u} + b\mathbf{v}$, where a and b are any real numbers. If these are all the points, then we just have a plane through the origin. If there is a point $\mathbf{w}$ off the plane in the subspace, then we have the whole space because the subspace then contains all points of the form $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, where a, b, and c are any real numbers.

EXAMPLE 3.3.6 Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ be a finite number of vectors from a vector space V. Consider the subset U of all vectors of the form

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + \cdots + c_n\mathbf{u}_n$$

where $c_1, c_2, c_3, \ldots, c_n$ is any set of scalars (real if V is a real vector space or complex if V is a complex vector space). Show that U is a subspace. Let $\mathbf{u}$ be as shown above and

$$\mathbf{v} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \gamma_3\mathbf{u}_3 + \cdots + \gamma_n\mathbf{u}_n$$

Then

$$a\mathbf{u} + b\mathbf{v} = (ac_1 + b\gamma_1)\mathbf{u}_1 + (ac_2 + b\gamma_2)\mathbf{u}_2 + \cdots + (ac_n + b\gamma_n)\mathbf{u}_n$$

so that $a\mathbf{u} + b\mathbf{v}$ is in V. In this case, we say that the set $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ *spans* the subspace U.

EXERCISES 3.3

1 (*a*) Consider a vector space consisting of one vector $\mathbf{0}$ with addition and multiplication defined by (i) $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and (ii) $a\mathbf{0} = \mathbf{0}$. Prove that this space (the *zero space*) is a vector space.
(*b*) Show that all other vector spaces contain an infinite number of vectors.

2 Show that the system of $m \times n$ matrices with complex elements is a complex vector space where addition and multiplication by complex scalars are the usual matrix operations.

3 Show that the system of n-tuples of complex numbers is a complex vector space where addition and multiplication by a scalar are defined as in Example 3.3.2.

4 Show that the collection of all polynomials of degree n or less in the complex variable z with complex coefficients is a complex vector space, with addition and multiplication by a complex scalar as defined in Example 3.3.3.

5 Consider the collection of all real-valued Riemann-integrable functions of the real variable x defined on the interval $\{x \mid 0 \le x \le 1\}$. Show that this is a real vector space with addition and multiplication by a scalar as defined in Example 3.3.4. Is the space of Example 3.3.4 a subspace of this space? Is it a proper subspace?

6 Consider the collection of all real-valued differentiable functions of the real variable x defined on the interval $\{x \mid a \le x \le b\}$. Show that this is a real vector space with addition and multiplication by a scalar as defined in Example 3.3.4. Is this a subspace of real-valued continuous functions on $\{x \mid a \le x \le b\}$? Is it a proper subspace?

7 Given a vector space V. Prove that in V, $a\mathbf{u} = \mathbf{0}$ implies $a = 0$, $\mathbf{u} = \mathbf{0}$, or both.

8 Characterize all the subspaces of R^2.

9 Consider the system of homogeneous linear algebraic equations $AX = 0$ in the real variables $(x_1, x_2, \ldots, x_n)$ with real coefficients a_{ij}, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Any solutions will be found in R^n. Prove that the set of all solutions is a subspace of R^n.

3.4 DEPENDENCE AND INDEPENDENCE OF VECTORS

We now come to the important concepts of dependence and independence of vectors. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is some finite set of vectors from a vector space V. A *linear combination* of these vectors is a sum of the form

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \sum_{i=1}^{k} c_i\mathbf{u}_i$$

where $c_1, c_2, \ldots, c_k$ are scalars. Obviously such a linear combination is $\mathbf{0}$ if all the c's are zero. We say that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are *dependent* if $\sum_{i=1}^{k} c_i\mathbf{u}_i = \mathbf{0}$ for some set of scalars, not all zero. If this is impossible, then we say that the set of vectors is *independent*.

Definition 3.4.1 A set of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in V is dependent if there is a linear combination $\sum_{i=1}^{k} c_i\mathbf{u}_i = \mathbf{0}$ with the scalars $c_1, c_2, \ldots, c_k$ not all zero. If $\sum_{i=1}^{k} c_i\mathbf{u}_i = \mathbf{0}$ only for $c_1 = c_2 = \cdots = c_k = 0$, then the set of vectors is independent.

EXERCISES 3.4

1 Show that any set of vectors from a vector space V is dependent if the set contains the zero vector.

2 Show that in R^3 a set of two nonzero vectors is dependent if and only if they are parallel.

3 Show that in R^3 a set of three nonzero vectors is dependent if and only if they are all parallel to a given plane.

4 Show that in R^3 any set of three mutually perpendicular vectors is independent. This shows by Theorem 3.4.2 that any vector in R^3 can be written as a linear combination of a given set of three mutually perpendicular vectors.

5 Determine whether the vectors (1,0,1), (0,1,1), (1,−1,1) are dependent or independent in R^3. Can the vector (1,2,3) be expressed as a linear combination of these vectors?

6 Determine whether the vectors (1,−1,1,−1), (2,0,−3,1), (0,1,2,−1), (4,−3,−3,0) are dependent or independent in R^4. Can the vector (1,2,3,4) be expressed as a linear combination of these vectors?

7 Determine whether the vectors (1,0,1,0), (0,2,−1,3), (1,4,2,−1) are dependent or independent in R^4. Can the vector (4,6,7,−5) be expressed as a linear combination of these vectors?

8 Determine whether the vectors $(1,i,-1)$, $(1+i, 0, 1-i)$, $(i,-1,-i)$ are dependent or independent in C^3.

9 In the space of continuous real-valued functions defined on the interval $\{x \mid 0 \le x \le 1\}$, are the functions x, $x^2 - 1$, and $x^2 + 2x + 1$ dependent or independent?

10 In the space of real-valued polynomials of degree 3 or less, show that the polynomials

$$\begin{aligned} p_0(x) &= -\tfrac{1}{6}(x-1)(x-2)(x-3) \\ p_1(x) &= \tfrac{1}{2}x(x-2)(x-3) \\ p_2(x) &= -\tfrac{1}{2}x(x-1)(x-3) \\ p_3(x) &= \tfrac{1}{6}x(x-1)(x-2) \end{aligned}$$

are independent. Show that any real-valued polynomial $p(x)$ of degree 3 or less can be expressed uniquely by

$$p(x) = p(0)p_0(x) + p(1)p_1(x) + p(2)p_2(x) + p(3)p_3(x)$$

11 Let A be an $n \times n$ matrix with real elements. Show that the following statements are all equivalent:

(*a*) A is nonsingular.
(*b*) $AX = 0$ has only the trivial solution.
(*c*) The columns of A are independent in R^n.
(*d*) The rows of A are independent in R^n.

12 Show that the functions $1, x + 1, x^2 + x + 1, \ldots, x^k + x^{k-1} + \cdots + x + 1$ are independent on the interval $\{x \mid 0 \le x \le 1\}$. Does the result depend on k?

13 Show that the functions e^x, e^{2x}, e^{3x} are independent on the interval $\{x \mid 0 \le x \le 1\}$.

14 Are the functions e^x, xe^x, x^2e^x dependent or independent on the interval $\{x \mid 0 \le x \le 1\}$?

15 Are the functions $\sin x$, $\cos x$, $x \sin x$, $x \cos x$ dependent or independent on the interval $\{x \mid 0 \le x \le 2\pi\}$?

16 Suppose $f(x)$ and $g(x)$ satisfy the differential equation $y'' + p(x)y = 0$ on the interval $\{x \mid 0 \le x \le 1\}$, where $p(x)$ is continuous. All functions are real-valued. Show that the Wronskian of f and g is constant. If $f(0) = 1, f'(0) = 0, g(0) = 0, g'(0) = 1$, are f and g independent? *Hint*: Compute $(f'g - g'f)'$.

3.5 BASIS AND DIMENSION

We have already seen several examples of vector spaces in which every vector can be expressed as a linear combination of some finite set of vectors. The collection of polynomials of Example 3.3.3 can all be expressed as linear combinations of the polynomials $1, x, x^2, \ldots, x^n$. Theorem 3.4.2 shows that any vector in R^n can be expressed as a linear combination of some independent set of n vectors. In Example 3.3.6, we showed that the collection of all linear combinations of a given set of vectors in V forms a subspace of V. But a subspace is a vector space, so this is another example of a vector space with such a representation. We formalize this situation by giving the following definition.

Definition 3.5.1 A given set $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ from a vector space V is said to *span* V if every vector in V can be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.

Theorem 3.5.1 If V is not the zero space and is spanned by a set $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$, then there is an independent subset which also spans V.

PROOF If V is not the zero space (consisting of the zero vector only), there is at least one nonzero vector in V. Therefore, there is at least one nonzero vector in the given spanning set. Hence, there are subsets of the spanning set which are independent. Now suppose the given set $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is dependent. Then $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$ with the c's not all zero. Suppose $c_k \ne 0$ (if $c_k = 0$, we can relabel the vectors so that the kth scalar is different from zero). Then

$$\mathbf{u}_k = \frac{-c_1}{c_k}\mathbf{u}_1 + \frac{-c_2}{c_k}\mathbf{u}_2 + \cdots + \frac{-c_{k-1}}{c_k}\mathbf{u}_{k-1}$$

Now since any vector in V can be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$, and since $\mathbf{u}_k$ can be written in terms of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$, the which shows that $\mathbf{v}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ span V. Proceeding as in the proof of Theorem 3.5.3, casting out u's and replacing them with v's, we eventually end up with n of the v's as a spanning set. But then, since $m > n$, there are v's which can be expressed as linear combinations of n of the v's, contradicting the independence of the v's. This completes the proof.

There are vector spaces with arbitrarily large† independent sets of vectors (see Examples 3.4.4 and 3.4.5). These vector spaces cannot be finite-dimensional. We simply say that such spaces are infinite-dimensional.

Definition 3.5.5 A vector space with independent sets of arbitrarily many vectors is said to be infinite-dimensional.

We conclude this section with a theorem which will simplify the search for bases of finite-dimensional vector spaces.

Theorem 3.5.5 In an n-dimensional vector space ($n \geq 1$) a set of n vectors is a basis if (i) it spans the space or (ii) it is independent.

PROOF (i) If a set of vectors spans the space but is dependent, then there is a subset of m vectors which spans the space and is independent with $m < n$. But this implies that there is a basis with fewer than n vectors, contradicting Theorem 3.5.3.

(ii) If a set of n vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is independent but does not span the space, there is at least one vector $\mathbf{v}$ which cannot be written as a linear combination of the u's. Consider a linear combination

$$c\mathbf{v} + c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$$

If $c \neq 0$, then $\mathbf{v}$ is a linear combination of the u's. Therefore, $c = 0$. If any of $c_1, c_2, \ldots, c_n$ is not zero, then the u's are dependent. Therefore, the set $\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is independent. But this contradicts Theorem 3.5.4. Hence, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span the space.

EXERCISES 3.5

1 Determine which of the following sets of vectors, if any, is a basis for R^3:
(*a*) (1,1,1), (1,−1,1), (0,1,0)
(*b*) (1,2,3), (1,0,1), (0,−1,2)
(*c*) (0,0,1), (0,1,−1), (0,−1,1)

† Here "large" refers to the number of vectors in the set.

2 Each of the following sets of vectors spans some subspace of R^4. Find the dimension of the subspace in each case.
(*a*) (1,1,1,1), (1,0,1,0), (0,1,0,1), (1,−1,1,−1)
(*b*) (1,2,3,4), (−1,0,1,3), (0,1,−1,2), (1,2,−1,4)
(*c*) (1,2,3,0), (1,0,1,0), (0,−1,2,0), (−1,1,3,0)
(*d*) (−1,3,4,2), (1,−3,−4,−2), (−2,6,8,4), (2,−6,−8,−4)

3 Show that the vectors (1,1,1), (1,−1,1), (2,0,3) form a basis for R^3. Find the coordinates of (4,5,6) with respect to this basis.

4 The vectors (1,1,1,1), (1,0,1,0), (0,1,0,1), (1,−1,1,−1) span a subspace of R^4. Is the vector (4,−2,4,−2) in that subspace? If so, express the vector as a linear combination of the given vectors.

5 Which of the following sets of vectors is a basis for C^4?
(*a*) (i,0,0,0), (0,i,0,0), (0,0,i,0), (0,0,0,i)
(*b*) (1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)
(*c*) (1,1,1,1), (i,i,i,i), (0,1,0,1), (i,0,i,0)

6 Show that the space of differentiable real-valued functions defined on the interval $\{x \mid 0 \leq x \leq 1\}$ is infinite-dimensional.

7 Show that the space of Riemann-integrable real-valued functions defined on the interval $\{x \mid 0 \leq x \leq 1\}$ is infinite-dimensional.

8 Prove that a vector space with an infinite-dimensional subspace is infinite-dimensional. Is the converse true?

9 Show that the set of vectors (1,1,1,1), (0,1,0,1), (1,0,2,0) is independent in R^4. Construct a basis in R^4 containing the three vectors.

10 Let V be an n-dimensional vector space. Given a set of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$, $k \leq n$, which are independent, prove that there is a basis for V containing the given set.

3.6 SCALAR PRODUCT

We have already seen a couple of vector spaces in which it was useful to introduce a kind of scalar-valued product between pairs of vectors. We did this in the systems of two- and three-dimensional euclidean vectors when we defined a scalar product† (dot product). The concept is, in fact, so useful that we shall now postulate a set of properties for a scalar product in general and study the properties of such a product. Then any particular vector space which has a suitable scalar product will have these additional properties. It is not necessary to have a scalar product defined in the space in order to have a vector space, but in most cases of interest to us we shall have a scalar product.

† This is not to be confused with multiplication by a scalar, where the product is between a scalar and a vector with the result a vector.